

Lower Bounds on Signed Total Double Roman k -domination in Graphs

Laila Shahbazi^a, Hossein Abdollahzadeh Ahangar^{b*}, Rana Khoeilar^a,
Seyed Mahmoud Sheikholeslami^a

^aDepartment of Mathematics, Azarbaijan Shahid Madani University, Tabriz,
I.R. Iran

^bDepartment of Mathematics, Babol Noshirvani University of Technology,
Shariati Ave., Babol, I.R. Iran, Postal Code: 47148-71167

E-mail: l.shahbazi@azaruniv.ac.ir

E-mail: ha.ahangar@nit.ac.ir

E-mail: khoeilar@azaruniv.ac.ir

E-mail: s.m.sheikholeslami@azaruniv.ac.ir

ABSTRACT. A signed total double Roman k -dominating function (STDRkDF) on an isolated-free graph $G = (V, E)$ is a function $f : V(G) \rightarrow \{-1, 1, 2, 3\}$ such that (i) every vertex v with $f(v) = -1$ has at least two neighbors assigned 2 under f or at least one neighbor w with $f(w) = 3$, (ii) every vertex v with $f(v) = 1$ has at least one neighbor w with $f(w) \geq 2$ and (iii) $\sum_{u \in N(v)} f(u) \geq k$ holds for any vertex v . The weight of an STDRkDF is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$. The signed total double Roman k -domination number $\gamma_{stdR}^k(G)$ is the minimum weight among all signed total double Roman k -dominating functions on G . In this paper we present sharp lower bounds for $\gamma_{stdR}^2(G)$ and $\gamma_{stdR}^3(G)$ in terms of the order and the size of the graph G .

Keywords: Signed total double Roman k -dominating function, Signed total double Roman k -domination number.

2000 Mathematics subject classification: 05C69, 05C05.

*Corresponding Author

Received 21 November 2020; Accepted 25 May 2022

©2025 Academic Center for Education, Culture and Research TMU

1. INTRODUCTION

In this paper we only consider finite isolated free graphs without loops and multiple edges. For notation and graph theory terminology we follow [15] in general. Let $G = (V, E)$ be a simple graphs without isolated vertices with vertex set $V = V(G)$ and edge set $E = E(G)$. The order $|V|$ of G is denoted by $n = n(G)$. For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in V(G) \mid uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $\deg_G(v) = |N(v)|$. The *minimum degree* and the *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. For any set S of vertices of a graph G and any vertex $v \in V(G)$, we denoted $\deg_S(v)$, for the number of neighbors of v in S . We write P_n for the *path* of order n , C_n for the *cycle* of length n and K_n for the complete graph of order n . For two disjoint subsets S and T of $V(G)$, we write $[S, T]$ for the set of edges of G joining S to T . If K is a subset of $\mathbb{Z}$ and f is a function from $V(G)$ into K , then we write $V_i = \{v \in V(G) \mid f(v) = i\}$ for each $i \in K$.

In 2016, Beeler et al. [10] defined the double Roman domination as follows. A function $f : V \rightarrow \{0, 1, 2, 3\}$ is a *double Roman dominating function* (DRDF) on a graph G if the following conditions hold.

- (i) If $f(v) = 0$, then v must have at least one neighbor in V_3 or at least two neighbors in V_2 .
- (ii) If $f(v) = 1$, then v must have at least one neighbor in $V_2 \cup V_3$.

The *double Roman domination number* $\gamma_{dR}(G)$ equals the minimum weight of a double Roman dominating function on G . The double Roman domination has been studied by several authors [1, 2, 4, 5]. For further results on several new variations of Roman domination see [6, 7, 8, 11, 14].

Amjadi et al. [9], introduced a new variation of double Roman domination as signed double Roman k -domination number. A *signed double Roman k -dominating function* (SDRkDF) on a graph $G = (V, E)$ is a function $f : V(G) \rightarrow \{-1, 1, 2, 3\}$ such that (i) every vertex v with $f(v) = -1$ is adjacent to at least two vertices assigned a 2 or to at least one vertex w with $f(w) = 3$, (ii) every vertex v with $f(v) = 1$ is adjacent to at least one vertex w with $f(w) \geq 2$ and (iii) $f(v) = \sum_{u \in N[v]} f(u) \geq k$ holds for any vertex v . The weight of an SDRkDF f is the value $\omega(f) = \sum_{u \in V(G)} f(u)$. The *signed double Roman k -domination number* $\gamma_{sdR}^k(G)$ is the minimum weight of an SDRkDF on G . For further results on signed double Roman k -domination see [7, 16].

A *signed total double Roman k -dominating function* (STDRkDF) on a graph $G = (V, E)$ is a function $f : V(G) \rightarrow \{-1, 1, 2, 3\}$ such that (i) every vertex v with $f(v) = -1$ is adjacent to at least two vertices assigned a 2 or to at least one vertex w with $f(w) = 3$, (ii) every vertex v with $f(v) = 1$ is adjacent to at least one vertex w with $f(w) \geq 2$ and (iii) $f(v) = \sum_{u \in N(v)} f(u) \geq k$ holds for

any vertex v . The weight of an STDRkDF f is the value $\omega(f) = \sum_{u \in V(G)} f(u)$. The *signed total double Roman k -domination number* $\gamma_{stdR}^k(G)$ is the minimum weight of an STDRkDF on G . For an STDRkDF f , let $V_i(f) = \{v \in V \mid f(v) = i\}$. In the context of a fixed STDRkDF, we suppress the argument and simply write V_{-1} , V_1 , V_2 and V_3 . Since this partition determines f , we can equivalently write $f = (V_{-1}, V_1, V_2, V_3)$. The concept of signed total double Roman k -domination was introduced and investigated by Shahbazi et al. [12]. The special case $k = 1$ is the usual signed total double Roman domination which has been investigated in [13]. Shahbazi et al. [13] proved that for any connected graph G of order $n \geq 3$ and size m , $\gamma_{stdR}^t(G) \geq \frac{11n-12m}{3}$.

Following the same idea, in this paper we present sharp lower bounds for $\gamma_{stdR}^2(G)$ and $\gamma_{stdR}^3(G)$ in terms of the order and the size of the graph G .

We make use of the following results in this paper.

