

Generalization of Ostrowski's Inequality for Differentiable Functions and its Applications

Nazia Irshad^a, Asif R. Khan^b, Muhammad Awais Shaikh^{c*}

^aDepartment of Mathematics, Dawood University of Engineering and Technology-74800, New M. A. Jinnah Road, Karachi, Pakistan

^bDepartment of Mathematics, University of Karachi, University Road, Karachi, Pakistan

^c Department of Mathematics, Nabi Bagh Z. M. Science College, Saddar, Karachi, Pakistan

E-mail: nazia.irshad@duet.edu.pk

E-mail: asifrk@uok.edu.pk

E-mail: m.awaisshaikh2014@gmail.com

ABSTRACT. We first establish weighted Ostrowski type inequalities for bounded differentiable functions. This inequality is also obtained for bounded above and bounded below differentiable functions. Some applications of the proposed results are presented to numerical standard and non standard quadrature rules. We recapture known results as well as obtain new results.

Keywords: Ostrowski's inequality, Numerical integration, Differentiable functions.

2000 Mathematics subject classification: 26D10, 26D20, 26D99.

1. INTRODUCTION

Our work mostly deals with integral inequalities. To highlight its importance we quote here from [20], "Among the many types of inequalities, integral

*Corresponding Author

inequalities are of supreme importance because over the last few decades this field has proven to be an extensively applicable field. The integral inequalities of various types have been widely studied in most subjects involving mathematical analysis. These inequalities are particularly useful in approximation theory and in numerical analysis where estimates of approximation errors are involved." Ostrowski inequality [25] is one of the most famous inequality, first presented by Alexander Markowich Ostrowski in 1938. It can be used to determine absolute deviation of functional value from its mean value. It is extremely important because of its wide range of applications in different areas of mathematics such as numerical integration, integral operator theory, probability theory and statistics. This inequality states that:

Proposition 1.1. *Let $\rho : I \rightarrow \mathbb{R}$ be a differentiable function on I^o such that $\rho \in L[j, k]$ where $j < k$ whose derivative ρ' is bounded on interior of I , i.e., $\|\rho'\|_\infty := \sup_{t \in (j, k)} |\rho'(t)| < \infty$. Then*

$$\left| \rho(\theta) - \frac{1}{k-j} \int_j^k \rho(t) dt \right| \leq (k-j) \left[\frac{1}{4} + \frac{\left(\theta - \frac{j+k}{2}\right)^2}{(k-j)^2} \right] \|\rho'(x)\|_\infty. \quad (1.1)$$

The constant $\frac{1}{4}$ is the best possible constant that it cannot be replaced by smaller one.

Ostrowski inequality for differentiable functions has been generalized in many times as stated in [6, 18, 19, 27]. For latest work related to Ostrowski inequality we refer the reader to following articles [1, 2, 3, 5, 7, 12, 13, 14, 15, 16, 17, 21, 22, 23, 24, 26]

To prove our main results we need the following two lemmas from [8] and [10].

Lemma 1.2. *Let $\rho : I \rightarrow \mathbb{R}$, be a differentiable function in interior I^o of interval I and also let $[j, k] \subset I^o$. Then the following identity holds*

$$\begin{aligned} \int_j^k K(\theta, t) \rho'(t) dt &= \rho(k) \int_\beta^k \omega(t) dt - \rho(j) \int_\alpha^j \omega(t) dt + \rho(\theta) \int_\alpha^{\frac{\alpha+\beta}{2}} \omega(t) dt \\ &+ \rho(j+k-\theta) \int_{\frac{\alpha+\beta}{2}}^\beta \omega(t) dt - \int_j^k \rho(t) \omega(t) dt, \end{aligned} \quad (1.2)$$

where $K(\theta, t)$ defined as:

$$K(\theta, t) = \begin{cases} \int_{\alpha}^t \omega(t) dt, & \text{if } t \in [j, \theta]; \\ \int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt, & \text{if } t \in (\theta, j+k-\theta]; \\ \int_{\beta}^t \omega(t) dt, & \text{if } t \in (j+k-\theta, k]; \end{cases} \quad (1.3)$$

where $\forall \theta \in [j, k]$, $\alpha = j + \lambda \frac{k-j}{2}$, $\beta = k - \lambda \frac{k-j}{2}$ and $\lambda \in [0, 1]$.

Lemma 1.3. Let $\rho : [j, k] \rightarrow \mathbb{R}$ be a differentiable function such that $\gamma(\theta) \leq \rho'(\theta) \leq \Gamma(\theta)$ for any $\gamma, \Gamma \in C[j, k]$ and $\theta \in [j, k]$. Then we have

$$\left| \rho'(t) - \frac{\gamma(t) + \Gamma(t)}{2} \right| \leq \frac{\Gamma(t) - \gamma(t)}{2}. \quad (1.4)$$

We are ready to present our main theorem which will be generalized in two ways: first, by adding weights that are actually probability density functions, and second, by adding a parameter. In this way, we will capture variety of results from various articles as special cases.