Proposition A. [12] For $n \geq 2$,

$$\gamma_{stdR}^2(P_n) = \begin{cases} 4 & \text{if } n = 2, 3 \\ n & \text{if } n \equiv 0 \pmod{4} \\ n + 2 & \text{if } n \equiv 1, 3 \pmod{4} \\ n + 3 & \text{if } n \equiv 2 \pmod{4}. \end{cases}$$

Proposition B. [12] For $n \geq 2$,

$$\gamma_{stdR}^3(P_n) = \begin{cases} \frac{3n}{2} + 3 & \text{if } n \equiv 2 \pmod{4} \\ \lceil \frac{3n}{2} \rceil + 2 & \text{otherwise.} \end{cases}$$

Proposition C. [12] For $n \geq 3$,

$$\gamma_{stdR}^2(C_n) = \begin{cases} 4 & \text{if } n = 3 \\ n & \text{if } n \equiv 0 \pmod{4} \\ n + 2 & \text{if } n = 6, n \equiv 1 \text{ or } 3 \pmod{4} \text{ and } n \neq 3 \\ n + 4 & \text{if } n \equiv 2 \pmod{4}. \end{cases}$$

Proposition D. [12] If $n \geq 3$, then

$$\gamma_{stdR}^3(C_n) = \begin{cases} \lceil \frac{3n}{2} \rceil + 1 & \text{if } n \equiv 2 \pmod{4} \\ \lceil \frac{3n}{2} \rceil & \text{otherwise.} \end{cases}$$

Proposition E. [12] For $k \geq 2$ and $n \geq \lceil \frac{k}{2} \rceil + 1$, $\gamma_{stdR}^k(K_n) = k + 2$.

We close this section with two simple results.

Lemma 1.1. If G is a connected graph of order 4 and size m , then $\gamma_{stdR}^2(G) \geq \frac{92-24m}{5}$.

Proof. Let G be a connected graph of order 4. If $\Delta(G) = 2$, then $G \in \{P_4, C_4\}$ and the result follows from Propositions A and C. Assume that $\Delta(G) = 3$. If G is the complete graph K_4 , then the result follows from Proposition E. Suppose G is not the complete graph K_4 . Let $V(G) = \{v_1, v_2, v_3, v_4\}$, $\deg(v_1) = 3$ and

f be a $\gamma_{stdR}^2(G)$ -function. If v_i is a leaf for some $i \in \{2, 3, 4\}$, say $i = 2$, then we have

$$\begin{aligned}\gamma_{stdR}^2(G) &= \omega(f) \\ &= f(v_1) + f(N(v_1)) \\ &= f(N(v_2)) + f(N(v_1)) \\ &\geq 4 \\ &\geq \frac{92-24m}{5}.\end{aligned}$$

Hence, we assume that $\delta(G) \geq 2$. This implies that $m \geq 5$ and so

$$\begin{aligned}\gamma_{stdR}^2(G) &= \omega(f) \\ &= f(v_1) + f(N(v_1)) \\ &\geq f(v_1) + 2 \\ &\geq 1 \\ &> \frac{92-24m}{5}.\end{aligned}$$

□

2. LOWER BOUNDS ON $\gamma_{stdR}^2(G)$ AND $\gamma_{stdR}^3(G)$

In this section we provide sharp bounds on $\gamma_{stdR}^k(G)$ for $k = 2, 3$, in terms of the order and the size of G . To this end, we introduce some notation.

If $f = (V_{-1}, V_1, V_2, V_3)$ is an STDR k DF of G , then for notational convenience, we assume that $V'_{-1} = \{v \in V_{-1} \mid N(v) \cap V_3 \neq \emptyset\}$ and $V''_{-1} = V_{-1} - V'_{-1}$. Also, we let $V_{12} = V_1 \cup V_2$, $V_{13} = V_1 \cup V_3$, $V_{123} = V_1 \cup V_2 \cup V_3$, $|V_{12}| = n_{12}$, $|V_{13}| = n_{13}$, $|V_{123}| = n_{123}$, $|V_1| = n_1$, $|V_2| = n_2$, $|V_3| = n_3$ and $|V_{-1}| = n_{-1}$. Then $n_{123} = n_1 + n_2 + n_3$ and $n_{-1} = n - n_{123}$. Let $G_{123} = G[V_{123}]$ be the subgraph induced by the set V_{123} and let G_{123} have size m_{123} . For $i = 1, 2, 3$, if $V_i \neq \emptyset$, let $G_i = G[V_i]$ be the subgraph induced by the set V_i and let G_i have size m_i . Hence, $m_{123} = m_1 + m_2 + m_3 + |[V_1, V_2]| + |[V_1, V_3]| + |[V_2, V_3]|$.

Theorem 2.1. *Let G be a connected graph of order $n \geq 4$ and size m . Then*

$$\gamma_{stdR}^2(G) \geq \frac{23n - 24m}{5}.$$

Proof. Let $f = (V_{-1}, V_1, V_2, V_3)$ be a $\gamma_{stdR}^3(G)$ -function such that (i) $|V_3|$ is maximized and (ii) subject to (i), $|V_3 \cap L|$ is minimized where $L = \{v \in V(G) \mid \deg(v) = 1\}$. The result is immediate for $n = 4$ by Lemma 1.1. Assume that $n \geq 5$.

If $V_{-1} = \emptyset$, then clearly $\gamma_{stdR}^2(G) \geq n + 1 \geq \frac{23n-24m}{5}$ since $n \geq 5$ and $m \geq n - 1$. Henceforth, we assume $V_{-1} \neq \emptyset$. We consider the following cases.

Case 1. $V_3 \neq \emptyset$.

We distinguish the following situation.

Subcase 1.1. $V_2 \neq \emptyset$.