2. MAIN RESULTS

Theorem 2.1. Let $\rho : I \rightarrow \mathbb{R}$, be a differentiable function on I^0 and also let $\gamma(\theta) \leq \rho'(\theta) \leq \Gamma(\theta)$ for any $\gamma, \Gamma \in C[j, k]$ and $\theta \in [j, k]$. Then

$$\begin{aligned} m(\theta, \lambda) \leq & \rho(k) \int_{\beta}^k \omega(t) dt - \rho(j) \int_{\alpha}^j \omega(t) dt + \rho(\theta) \int_{\alpha}^{\frac{\alpha+\beta}{2}} \omega(t) dt \\ & + \rho(j+k-\theta) \int_{\frac{\alpha+\beta}{2}}^{\beta} \omega(t) dt - \int_j^k \rho(t) \omega(t) dt \leq M(\theta, \lambda), \end{aligned} \quad (2.1)$$

where

$$\begin{aligned} m(\theta, \lambda) = & \int_j^{\theta} \left(\left(\int_{\alpha}^t \omega(t) dt - \left| \int_{\alpha}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ & + \left. \left(\int_{\alpha}^t \omega(t) dt + \left| \int_{\alpha}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\ & + \int_{\theta}^{j+k-\theta} \left(\left(\int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt - \left| \int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ & + \left. \left(\int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt + \left| \int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\ & + \int_{j+k-\theta}^k \left(\left(\int_{\beta}^t \omega(t) dt - \left| \int_{\beta}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ & + \left. \left(\int_{\beta}^t \omega(t) dt + \left| \int_{\beta}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \end{aligned}$$

and

$$\begin{aligned}
 M(\theta, \lambda) &= \int_j^\theta \left(\left(\int_\alpha^t \omega(t) dt + \left| \int_\alpha^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\
 &+ \left. \left(\int_\alpha^t \omega(t) dt - \left| \int_\alpha^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\
 &+ \int_\theta^{j+k-\theta} \left(\left(\int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt + \left| \int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\
 &+ \left. \left(\int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt - \left| \int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\
 &+ \int_{j+k-\theta}^k \left(\left(\int_\beta^t \omega(t) dt + \left| \int_\beta^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\
 &+ \left. \left(\int_\beta^t \omega(t) dt - \left| \int_\beta^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt.
 \end{aligned}$$

Proof. Using (1.2) and (1.4), we obtain

$$\begin{aligned}
 &\int_j^k K(\theta, t) \left(\rho'(t) - \frac{\gamma(t) + \Gamma(t)}{2} \right) dt \\
 &= \int_j^k K(\theta, t) \rho'(t) dt - \frac{1}{2} \left(\int_j^k K(\theta, t) (\gamma(t) + \Gamma(t)) dt \right) \\
 &= \rho(k) \int_\beta^k \omega(t) dt - \rho(j) \int_\alpha^j \omega(t) dt + \rho(\theta) \int_\alpha^{\frac{\alpha+\beta}{2}} \omega(t) dt \\
 &+ \rho(j+k-\theta) \int_{\frac{\alpha+\beta}{2}}^\beta \omega(t) dt - \int_j^k \omega(t) \rho(t) dt \\
 &- \frac{1}{2} \left[\int_j^\theta \int_\alpha^t \omega(t) dt (\gamma(t) + \Gamma(t)) dt + \int_\theta^{j+k-\theta} \int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt (\gamma(t) + \Gamma(t)) dt \right. \\
 &\left. + \int_{j+k-\theta}^k \int_\beta^t \omega(t) dt (\gamma(t) + \Gamma(t)) dt \right]. \tag{2.2}
 \end{aligned}$$

From (1.4) we have

$$\begin{aligned}
 &\left| \int_\beta^k \omega(t) dt \rho(k) - \rho(j) \int_\alpha^j \omega(t) dt + \rho(\theta) \int_\alpha^{\frac{\alpha+\beta}{2}} \omega(t) dt + \rho(j+k-\theta) \right. \\
 &\times \left. \int_{\frac{\alpha+\beta}{2}}^\beta \omega(t) dt - \int_j^k \omega(t) \rho(t) dt - \frac{1}{2} \left[\int_j^\theta \int_\alpha^t \omega(t) dt (\gamma(t) + \Gamma(t)) dt \right. \right. \\
 &+ \left. \int_\theta^{j+k-\theta} \int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt (\gamma(t) + \Gamma(t)) dt \right. \\
 &\left. \left. + \int_{j+k-\theta}^k \int_\beta^t \omega(t) dt (\gamma(t) + \Gamma(t)) dt \right] \right|
 \end{aligned}$$

$$\begin{aligned}
&= \left| \int_j^k K(\theta, t) \left(\rho'(t) - \frac{\gamma(t) + \Gamma(t)}{2} \right) dt \right| \\
&\leq \int_j^k |K(\theta, t)| \left| \left(\rho'(t) - \frac{\gamma(t) + \Gamma(t)}{2} \right) \right| dt \\
&\leq \int_j^k |K(\theta, t)| \left(\frac{\Gamma(t) - \gamma(t)}{2} \right) dt \\
&= \frac{1}{2} \left[\int_j^\theta \left| \int_\alpha^t \omega(t) dt \right| (\Gamma(t) - \gamma(t)) dt + \int_\theta^{j+k-\theta} \left| \int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt \right| \right. \\
&\quad \left. \times (\Gamma(t) - \gamma(t)) dt + \int_{j+k-\theta}^k \left| \int_\beta^t \omega(t) dt \right| (\Gamma(t) - \gamma(t)) dt \right]. \quad (2.3)
\end{aligned}$$