Since each vertex in V_{-1} is adjacent to at least one vertex in V_3 or to at least

two vertices in V_2 , we have

$$|[V_{-1}, V_3]| + |[V_{-1}, V_2]| \geq |V'_{-1}| + 2|V''_{-1}| \geq |V_{-1}| + |V''_{-1}| \geq n_{-1}.$$

Furthermore we have

$$2n_{-1} = 2|V'_{-1}| + 2|V''_{-1}| \leq 2|[V_{-1}, V_3]| + |[V_{-1}, V_2]| = 2 \sum_{v \in V_3} \deg_{V_{-1}}(v) + \sum_{u \in V_2} \deg_{V_{-1}}(u).$$

For each vertex $v \in V_2 \cup V_3$, we have that $3 \deg_{V_3}(v) + 2 \deg_{V_2}(v) + \deg_{V_1}(v) - \deg_{V_{-1}}(v) = f(N(v)) \geq 2$, and so

$$\deg_{V_{-1}}(v) \leq 3 \deg_{V_3}(v) + 2 \deg_{V_2}(v) + \deg_{V_1}(v) - 2.$$

Now, we have

$$\begin{aligned} 2n_{-1} &\leq 2 \sum_{v \in V_3} \deg_{V_{-1}}(v) + \sum_{u \in V_2} \deg_{V_{-1}}(u) \\ &\leq 2 \sum_{v \in V_3} (3 \deg_{V_3}(v) + 2 \deg_{V_2}(v) + \deg_{V_1}(v) - 2) \\ &\quad + \sum_{u \in V_2} (3 \deg_{V_3}(u) + 2 \deg_{V_2}(u) + \deg_{V_1}(u) - 2) \\ &= (12m_3 + 4|[V_2, V_3]| + 2|[V_1, V_3]| - 4n_3) + (3|[V_2, V_3]| + 4m_2 + |[V_1, V_2]| - 2n_2) \\ &= 12m_3 + 4m_2 + 7|[V_2, V_3]| + 2|[V_1, V_3]| + |[V_1, V_2]| - 4n_3 - 2n_2 \\ &= 12m_{123} - 12m_1 - 8m_2 - 5|[V_2, V_3]| - 10|[V_1, V_3]| - 11|[V_1, V_2]| - 4n_3 - 2n_2, \end{aligned}$$

and this implies that

$$m_{123} \geq \frac{1}{12}(2n_{-1} + 12m_1 + 8m_2 + 5|[V_2, V_3]| + 10|[V_1, V_3]| + 11|[V_1, V_2]| + 4n_3 + 2n_2).$$

Therefore,

$$\begin{aligned} m &\geq m_{123} + |[V_{-1}, V_{123}]| + m_{-1} \\ &\geq m_{123} + |[V_{-1}, V_{123}]| \\ &\geq \frac{1}{12}(2n_{-1} + 12m_1 + 8m_2 + 5|[V_2, V_3]| + 10|[V_1, V_3]| + 11|[V_1, V_2]| + 4n_3 + 2n_2) \\ &\quad + |[V_{-1}, V_1]| + n_{-1} \\ &= \frac{1}{12}(14n_{-1} + 4n_{123} - 4n_1 - 2n_2 + 12m_1 + 8m_2 + 5|[V_2, V_3]| + 10|[V_1, V_3]| \\ &\quad + 11|[V_1, V_2]| + 12|[V_{-1}, V_1]|) \\ &= \frac{1}{12}(14n - 10n_{123} - 4n_1 - 2n_2 + 12m_1 + 8m_2 + 5|[V_2, V_3]| + 10|[V_1, V_3]| \\ &\quad + 11|[V_1, V_2]| + 12|[V_{-1}, V_1]|), \end{aligned}$$

and so

$$\begin{aligned} n_{123} &\geq \frac{1}{10}(-12m + 14n - 4n_1 - 2n_2 + 12m_1 + 8m_2 + 5|[V_2, V_3]| \\ &\quad + 10|[V_1, V_3]| + 11|[V_1, V_2]| + 12|[V_{-1}, V_1]|). \end{aligned}$$

Now, we have

$$\begin{aligned}
\gamma_{stdR}^2(G) &= 3n_3 + 2n_2 + n_1 - n_{-1} \\
&= 4n_3 + 3n_2 + 2n_1 - n \\
&= 4n_{123} - n - n_2 - 2n_1 \\
&\geq \frac{4}{10}(-12m + 14n - 4n_1 - 2n_2 + 12m_1 + 8m_2 + 5|[V_2, V_3]| \\
&\quad + 10|[V_1, V_3]| + 11|[V_1, V_2]| + 12|[V_{-1}, V_1]|) - n - n_2 - 2n_1 \\
&= \frac{2}{5}\left(\frac{23n}{2} - \frac{24m}{2}\right) + \frac{2}{5}\left(-9n_1 - \frac{9}{2}n_2 + 12m_1 + 8m_2 + 5|[V_2, V_3]| \right. \\
&\quad \left. + 10|[V_1, V_3]| + 11|[V_1, V_2]| + 12|[V_{-1}, V_1]|)\right).
\end{aligned}$$

Let $\Theta = -9n_1 - \frac{9}{2}n_2 + 12m_1 + 8m_2 + 5|[V_2, V_3]| + 10|[V_1, V_3]| + 11|[V_1, V_2]| + 12|[V_{-1}, V_1]|$. We show that $\Theta \geq 0$. First let $n_1 = 0$, then $\Theta = -\frac{9}{2}n_2 + 8m_2 + 5|[V_2, V_3]|$. Let V_2^1 be the set of vertices with label 2 having a neighbor in V_3 , V_2^2 be the subset of $V_2 - V_2^1$ with label 2 having a neighbor in V_2^1 , V_2^3 be the subset of $V_2 - (V_2^1 \cup V_2^2)$ and etc. Since any vertex in V_2 have a neighbor in $V_3 \cup V_2$, by repeating this process we obtain a partition $V_2^1 \cup V_2^2 \cup \dots \cup V_2^r$ of V_2 such that each vertex in V_2^i has a neighbor in V_2^{i-1} for each $2 \leq i \leq r-1$ and that $N(x) \cap V_2 \subseteq V_2^r$ for each $x \in V_2^r$. We claim that each vertex $v \in V_2^r$ has at least two neighbors in V_2^r . Suppose, to the contrary, that there exists a vertex $v \in V_2^r$ having exactly one neighbor u in V_2^r . Then $\deg_G(v) = 1$ and since G is connected and f is an STD2DF of G we deduce that u has a neighbor in $V_2^r - \{v\}$. But then the function g defined on G by $g(v) = 1, g(u) = 3$ and $g(x) = f(x)$ otherwise, is a $\gamma_{stdR}^2(G)$ -function which contradicts the choice of f . Hence each vertex $v \in V_2^r$ has at least two neighbors in V_2^r . Then

$$\begin{aligned}
\Theta &= -\frac{9}{2}n_2 + 8m_2 + 5|[V_2, V_3]| \\
&\geq -\frac{9}{2}n_2 + 5|[V_2^1, V_3]| + 8\left(\sum_{i=2}^{r-1} |[V_2^i, V_2^{i-1}]| \right) + 8|E(G[V_2^r])| \\
&\geq -\frac{9}{2}n_2 + 5|V_2^1| + 8\left(\sum_{i=2}^{r-1} |V_2^i| \right) + 8|V_2^r| \\
&\geq -\frac{9}{2}n_2 + 5|V_2| \\
&> 0.
\end{aligned}$$

Therefore $\gamma_{stdR}^2(G) > \frac{23n-24m}{5}$. Suppose now that $n_1 \geq 1$. Let V_1^1 be the set of vertices with label 1 having a neighbor in V_3 and V_2^1 be the set of vertices with label 2 having a neighbor in $V_3 \cup V_1^1$. Suppose V_1^2 is the subset of $V_1 - V_1^1$ having a neighbor in $V_1^1 \cup V_2^1$ and V_2^2 is the subset of $V_2 - V_2^1$ having a neighbor in $V_1^2 \cup V_2^1$. Since V_1 and V_2 are finite sets, by repeating this process we obtain disjoint subsets $V_1^1 \cup V_1^2 \cup \dots \cup V_1^r$ of V_1 (possibly some of V_1^i are empty) such

that each vertex in V_1^i has a neighbor in $V_1^{i-1} \cup V_2^{i-1}$ for each $2 \leq i \leq r$, and disjoint subsets $V_2^1 \cup V_2^2 \cup \dots \cup V_2^r$ of V_2 (possibly some of V_2^i are empty) so that every vertex in V_2^i has a neighbor in $V_1^i \cup V_2^{i-1}$ for each $2 \leq i \leq r$ and that $V_1^r = V_2^r = \emptyset$. Let $V_1^{r+1} = V_1 - (\cup_{i=1}^r V_1^i)$ and $V_2^{r+1} = V_2 - (\cup_{i=1}^r V_2^i)$. Clearly, $V_1^1 \cup V_1^2 \cup \dots \cup V_1^{r+1}$ is a weak partition of V_1 and $V_2^1 \cup V_2^2 \cup \dots \cup V_2^{r+1}$ is a weak partition of V_2 . Note that $N(x) \subseteq V_1^{r+1} \cup V_2^{r+1} \cup V_{-1}$ for each $x \in V_1^{r+1} \cup V_2^{r+1}$. Assume that $H_1, \dots, H_t$ be the components of $G[V_1^{r+1} \cup V_2^{r+1}]$. Since G is connected and f is a STDR2DF of G , we must have $|V(H_i)| \geq 3$ for each $1 \leq i \leq t$, if $V_1^{r+1} \cup V_2^{r+1} \neq \emptyset$. Then

$$\begin{aligned} \Theta &= -9n_1 - \frac{9}{2}n_2 + 12m_1 + 8m_2 + 5|[V_2, V_3]| + 10|[V_1, V_3]| + 11|[V_1, V_2]| + 12|[V_{-1}, V_1]| \\ &\geq (-9|V_1^1| + 10|[V_1^1, V_3]|) + \sum_{i=2}^r \left(-9|V_1^i| + 12|[V_1^i, V_1^{i-1}]| + 11|[V_1^i, V_2^{i-1}]| \right) + \\ &\quad \left(-\frac{9}{2}|V_2^1| + 5|[V_2^1, V_3]| + 11|[V_1^1, V_2^1]| \right) \\ &\quad + \sum_{i=2}^r \left(-\frac{9}{2}|V_2^i| + 8|[V_2^i, V_2^{i-1}]| + 11|[V_1^i, V_2^i]| \right) + \sum_{i=1}^t \left(-\frac{9}{2}n(H_i) + 8m(H_i) \right) \\ &\geq \sum_{i=1}^t \left(-\frac{9}{2}n(H_i) + 8(n(H_i) - 1) \right) \\ &\geq \sum_{i=1}^t \left(\frac{7}{2}n(H_i) - 8 \right) \\ &> 0. \end{aligned}$$

Therefore $\gamma_{stdR}^2(G) > \frac{23n-24m}{5}$.

Subcase 1.2. $V_2 = \emptyset$.

By definition of STDR2DF, each vertex in V_{-1} is adjacent to one vertex in V_3 , and so

$$\sum_{v \in V_3} \deg_{V_{-1}}(v) = |[V_{-1}, V_3]| \geq |V_{-1}| = n_{-1}.$$

As in Subcase 1.1, for each $v \in V_3$ we have $3 \deg_{V_3}(v) + \deg_{V_1}(v) - \deg_{V_{-1}}(v) = f(N(v)) \geq 2$, and so $\deg_{V_{-1}}(v) \leq 3 \deg_{V_3}(v) + \deg_{V_1}(v) - 2$. Now, we have

$$\begin{aligned} n_{-1} &\leq \sum_{v \in V_3} \deg_{V_{-1}}(v) \\ &\leq \sum_{v \in V_3} (3 \deg_{V_3}(v) + \deg_{V_1}(v) - 2) \\ &= 6m_3 + |[V_1, V_3]| - 2n_3 \\ &= 6m_{13} - 6m_1 - 5|[V_1, V_3]| - 2n_3, \end{aligned}$$

which implies that $m_{13} \geq \frac{1}{6}(n_{-1} + 6m_1 + 5|[V_1, V_3]| + 2n_3)$. Hence,

$$\begin{aligned}
 m &= m_{13} + |[V_{-1}, V_3]| + |[V_{-1}, V_1]| + m_{-1} \\
 &\geq m_{13} + |[V_{-1}, V_3]| + |[V_{-1}, V_1]| \\
 &\geq \frac{1}{6}(n_{-1} + 6m_1 + 5|[V_1, V_2]| + 2n_3) + n_{-1} + |[V_{-1}, V_1]| \\
 &= \frac{1}{6}(7n_{-1} + 2n_3 + 6m_1 + 5|[V_1, V_3]| + 6|[V_{-1}, V_1]|) \\
 &= \frac{1}{6}(7n_{-1} + 2n_{13} - 2n_1 + 6m_1 + 5|[V_1, V_3]| + 6|[V_{-1}, V_1]|) \\
 &= \frac{1}{6}(7n - 5n_{13} - 2n_1 + 6m_1 + 5|[V_1, V_3]| + 6|[V_{-1}, V_1]|),
 \end{aligned}$$

and so

$$n_{13} \geq \frac{1}{5}(-6m + 7n - 2n_1 + 6m_1 + 5|[V_1, V_3]| + 6|[V_{-1}, V_1]|).$$