After rearranging (2.3), we get the required result. $\square$

Remark 2.2. It is worth mentioning that if we put $\omega(t) = \frac{1}{k-j}$ in our main result we will get the following result.

Corollary 2.3. *Let all the assumptions of Theorem 2.1 be valid. Then*

$$\begin{aligned}
m_0(\theta, \lambda) &\leq \lambda \frac{\rho(j) + \rho(k)}{2} + (1 - \lambda) \frac{\rho(\theta) + \rho(j + k - \theta)}{2} - \int_j^k \rho(t) d(t) \\
&\leq M_0(\theta, \lambda) \quad (2.4)
\end{aligned}$$

$$m_0(t, \lambda) =$$

$$\begin{aligned}
&\frac{1}{k-j} \left[\int_{-\lambda \frac{k-j}{2}}^{t-(j+\lambda \frac{k-j}{2})} \left(\frac{\eta + |\eta|}{2} \mu \left(\eta + j + \lambda \frac{k-j}{2} \right) + \frac{\eta - |\eta|}{2} \nu \left(\eta + j + \lambda \frac{k-j}{2} \right) \right) d\eta \right. \\
&+ \int_{t-\frac{j+k}{2}}^{\frac{j+k}{2}-t} \left(\frac{\eta + |\eta|}{2} \mu \left(\eta + \frac{j+k}{2} \right) + \frac{\eta - |\eta|}{2} \nu \left(\eta + \frac{j+k}{2} \right) \right) d\eta \\
&\left. + \int_{j+\lambda \frac{k-j}{2}-t}^{\lambda \frac{k-j}{2}} \left(\frac{\eta + |\eta|}{2} \mu \left(\eta + j - \lambda \frac{k-j}{2} \right) + \frac{\eta - |\eta|}{2} \nu \left(\eta + j - \lambda \frac{k-j}{2} \right) \right) d\eta \right]
\end{aligned}$$

and

$$\begin{aligned}
M_0(t, \lambda) &= \frac{1}{k-j} \left[\int_{-\lambda \frac{k-j}{2}}^{t-(j+\lambda \frac{k-j}{2})} \left(\frac{\eta - |\eta|}{2} \mu \left(\eta + j + \lambda \frac{k-j}{2} \right) + \frac{\eta + |\eta|}{2} \nu \left(\eta + j + \lambda \frac{k-j}{2} \right) \right) d\eta \right. \\
&+ \int_{t-\frac{j+k}{2}}^{\frac{j+k}{2}-t} \left(\frac{\eta - |\eta|}{2} \mu \left(\eta + \frac{j+k}{2} \right) + \frac{\eta + |\eta|}{2} \nu \left(\eta + \frac{j+k}{2} \right) \right) d\eta \\
&\left. + \int_{j+\lambda \frac{k-j}{2}-t}^{\lambda \frac{k-j}{2}} \left(\frac{\eta - |\eta|}{2} \mu \left(\eta + j - \lambda \frac{k-j}{2} \right) + \frac{\eta + |\eta|}{2} \nu \left(\eta + j - \lambda \frac{k-j}{2} \right) \right) d\eta \right],
\end{aligned}$$

which is Theorem 2.3 of [7] and hence all its Corollaries and Remarks and further consequences would become our special cases. Throughout the section $\gamma_0, \gamma_1, \Gamma_0, \Gamma_1$ are real constants.

Remark 2.4. If we put $\lambda = 1$ in (2.1), we obtain following result.

Corollary 2.5. *Let all the assumptions of Theorem 2.1 be valid. Then*

$$m_1 \leq \rho(k) \int_{\frac{j+k}{2}}^k \omega(t) dt + \rho(j) \int_j^{\frac{j+k}{2}} \omega(t) dt - \int_j^k \rho(t) \omega(t) dt \leq M_1, \quad (2.5)$$

where

$$\begin{aligned} m_1 &= \int_j^k \left[\left(\int_{\frac{j+k}{2}}^t \omega(t) dt - \left| \int_{\frac{j+k}{2}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &\quad \left. + \left(\int_{\frac{j+k}{2}}^t \omega(t) dt + \left| \int_{\frac{j+k}{2}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right] dt \end{aligned}$$

and

$$\begin{aligned} M_1 &= \int_j^k \left[\left(\int_{\frac{j+k}{2}}^t \omega(t) dt + \left| \int_{\frac{j+k}{2}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &\quad \left. + \left(\int_{\frac{j+k}{2}}^t \omega(t) dt - \left| \int_{\frac{j+k}{2}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right] dt. \end{aligned}$$

Special Case 2.5.(a) If we take, $\gamma(t) = \gamma_0 \neq 0$, $\Gamma(t) = \Gamma_0 \neq 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.5), then

$$\frac{(k-j)}{8}(\gamma_0 - \Gamma_0) \leq \frac{\rho(j) + \rho(k)}{2} - \frac{1}{k-j} \int_j^k \rho(t) dt \leq \frac{(k-j)}{8}(\Gamma_0 - \gamma_0),$$

which is the Corollary 2 of [27] and Special Case 2.4.1 of [7].