Now, we have

$$\begin{aligned}
 \gamma_{stdR}^2(G) &= 3n_3 + n_1 - n_{-1} \\
 &= 4n_3 + 2n_1 - n \\
 &= 4n_{13} - n - 2n_1 \\
 &\geq \frac{4}{5}(-6m + 7n - 2n_1 + 6m_1 + 5|[V_1, V_3]| + 6|[V_{-1}, V_1]|) - n - 2n_1 \\
 &= \frac{4}{5}(-6m + 7n - \frac{5}{4}n - 2n_1 - \frac{5}{2}n_1 + 6m_1 + 5|[V_1, V_3]| + 6|[V_{-1}, V_1]|) \\
 &= \frac{4}{5}(\frac{23}{4}n - 6m) + \frac{4}{5}(-\frac{9}{2}n_1 + 6m_1 + 5|[V_1, V_3]| + 6|[V_{-1}, V_1]|).
 \end{aligned}$$

Let $\Theta = -\frac{9}{2}n_1 + 6m_1 + 5|[V_1, V_3]| + 6|[V_{-1}, V_1]|$. We show that $\Theta \geq 0$. If $n_1 = 0$, then $\Theta = 0$. Suppose that $n_1 \geq 1$. Since each vertex of V_1 is adjacent to a vertex of V_3 , we have $|[V_1, V_3]| \geq n_1$. It follows that

$$\begin{aligned}
 \Theta &= -\frac{9}{2}n_1 + 6m_1 + 5|[V_1, V_3]| + 6|[V_{-1}, V_1]| \\
 &\geq -\frac{9}{2}n_1 + 6m_1 + 5n_1 + 6|[V_{-1}, V_1]| \\
 &> 0.
 \end{aligned}$$

Therefore $\gamma_{stdR}^2(G) > \frac{23n-24m}{5}$.

Case 2. $V_3 = \emptyset$.

Since $V_{-1} \neq \emptyset$, we conclude that $V_2 \neq \emptyset$. By definition of STDR2DF, each vertex in V_{-1} is adjacent to at least two vertices in V_2 , and so

$$\sum_{v \in V_2} \deg_{V_{-1}}(v) = |[V_{-1}, V_2]| \geq 2|V_{-1}| = 2n_{-1}.$$

As in Subcase 1.1, for each $v \in V_2$ we have that $2\deg_{V_2}(v) + \deg_{V_1}(v) - \deg_{V_{-1}}(v) = f(N(v)) \geq 2$, and so $\deg_{V_{-1}}(v) \leq 2\deg_{V_2}(v) + \deg_{V_1}(v) - 2$.

Now, we have

$$\begin{aligned}
 2n_{-1} &\leq \sum_{v \in V_2} \deg_{V_{-1}}(v) \\
 &\leq \sum_{v \in V_2} (2 \deg_{V_2}(v) + \deg_{V_1}(v) - 2) \\
 &= 4m_2 + |[V_1, V_2]| - 2n_2 \\
 &= 4m_{12} - 4m_1 - 3|[V_1, V_2]| - 2n_2,
 \end{aligned}$$

which implies that

$$m_{12} \geq \frac{1}{4}(2n_{-1} + 4m_1 + 3|[V_1, V_2]| + 2n_2).$$

Hence,

$$\begin{aligned}
 m &= m_{12} + |[V_{-1}, V_{12}]| + m_{-1} \\
 &\geq m_{12} + |[V_{-1}, V_{12}]| \\
 &\geq \frac{1}{4}(2n_{-1} + 4m_1 + 3|[V_1, V_2]| + 2n_2) + 2n_{-1} + |[V_1, V_{-1}]| \\
 &= \frac{1}{4}(10n_{-1} + 2n_{12} - 2n_1 + 4m_1 + 3|[V_1, V_2]| + 4|[V_1, V_{-1}]|) \\
 &= \frac{1}{4}(10n - 8n_{12} - 2n_1 + 4m_1 + 3|[V_1, V_2]| + 4|[V_1, V_{-1}]|)
 \end{aligned}$$

and so $n_{12} \geq \frac{1}{8}(-4m + 10n - 2n_1 + 4m_1 + 3|[V_1, V_2]| + 4|[V_1, V_{-1}]|)$. Now, we have

$$\begin{aligned}
 \gamma_{stdR}^2(G) &= 2n_2 + n_1 - n_{-1} \\
 &= 3n_2 + 2n_1 - n \\
 &= 3n_{12} - n - n_1 \\
 &\geq \frac{3}{8}(-4m + 10n - 2n_1 + 4m_1 + 3|[V_1, V_2]| + 4|[V_1, V_{-1}]|) - n - n_1 \\
 &= \frac{3}{8}(-4m + 10n - \frac{8}{3}n) + \frac{3}{8}(-\frac{14}{3}n_1 + 4m_1 + 3|[V_1, V_2]| \\
 &\quad + 4|[V_1, V_{-1}]|) \\
 &\geq \frac{3}{8}(-4m + \frac{22}{3}n) - \frac{5}{8}m + \frac{5}{8}m + \frac{3}{8}(-\frac{14}{3}n_1 + 4m_1 + 3|[V_1, V_2]| \\
 &\quad + 4|[V_1, V_{-1}]|) \\
 &= \frac{1}{8}(-17m + 22n) + \frac{3}{8}(-\frac{14}{3}n_1 + 4m_1 + \frac{5}{3}m + 3|[V_1, V_2]| \\
 &\quad + 4|[V_1, V_{-1}]|).
 \end{aligned}$$

Let $\Theta = -\frac{14}{3}n_1 + 4m_1 + \frac{5}{3}m + 3|[V_1, V_2]| + 4|[V_1, V_{-1}]|$. If $n_1 = 0$, then $\Theta > 0$. Suppose that $n_1 \geq 1$. Since any vertex in V_1 is adjacent to a vertex in V_2 , we have

$$\begin{aligned}\Theta &= -\frac{14}{3}n_1 + 4m_1 + \frac{5}{3}m + 3|[V_1, V_2]| + 4|[V_1, V_{-1}]| \\ &\geq -\frac{14}{3}n_1 + \frac{17}{3}m_1 + \frac{14}{3}|[V_1, V_2]|. \\ &\geq 0\end{aligned}$$

Therefore $\gamma_{stdR}^2(G) \geq \frac{1}{8}(22n - 17m) > \frac{1}{5}(23n - 24m)$. This completes the proof. $\square$

In the next example, we present an infinite family of graphs that attain the bound of Theorem 2.1.

EXAMPLE 2.2. For any connected graph F of order $t \geq 2$, let F_t be the graph obtained from F by adding $3 \deg_F(v) - 2$ pendant edges to each vertex v of F . Then

$$n(F_t) = n(F) + \sum_{v \in V(F)} (3 \deg_F(v) - 2) = 6m(F) - n(F)$$

and

$$m(F_t) = m(F) + \sum_{v \in V(F)} (3 \deg_F(v) - 2) = 7m(F) - 2n(F).$$

Assigning a 3 to every vertex in $V(F)$ and a -1 to every vertex in $V(F_t) - V(F)$ produces an STDR2DF of weight

$$3n(F) - \sum_{v \in V(F)} (3 \deg_F(v) - 2) = 5n(F) - 6m(F) = \frac{23n(F_t) - 24m(F_t)}{5},$$

and so $\gamma_{stdR}^2(F_t) \leq \frac{23n(F_t) - 24m(F_t)}{5}$. Applying Theorem 2.1, we have $\gamma_{stdR}^2(F_t) = \frac{23n(F_t) - 24m(F_t)}{5}$.

Next we present a sharp lower bound on $\gamma_{stdR}^3(G)$.

Theorem 2.3. *Let G be a connected graph of order $n \geq 5$ and size m . Then*

$$\gamma_{stdR}^3(G) \geq 6n - 6m.$$

Furthermore, this bound is sharp.

Proof. Let $f = (V_{-1}, V_1, V_2, V_3)$ be a $\gamma_{stdR}^2(G)$ -function such that (i) $|V_3|$ is maximized and (ii) subject to (i), $|V_3 \cap L|$ is minimized where $L = \{v \in V(G) \mid \deg(v) = 1\}$. If $V_{-1} = \emptyset$, then $\gamma_{stdR}^3(G) \geq n + 1 > 6n - 6m$. Suppose that $V_{-1} \neq \emptyset$. Consider the following cases.

Case 1. $V_3 \neq \emptyset$.

First let $V_2 \neq \emptyset$. As in the proof of Theorem 2.1, we have

$$\sum_{v \in V_3} \deg_{V_{-1}}(v) + \frac{1}{2} \sum_{u \in V_2} \deg_{V_{-1}}(u) = |[V_{-1}, V_3]| + \frac{1}{2} |[V_{-1}, V_2]| \geq |V'_{-1}| + |V''_{-1}| = n_{-1},$$

and for each vertex $v \in V_2 \cup V_3$, $\deg_{V_{-1}}(v) \leq 3 \deg_{V_3}(v) + 2 \deg_{V_2}(v) + \deg_{V_1}(v) - 3$.

3. Now, we have

$$\begin{aligned}
 3n_{-1} &\leq 3 \sum_{v \in V_3} \deg_{V_{-1}}(v) + \frac{3}{2} \sum_{u \in V_2} \deg_{V_{-1}}(u) \\
 &\leq 3 \sum_{v \in V_3} (3 \deg_{V_3}(v) + 2 \deg_{V_2}(v) + \deg_{V_1}(v) - 3) \\
 &\quad + \frac{3}{2} \sum_{u \in V_2} (3 \deg_{V_3}(u) + 2 \deg_{V_2}(u) + \deg_{V_1}(u) - 3) \\
 &= (18m_3 + 6|[V_2, V_3]| + 3|[V_1, V_3]| - 9n_3) + \left(\frac{9}{2}[|V_2, V_3]| + 6m_2 + \frac{3}{2}[|V_1, V_2]| - \frac{9}{2}n_2\right) \\
 &= 18m_3 + 6m_2 + \frac{21}{2}[|V_2, V_3]| + 3|[V_1, V_3]| + \frac{3}{2}[|V_1, V_2]| - 9n_3 - \frac{9}{2}n_2 \\
 &= 18m_{123} - 18m_1 - 12m_2 - \frac{15}{2}[|V_2, V_3]| - 15|[V_1, V_3]| - \frac{33}{2}[|V_1, V_2]| - 9n_3 - \frac{9}{2}n_2,
 \end{aligned}$$

and so

$$m_{123} \geq \frac{1}{18}(3n_{-1} + 18m_1 + 12m_2 + \frac{15}{2}[|V_2, V_3]| + 15|[V_1, V_3]| + \frac{33}{2}[|V_1, V_2]| + 9n_3 + \frac{9}{2}n_2).$$

Using an argument similar to that described in the proof of Theorem 2.1, we obtain

$$\begin{aligned}
 n_{123} &\geq \frac{1}{12}(-18m + 21n - 9n_1 - \frac{9}{2}n_2 + 18m_1 + 12m_2 + \frac{15}{2}[|V_2, V_3]| + 15|[V_1, V_3]| \\
 &\quad + \frac{33}{2}[|V_1, V_2]| + 18|[V_{-1}, V_1]|).
 \end{aligned}$$

Now, we have

$$\begin{aligned}
 \gamma_{stdR}^3(G) &= 3n_3 + 2n_2 + n_1 - n_{-1} \\
 &= 4n_3 + 3n_2 + 2n_1 - n \\
 &= 4n_{123} - n - n_2 - 2n_1 \\
 &\geq \frac{4}{12}(-18m + 21n - 9n_1 - \frac{9}{2}n_2 + 18m_1 + 12m_2 + \frac{15}{2}[|V_2, V_3]| \\
 &\quad + 15|[V_1, V_3]| + \frac{33}{2}[|V_1, V_2]| + 18|[V_{-1}, V_1]|) - n - n_2 - 2n_1 \\
 &= \frac{1}{3}(-18m + 18n - 15n_1 - \frac{15}{2}n_2 + 18m_1 + 12m_2 + \frac{15}{2}[|V_2, V_3]| \\
 &\quad + 15|[V_1, V_3]| + \frac{33}{2}[|V_1, V_2]| + 18|[V_{-1}, V_1]|) \\
 &= 6n - 6m + \frac{1}{3}(-15n_1 - \frac{15}{2}n_2 + 18m_1 + 12m_2 + \frac{15}{2}[|V_2, V_3]| \\
 &\quad + 15|[V_1, V_3]| + \frac{33}{2}[|V_1, V_2]| + 18|[V_{-1}, V_1]|).
 \end{aligned}$$

Let $\Theta = -15n_1 - \frac{15}{2}n_2 + 18m_1 + 12m_2 + \frac{15}{2}[|V_2, V_3]| + 15|[V_1, V_3]| + \frac{33}{2}[|V_1, V_2]| + 18|[V_{-1}, V_1]|$. We show that $\Theta \geq 0$. If $n_1 = 0$, then $\Theta = -\frac{15}{2}n_2 + 12m_2 + \frac{15}{2}[|V_2, V_3]|$ and as in the proof of Theorem 2.1 we can see that $\Theta > 0$ implying that $\gamma_{sdR}^t(G) > 6n - 6m$. Suppose now that $n_1 \geq 1$.