Special Case 2.5.(b) If we take, $\gamma(t) = \gamma_1 t + \gamma_0 \neq 0$, $\Gamma(t) = \Gamma_1 t + \Gamma_0 \neq 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.5), then

$$m_2 \leq \frac{\rho(j) + \rho(k)}{2} - \frac{1}{k-j} \int_j^k \rho(t) dt \leq M_2,$$

where

$$m_2 = \frac{(k-j)}{8} \left[\frac{(k-j)}{3}(\gamma_1 + \Gamma_1) + \frac{(j+k)}{2}(\gamma_1 - \Gamma_1) + \gamma_0 - \Gamma_0 \right]$$

and

$$M_2 = \frac{(k-j)}{8} \left[\frac{(k-j)}{3}(\gamma_1 + \Gamma_1) + \frac{(j+k)}{2}(\Gamma_1 - \gamma_1) + \Gamma_0 - \gamma_0 \right],$$

which is Special Case 2.4.2 of [7].

Remark 2.6. If we choose $\theta = \frac{j+k}{2}$ in (2.1), we obtain the following result.

Corollary 2.7. *Let all the assumptions of Theorem 2.1 be valid. Then*

$$\begin{aligned} m_3(\lambda) &\leq \rho(k) \int_{\beta}^k \omega(t) dt - \rho(j) \int_{\alpha}^j \omega(t) dt + \rho\left(\frac{k+j}{2}\right) \int_{\alpha}^{\beta} \omega(t) dt \\ &\quad - \int_j^k \rho(t) \omega(t) dt \leq M_3(\lambda), \quad (2.6) \end{aligned}$$

where

$$\begin{aligned} m_3(\lambda) &= \int_j^{\frac{j+k}{2}} \left(\left(\int_\alpha^t \omega(t) dt - \left| \int_\alpha^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &\quad + \left. \left(\int_\alpha^t \omega(t) dt + \left| \int_\alpha^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\ &\quad + \int_{\frac{j+k}{2}}^k \left(\left(\int_\beta^t \omega(t) dt - \left| \int_\beta^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &\quad + \left. \left(\int_\beta^t \omega(t) dt + \left| \int_\beta^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \end{aligned}$$

and

$$\begin{aligned} M_3(\lambda) &= \int_j^{\frac{j+k}{2}} \left(\left(\int_\alpha^t \omega(t) dt + \left| \int_\alpha^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &\quad + \left. \left(\int_\alpha^t \omega(t) dt - \left| \int_\alpha^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\ &\quad + \int_{\frac{j+k}{2}}^k \left(\left(\int_\beta^t \omega(t) dt + \left| \int_\beta^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &\quad + \left. \left(\int_\beta^t \omega(t) dt - \left| \int_\beta^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt. \end{aligned}$$

Remark 2.8. If we choose $\lambda = 0$ in (2.6), we obtain the following result.

Corollary 2.9. *Let all the assumptions of Theorem 2.1 be valid. Then*

$$m_4 \leq \rho \left(\frac{k+j}{2} \right) \int_j^k \omega(t) dt - \int_j^k \rho(t) \omega(t) dt \leq M_4(t)$$

where

$$\begin{aligned} m_4 &= \int_j^{\frac{j+k}{2}} \left(\left(\int_j^t \omega(t) dt - \left| \int_j^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &\quad + \left. \left(\int_j^t \omega(t) dt + \left| \int_j^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\ &\quad + \int_{\frac{j+k}{2}}^k \left(\left(\int_k^t \omega(t) dt - \left| \int_k^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &\quad + \left. \left(\int_k^t \omega(t) dt + \left| \int_k^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \end{aligned}$$

and

$$\begin{aligned} M_4 &= \int_j^{\frac{j+k}{2}} \left(\left(\int_j^t \omega(t) dt + \left| \int_j^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &\quad \left. + \left(\int_j^t \omega(t) dt - \left| \int_j^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\ &\quad + \int_{\frac{j+k}{2}}^k \left(\left(\int_k^t \omega(t) dt + \left| \int_k^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &\quad \left. + \left(\int_k^t \omega(t) dt - \left| \int_k^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt. \end{aligned}$$

Special Case 2.9.(a) If we take, $\gamma(\theta) = \gamma_0 \neq 0$, $\Gamma(\theta) = \Gamma_0 \neq 0$, $\theta = \frac{j+k}{2}$ and $\omega(t) = \frac{1}{k-j}$, in (2.7), then

$$\frac{(k-j)}{8}(\gamma_0 - \Gamma_0) \leq \rho \left(\frac{j+k}{2} \right) - \frac{1}{k-j} \int_j^k \rho(t) dt \leq \frac{(k-j)}{8}(\Gamma_0 - \gamma_0),$$

which is in fact the Special Case 1 of Theorem 1 presented in [18], Corollary 1 of [27] and Special Case 2.6.1 of [7].

Special Case 2.9.(b) If we take, $\gamma(\theta) = \gamma_1 \theta + \gamma_0 \neq 0$, $\Gamma(\theta) = \Gamma_1 \theta + \Gamma_0 \neq 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.7), then

$$m_5 \leq \rho \left(\frac{j+k}{2} \right) - \frac{1}{k-j} \int_j^k \rho(t) dt \leq M_5,$$

where

$$m_5 = \frac{(k-j)}{8} \left(\frac{k-j}{3} (\gamma_1 + \Gamma_1) + j\gamma_1 - k\Gamma_1 + \gamma_0 - \Gamma_0 \right)$$

and

$$M_5 = \frac{(k-j)}{8} \left(\frac{k-j}{3} (\gamma_1 + \Gamma_1) + j\Gamma_1 - k\gamma_1 + \Gamma_0 - \gamma_0 \right),$$

which is example of Corollary 1 of [18] and Special Case 2.6.2 of [7].

Remark 2.10. By choosing $\lambda = \frac{1}{3}$ in (2.6), then we get the bounds for $\frac{1}{3}$ Simpson's rule.

Corollary 2.11. *Let all the assumptions of Theorem 2.1 be valid. Then*

$$\begin{aligned} m_6 &\leq \rho(k) \int_{\frac{j+5k}{6}}^k \omega(t) dt - \rho(j) \int_{\frac{5j+k}{6}}^j \omega(t) dt + \rho \left(\frac{k+j}{2} \right) \int_{\frac{5j+k}{6}}^{\frac{j+k}{2}} \omega(t) dt \\ &\quad + \rho \left(\frac{k+j}{2} \right) \int_{\frac{j+k}{2}}^{\frac{j+5k}{6}} \omega(t) dt - \int_j^k \rho(t) \omega(t) dt \leq M_6, \quad (2.7) \end{aligned}$$

where

$$\begin{aligned} m_6 &= \int_{-\frac{k-j}{6}}^{\frac{j+k}{2}} \left(\left(\int_{\frac{5j+k}{6}}^t \omega(t) dt - \left| \int_{\frac{5j+k}{6}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &+ \left. \left(\int_{\frac{5j+k}{6}}^t \omega(t) dt + \left| \int_{\frac{5j+k}{6}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\ &+ \int_{\frac{j+k}{2}}^{\frac{k-j}{6}} \left(\left(\int_{\frac{j+5k}{6}}^t \omega(t) dt - \left| \int_{\frac{j+5k}{6}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &+ \left. \left(\int_{\frac{j+5k}{6}}^t \omega(t) dt + \left| \int_{\frac{j+5k}{6}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \end{aligned}$$

and

$$\begin{aligned} M_6 &= \int_{-\frac{k-j}{6}}^{\frac{j+k}{2}} \left(\left(\int_{\frac{5j+k}{6}}^t \omega(t) dt + \left| \int_{\frac{5j+k}{6}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &+ \left. \left(\int_{\frac{5j+k}{6}}^t \omega(t) dt - \left| \int_{\frac{5j+k}{6}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\ &+ \int_{\frac{j+k}{2}}^{\frac{k-j}{6}} \left(\left(\int_{\frac{j+5k}{6}}^t \omega(t) dt + \left| \int_{\frac{j+5k}{6}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &+ \left. \left(\int_{\frac{j+5k}{6}}^t \omega(t) dt - \left| \int_{\frac{j+5k}{6}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt. \end{aligned}$$

Special Case 2.11.(a) If we take, $\gamma(t) = \gamma_0 \neq 0$, $\Gamma(t) = \Gamma_0 \neq 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.7), then

$$\begin{aligned} \frac{(k-j)}{72}(\gamma_0 - \Gamma_0) &\leq \frac{1}{3} \left[\frac{\rho(j) + \rho(k)}{2} + 2\rho\left(\frac{j+k}{2}\right) \right] - \frac{1}{k-j} \int_j^k \rho(t) dt \\ &\leq \frac{(k-j)}{72}(\Gamma_0 - \gamma_0), \end{aligned}$$

which is Corollary 4 of [27] and Special Case 2.7.1 of [7].