Now we use the notations defined in the proof of Theorem 2.1 (Subcase 1.1). Since G is connected and f is a STDR3DF of G , we must have $|V(H_i)| \geq 3$

and $\delta(H_i) \geq 2$ for each $1 \leq i \leq t$. It follows that $|E(H_i)| \geq |V(H_i)|$ for each $1 \leq i \leq t$. Thus

$$\begin{aligned}
\Theta &= -15n_1 - \frac{15}{2}n_2 + 18m_1 + 12m_2 + \frac{15}{2}||[V_2, V_3]| + 15|[V_1, V_3]| \\
&\quad + \frac{33}{2}||[V_1, V_2]| + 18|[V_{-1}, V_1]| \\
&\geq (-15|V_1^1| + 15|[V_1^1, V_3]|) + \sum_{i=2}^r \left(-15|V_1^i| + 18|[V_1^i, V_1^{i-1}]| + \frac{33}{2}|[V_1^i, V_2^{i-1}]| \right) \\
&\quad + \left(-\frac{15}{2}|V_2^1| + \frac{15}{2}||[V_2^1, V_3]| + \frac{33}{2}||[V_1^1, V_2^1]| \right) \\
&\quad + \sum_{i=2}^{r+1} \left(-\frac{15}{2}|V_2^i| + 12|[V_2^i, V_2^{i-1}]| + \frac{33}{2}||[V_1^i, V_2^i]| \right) + \sum_{i=1}^t \left(-\frac{15}{2}n(H_i) + 12m(H_i) \right) \\
&\geq \sum_{i=1}^t \left(-\frac{15}{2}n(H_i) + 12n(H_i) \right) \\
&> 0.
\end{aligned}$$

Therefore $\gamma_{stdR}^3(G) \geq 6n - 6m$.

Now let $V_2 = \emptyset$. As above, we have $\sum_{v \in V_3} \deg_{V_{-1}}(v) = |[V_{-1}, V_3]| \geq n_{-1}$ and $\deg_{V_{-1}}(v) \leq 3\deg_{V_3}(v) + \deg_{V_1}(v) - 3$ for each vertex $v \in V_3$. Now, we have

$$\begin{aligned}
n_{-1} &\leq \sum_{v \in V_3} \deg_{V_{-1}}(v) \\
&\leq \sum_{v \in V_3} (3\deg_{V_3}(v) + \deg_{V_1}(v) - 3) \\
&= 6m_3 + |[V_1, V_3]| - 3n_3 \\
&= 6m_{13} - 6m_1 - 5|[V_1, V_3]| - 3n_3,
\end{aligned}$$

which implies that $m_{13} \geq \frac{1}{6}(n_{-1} + 6m_1 + 5|[V_1, V_3]| + 3n_3)$. Hence,

$$\begin{aligned}
m &= m_{13} + |[V_{-1}, V_3]| + |[V_{-1}, V_1]| + m_{-1} \\
&\geq m_{13} + |[V_{-1}, V_3]| + |[V_{-1}, V_1]| \\
&\geq \frac{1}{6}(n_{-1} + 6m_1 + 5|[V_1, V_3]| + 3n_3) + n_{-1} + |[V_{-1}, V_1]| \\
&= \frac{1}{6}(7n_{-1} + 3n_3 + 6m_1 + 5|[V_1, V_3]| + 6|[V_{-1}, V_1]|) \\
&= \frac{1}{6}(7n_{-1} + 3n_{13} - 3n_1 + 6m_1 + 5|[V_1, V_3]| + 6|[V_{-1}, V_1]|) \\
&= \frac{1}{6}(7n - 4n_{13} - 3n_1 + 6m_1 + 5|[V_1, V_3]| + 6|[V_{-1}, V_1]|)
\end{aligned}$$

and this implies that

$$n_{13} \geq \frac{1}{4}(-6m + 7n - 3n_1 + 6m_1 + 5|[V_1, V_3]| + 6|[V_{-1}, V_1]|).$$

Now, we have

$$\begin{aligned}
 \gamma_{stdR}^3(G) &= 3n_3 + n_1 - n_{-1} \\
 &= 4n_3 + 2n_1 - n \\
 &= 4n_{13} - n - 2n_1 \\
 &\geq \frac{4}{4}(-6m + 7n - 3n_1 + 6m_1 + 5|[V_1, V_3]| + 6|[V_{-1}, V_1]|) - n - 2n_1 \\
 &= (-6m + 6n) + (-5n_1 + 6m_1 + 5|[V_1, V_3]| + 6|[V_{-1}, V_1]|) \\
 &> (-6m + 6n).
 \end{aligned}$$

Case 2. $V_3 = \emptyset$.

Since f is a STDR3DF of G , we conclude that $\delta(G) \geq 2$ and so $m \geq n$. Now $V_{-1} \neq \emptyset$ implies that $V_2 \neq \emptyset$. By definition of STDR3DF, each vertex in V_{-1} is adjacent to at least two vertices in V_2 , and so

$$|[V_{-1}, V_{12}]| \geq |[V_{-1}, V_2]| \geq 2|V_{-1}| = 2n_{-1}.$$

As above, we have $2n_{-1} \leq 4m_{12} - 4m_1 - 3|[V_1, V_2]| - 3n_2$ and hence

$$m_{12} \geq \frac{1}{4}(2n_{-1} + 4m_1 + 3|[V_1, V_2]| + 3n_2).$$

Now we have

$$\begin{aligned}
 m &= m_{12} + |[V_{-1}, V_{12}]| + m_{-1} \\
 &\geq m_{12} + |[V_{-1}, V_{12}]| \\
 &\geq \frac{1}{4}(2n_{-1} + 4m_1 + 3|[V_1, V_2]| + 3n_2) + 2n_{-1} + |[V_1, V_{-1}]| \\
 &\geq \frac{1}{4}(10n_{-1} + 3n_{12} + 4m_1 + 3|[V_1, V_2]| - 3n_1 + 4|[V_1, V_{-1}]|) \\
 &= \frac{1}{4}(10n - 7n_{12} + 4m_1 + 3|[V_1, V_2]| - 3n_1 + 4|[V_1, V_{-1}]|)
 \end{aligned}$$

and so

$$n_{12} \geq \frac{1}{7}(-4m + 10n + 4m_1 + 3|[V_1, V_2]| - 3n_1 + 4|[V_1, V_{-1}]|).$$