Special Case 2.11.(b) If we take, $\gamma(t) = \gamma_1 t + \gamma_0 \neq 0$ and $\Gamma(t) = \Gamma_1 t + \Gamma_0 \neq 0$, $\omega(t) = \frac{1}{k-j}$ in (2.7), then

$$m_7 \leq \frac{1}{3} \left[\frac{\rho(j) + \rho(k)}{2} + 2\rho\left(\frac{j+k}{2}\right) \right] - \frac{1}{k-j} \int_j^k \rho(t) dt \leq M_7,$$

where

$$m_7 = \frac{(k-j)}{72} \left[(k-j)(\gamma_1 + \Gamma_1) + \frac{j}{2}(7\gamma_1 - 3\Gamma_1) + \frac{k}{2}(3\gamma_1 - 7\Gamma_1) + 5(\gamma_0 - \Gamma_0) \right]$$

and

$$M_7 = \frac{(k-j)}{72} \left[(k-j)(\gamma_1 + \Gamma_1) + \frac{j}{2}(7\Gamma_1 - 3\gamma_1) + \frac{k}{2}(3\Gamma_1 - 7\gamma_1) + 5(\Gamma_0 - \gamma_0) \right],$$

which is Special Case 2.7.2 of [7].

Remark 2.12. If we choose $\lambda = \frac{1}{2}$ in (2.6), we obtain bound of average midpoint and trapezoidal inequality in the following result.

Corollary 2.13. *Let all the assumptions of Theorem 2.1 be valid. Then*

$$m_8 \leq \rho(k) \int_{\frac{j+3k}{4}}^k \omega(t) dt - \rho(j) \int_{\frac{3j+k}{4}}^j \omega(t) dt + \rho\left(\frac{k+j}{2}\right) \int_{\frac{3j+k}{4}}^{\frac{j+k}{2}} \omega(t) dt \\ + \rho\left(\frac{k+j}{2}\right) \int_{\frac{j+k}{2}}^{\frac{j+3k}{4}} \omega(t) dt - \int_j^k \rho(t) \omega(t) dt \leq M_8, \quad (2.8)$$

where

$$m_8 = \int_{-\frac{k-j}{4}}^{\frac{j+k}{2}} \left(\left(\int_{\frac{3j+k}{4}}^t \omega(t) dt - \left| \int_{\frac{3j+k}{4}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ + \left. \left(\int_{\frac{3j+k}{4}}^t \omega(t) dt + \left| \int_{\frac{3j+k}{4}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\ + \int_{\frac{j+k}{2}}^{\frac{k-j}{4}} \left(\left(\int_{\frac{j+3k}{4}}^t \omega(t) dt - \left| \int_{\frac{j+3k}{4}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ + \left. \left(\int_{\frac{j+3k}{4}}^t \omega(t) dt + \left| \int_{\frac{j+3k}{4}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt$$

and

$$M_8 = \int_{-\frac{k-j}{4}}^{\frac{j+k}{2}} \left(\left(\int_{\frac{3j+k}{4}}^t \omega(t) dt + \left| \int_{\frac{3j+k}{4}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ + \left. \left(\int_{\frac{3j+k}{4}}^t \omega(t) dt - \left| \int_{\frac{3j+k}{4}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\ + \int_{\frac{j+k}{2}}^{\frac{k-j}{4}} \left(\left(\int_{\frac{j+3k}{4}}^t \omega(t) dt + \left| \int_{\frac{j+3k}{4}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ + \left. \left(\int_{\frac{j+3k}{4}}^t \omega(t) dt - \left| \int_{\frac{j+3k}{4}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt.$$

Special Case 2.13.(a) If we take, $\gamma(t) = \gamma_0 \neq 0$, $\Gamma(t) = \Gamma_0 \neq 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.8), then

$$\frac{(k-j)}{16} (\gamma_0 - \Gamma_0) \leq \frac{1}{2} \left[\frac{\rho(j) + \rho(k)}{2} + \rho\left(\frac{j+k}{2}\right) \right] - \frac{1}{k-j} \int_j^k \rho(t) dt \\ \leq \frac{(k-j)}{16} (\Gamma_0 - \gamma_0),$$

which is the Corollary 3 of [27] and Special Case 2.8.1 of [7].

Special Case 2.13.(b) If we take, $\gamma(t) = \gamma_1 t + \gamma_0 \neq 0$, $\Gamma(t) = \Gamma_1 t + \Gamma_0 \neq 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.8), then

$$m_9 \leq \frac{1}{2} \left[\frac{\rho(j) + \rho(k)}{2} + \rho \left(\frac{j+k}{2} \right) \right] - \frac{1}{k-j} \int_j^k \rho(t) dt \leq M_9,$$

where

$$m_9 = \frac{(k-j)}{16} \left[\frac{(k-j)}{6} (\gamma_1 + \Gamma_1) + \frac{j}{2} (\gamma_1 - \Gamma_1) + \frac{k}{2} (\gamma_1 - \Gamma_1) + \gamma_0 - \Gamma_0 \right]$$

and

$$M_9 = \frac{(k-j)}{16} \left[\frac{(k-j)}{6} (\gamma_1 + \Gamma_1) + \frac{j}{2} (\Gamma_1 - \gamma_1) + \frac{k}{2} (\Gamma_1 - \gamma_1) + \Gamma_0 - \gamma_0 \right],$$

which is Special Case 2.8.2 of [7].

Remark 2.14. If we choose $\theta = j$ in (2.1), for any value of $\lambda \in [0, 1]$ we obtain the following result.

Corollary 2.15. *Let all the assumptions of Theorem 2.1 be valid. Then*

$$\begin{aligned} m_{10}(\lambda) &\leq \rho(k) \int_{\beta}^k \omega(t) dt - \rho(j) \int_{\alpha}^j \omega(t) dt + \rho(p) \int_{\alpha}^{\frac{\alpha+\beta}{2}} \omega(t) dt \\ &\quad + \rho(k) \int_{\frac{\alpha+\beta}{2}}^{\beta} \omega(t) dt - \int_j^k \rho(t) \omega(t) dt \leq M_{10}(\lambda), \end{aligned} \quad (2.9)$$

where

$$\begin{aligned} m_{10}(\lambda) &= \int_j^k \left(\left(\int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt - \left| \int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &\quad \left. + \left(\int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt + \left| \int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \end{aligned}$$

and

$$\begin{aligned} M_{10}(\lambda) &= \int_j^k \left(\left(\int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt + \left| \int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &\quad \left. + \left(\int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt - \left| \int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt. \end{aligned}$$

Remark 2.16. If we take, $\gamma(t) = \gamma_0 \neq 0$ and $\Gamma(t) = \Gamma_0 \neq 0$, and $\gamma(t) = \gamma_1 t + \gamma_0 \neq 0$ and $\Gamma(t) = \Gamma_1 t + \Gamma_0 \neq 0$, and $\omega(t) = \frac{1}{k-j}$ in (2.9), then we obtain results similar to Special Cases 2.4 a and b respectively.

Remark 2.17. If we choose $\theta = k$, and $\lambda = 0$ in (2.1), we obtain the following result.

Corollary 2.18. *Let all suppositions of Theorem 2.1 be valid. Then*

$$m_{11} \leq \frac{\rho(j) + \rho(k)}{2} - \frac{1}{k-j} \int_j^k \rho(t) dt \leq M_{11}, \quad (2.10)$$

where

$$\begin{aligned} m_{11} &= \int_j^k \left(\left(\int_j^t \omega(t) dt - \left| \int_j^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &\quad + \left. \left(\int_j^t \omega(t) dt + \left| \int_j^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\ &\quad + \int_k^j \left(\left(\int_{\frac{j+k}{2}}^t \omega(t) dt - \left| \int_{\frac{j+k}{2}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &\quad + \left. \left(\int_{\frac{j+k}{2}}^t \omega(t) dt + \left| \int_{\frac{j+k}{2}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\ &\quad + \int_j^k \left(\left(\int_k^t \omega(t) dt - \left| \int_k^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &\quad + \left. \left(\int_k^t \omega(t) dt + \left| \int_k^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt. \end{aligned}$$

and

$$\begin{aligned} M_{11} &= \int_j^k \left(\left(\int_j^t \omega(t) dt + \left| \int_j^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &\quad + \left. \left(\int_j^t \omega(t) dt - \left| \int_j^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\ &\quad + \int_k^j \left(\left(\int_{\frac{j+k}{2}}^t \omega(t) dt + \left| \int_{\frac{j+k}{2}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &\quad + \left. \left(\int_{\frac{j+k}{2}}^t \omega(t) dt - \left| \int_{\frac{j+k}{2}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\ &\quad + \int_j^k \left(\left(\int_k^t \omega(t) dt + \left| \int_k^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &\quad + \left. \left(\int_k^t \omega(t) dt - \left| \int_k^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt. \end{aligned}$$

Special Case 2.18.(a) If we take, $\gamma(t) = \gamma_0 \neq 0$, $\Gamma(t) = \Gamma_0 \neq 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.10), then

$$\frac{3(k-j)}{8}(\gamma_0 - \Gamma_0) \leq \left[\frac{\rho(j) + \rho(k)}{2} \right] - \frac{1}{k-j} \int_j^k \rho(t) dt \leq \frac{3(k-j)}{8}(\Gamma_0 - \gamma_0),$$

which is Special Case 2.10.1 of [7].

Special Case 2.18.(b) If we take, $\gamma(t) = \gamma_1 t + \gamma_0 \neq 0$, $\Gamma(t) = \Gamma_1 t + \Gamma_0 \neq 0$

and $\omega(t) = \frac{1}{k-j}$ in (2.10), then

$$m_{12} \leq \left[\frac{\rho(j) + \rho(k)}{2} \right] - \frac{1}{k-j} \int_j^k \rho(t) dt \leq M_{12},$$

where

$$m_{12} = \frac{(k-j)}{2} \left[\frac{7(k-j)}{12} (\gamma_1 + \Gamma_1) + \frac{j}{8} (7\gamma_1 + \Gamma_1) - \frac{k}{8} (\gamma_1 + 7\Gamma_1) + \frac{3}{4} (\gamma_0 - \Gamma_0) \right]$$

and

$$M_{12} = \frac{(k-j)}{2} \left[\frac{7(k-j)}{12} (\gamma_1 + \Gamma_1) + \frac{j}{8} (\gamma_1 + 7\Gamma_1) - \frac{k}{8} (7\gamma_1 + \Gamma_1) + \frac{3}{4} (\Gamma_0 - \gamma_0) \right],$$

which is Special Case 2.10.2 of [7].