Thus

$$\begin{aligned}
 \gamma_{stdR}^3(G) &= 2n_2 + n_1 - n_{-1} \\
 &= 3n_2 + 2n_1 - n \\
 &= 3n_{12} - n - n_1 \\
 &\geq \frac{3}{7}(-4m + 10n + 4m_1 + 3|[V_1, V_2]| - 3n_1 + 4|[V_1, V_{-1}]|) - n - n_1 \\
 &= \frac{3}{7}(-4m + \frac{23}{3}n - \frac{16}{3}n_1 + 4m_1 + 3|[V_1, V_2]| + 4|[V_1, V_{-1}]|) \\
 &\geq \frac{3}{7}(-4m + \frac{23}{3}n) + \frac{3}{7}(-\frac{16}{3}n_1 + 4m_1 + 3|[V_1, V_2]| + 4|[V_1, V_{-1}]|) \\
 &= \frac{-12m + 23n}{7} + \frac{3}{7}(-\frac{16}{3}n_1 + 4m_1 + 3|[V_1, V_2]| + 4|[V_1, V_{-1}]|) \\
 &\geq \frac{-12m + 23n}{7} \\
 &> -6m + 6n.
 \end{aligned}$$

To prove the sharpness, let H_t ($t \geq 2$) be the graph obtained from a connected graph H of order t by adding $3 \deg_H(v) - 3$ pendant edges to each vertex v of H . Then

$$n(H_t) = n(H) + \sum_{v \in V(H)} (3 \deg_H(v) - 3) = 6m(H) - 2n(H)$$

and

$$m(H_t) = m(H) + \sum_{v \in V(H)} (3 \deg_H(v) - 3) = 7m(H) - 3n(H).$$

Assigning a 3 to every vertex in $V(H)$ and a -1 to every vertex in $V(H_t) - V(H)$ produces an STDR3DF f of weight

$$\omega(f) = 3n(H) - \sum_{v \in V(H)} (3 \deg_H(v) - 3) = 6n(H) - 6m(H) = 6n(H_t) - 6m(H_t),$$

and hence $\gamma_{stdR}^3(H_t) \leq 6n(H_t) - 6m(H_t)$. Thus $\gamma_{stdR}^3(H_t) = 6n(H_t) - 6m(H_t)$ and the proof is complete. $\square$

ACKNOWLEDGMENTS

The authors are deeply thankful to the reviewer for his/her valuable suggestions to improve the quality of this manuscript. H. Abdollahzadeh Ahangar was supported by the Babol Noshirvani University of Technology under research grant number BNUT/385001/1401.

REFERENCES

1. H. Abdollahzadeh Ahangar, J. Amjadi, M. Atapour, M. Chellali, S. M. Sheikholeslami, Double Roman Trees, *Ars Combin.*, **145**, (2019), 173–183.

2. H. Abdollahzadeh Ahangar, J. Amjadi, M. Chellali, S. Nazari-Moghaddam, S. M. Sheikholeslami, Trees with Double Roman Domination Number Twice the Domination Number Plus Two, *Iran. J. Sci. Technol. Trans. A, Sci.*, **43**, (2019), 1081–1088.
3. H. Abdollahzadeh Ahangar, M. Chellali, V. Samodivkin, Outer Independent Roman Dominating Functions in Graphs, *Int. J. Comput. Math.*, **44**, (2019), 2547–2557.
4. H. Abdollahzadeh Ahangar, M. Chellali, S. M. Sheikholeslami, Outer Independent Double Roman Domination, *Appl. Math. Comput.*, **364**, (2020), 124617 (9 pages).
5. H. Abdollahzadeh Ahangar, M. Chellali, S. M. Sheikholeslami, On the Double Roman Domination in Graphs, *Discrete Appl. Math.*, **232**, (2017), 1–7.
6. H. Abdollahzadeh Ahangar, M. Chellali, S. M. Sheikholeslami, J. C. Valenzuela-Tripodoro, Maximal Double Roman Domination in Graphs, *Appl. Math. Comput.*, **414**, (2022), 126662.
7. H. Abdollahzadeh Ahangar, M. Chellali, S. M. Sheikholeslami, Signed Double Roman Domination of Graphs, *Filomat*, **33**, (2019), 121–134.
8. H. Abdollahzadeh Ahangar, F. Nahani Pour, M. Chellali, S. M. Sheikholeslami, Outer Independent Signed Double Roman Domination, *J. Appl. Math. Comput.*, **68**, (2022), 705–720.
9. J. Amjadi, H. Yang, S. Nazari-Moghaddam, Z. Shao, S. M. Sheikholeslami, Signed Double Roman k -domination in Graphs, *Australas. J. Combin.*, **72**, (2018), 82–105.
10. R. A. Beeler, T. W. Haynes, S. T. Hedetniemi, Double Roman Domination, *Discrete Appl. Math.*, **211**, (2016), 23–29.
11. G. Hao, L. Volkmann, D. A. Mojdeh, Total Double Roman Domination in Graphs, *Commun. Comb. Optim.*, **5**, (2020), 27–39.
12. L. Shahbazi, H. Abdollahzadeh Ahangar, R. Khoeilar, S. M. Sheikholeslami, Signed Total Double Roman k -domination in Graphs, *Discrete Math. Algorithms Appl.*, **12**, (2020), ID: 2050009.
13. L. Shahbazi, H. Abdollahzadeh Ahangar, R. Khoeilar, S. M. Sheikholeslami, Bounds on Signed Total Double Roman Domination, *Commun. Comb. Optim.*, **5**, (2020), 191–206.
14. A. Teymourzadeh, D. A. Mojdeh, Covering Total Double Roman Domination in Graphs, *Commun. Comb. Optim.*, **8**(1), (2023), 115–125.
15. D. B. West, *Introduction to Graph Theory (Second Edition)*, Prentice Hall, USA, 2001.
16. H. Yang, P. Wu, S. Nazari-Moghaddam, S. M. Sheikholeslami, X. Zhang, Z. Shao, Y. Y. Tang, Bounds for Signed Double Roman k -domination in Trees, *RAIRO - Ope. Res.*, **53**, (2019), 627–643.