Remark 2.19. If we choose $\theta = k$ and $\lambda = \frac{1}{2}$ in (2.1), we obtain the following result.

Corollary 2.20. *Let all assumptions of Theorem 2.1 be valid. Then*

$$m_{13} \leq \rho(k) \int_{\frac{j+3k}{4}}^k \omega(t) dt - \rho(j) \int_{\frac{3j+k}{4}}^j \omega(t) dt + \rho(k) \int_{\frac{3j+k}{4}}^{\frac{j+k}{2}} \omega(t) dt + \rho(j) \int_{\frac{j+k}{2}}^{\frac{j+3k}{4}} \omega(t) dt - \int_j^k \rho(t) \omega(t) dt \leq M_{13}, \quad (2.11)$$

where

$$\begin{aligned} m_{13} &= \int_j^k \left(\left(\int_{\frac{3j+k}{4}}^t \omega(t) dt - \left| \int_{\frac{3j+k}{4}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &\quad + \left. \left(\int_{\frac{3j+k}{4}}^t \omega(t) dt + \left| \int_{\frac{3j+k}{4}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\ &\quad + \int_k^j \left(\left(\int_{\frac{j+k}{2}}^t \omega(t) dt - \left| \int_{\frac{j+k}{2}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &\quad + \left. \left(\int_{\frac{j+k}{2}}^t \omega(t) dt + \left| \int_{\frac{j+k}{2}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\ &\quad + \int_j^k \left(\left(\int_{\frac{j+3k}{4}}^t \omega(t) dt - \left| \int_{\frac{j+3k}{4}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &\quad + \left. \left(\int_{\frac{j+3k}{4}}^t \omega(t) dt + \left| \int_{\frac{j+3k}{4}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \end{aligned}$$

and

$$\begin{aligned}
 M_{13} = & \int_j^k \left(\left(\int_{\frac{3j+k}{4}}^t \omega(t) dt + \left| \int_{\frac{3j+k}{4}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\
 & + \left. \left(\int_{\frac{3j+k}{4}}^t \omega(t) dt - \left| \int_{\frac{3j+k}{4}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\
 & + \int_k^j \left(\left(\int_{\frac{j+k}{2}}^t \omega(t) dt + \left| \int_{\frac{j+k}{2}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\
 & + \left. \left(\int_{\frac{j+k}{2}}^t \omega(t) dt - \left| \int_{\frac{j+k}{2}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\
 & + \int_j^k \left(\left(\int_{\frac{j+3k}{4}}^t \omega(t) dt + \left| \int_{\frac{j+3k}{4}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\
 & + \left. \left(\int_{\frac{j+3k}{4}}^t \omega(t) dt - \left| \int_{\frac{j+3k}{4}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt.
 \end{aligned}$$

Special Case 2.20.(a) If we take, $\gamma(t) = \gamma_0 \neq 0$, $\Gamma(t) = \Gamma_0 \neq 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.10), then

$$\frac{9(k-j)}{32}(\gamma_0 - \Gamma_0) \leq \left[\frac{\rho(j) + \rho(k)}{2} \right] - \frac{1}{k-j} \int_j^k \rho(t) dt \leq \frac{9(k-j)}{32}(\Gamma_0 - \gamma_0),$$

which is Special Case 2.11.1 of [7].

Special Case 2.20.(b) If we take, $\gamma(t) = \gamma_1 t + \gamma_0 \neq 0$, $\Gamma(t) = \Gamma_1 t + \Gamma_0 \neq 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.10), then

$$m_{14} \leq \left[\frac{\rho(j) + \rho(k)}{2} \right] - \frac{1}{k-j} \int_j^k \rho(t) dt \leq M_{14},$$

where

$$\begin{aligned}
 m_{14} = & \frac{k-j}{16} \left[\frac{5(k-j)}{3}(\gamma_1 + \Gamma_1) + \frac{j}{2}(5\gamma_1 - \Gamma_1) - \frac{k}{2}(\gamma_1 - 5\Gamma_1) \right. \\
 & + \left. 3(\gamma_0 - \Gamma_0) \right]
 \end{aligned}$$

and

$$\begin{aligned}
 M_{14} = & \frac{k-j}{16} \left[\frac{5(k-j)}{3}(\gamma_1 + \Gamma_1) + \frac{j}{2}(5\Gamma_1 - \gamma_1) - \frac{k}{2}(\Gamma_1 - 5\gamma_1) \right. \\
 & + \left. 3(\Gamma_0 - \gamma_0) \right],
 \end{aligned}$$

which is Special Case 2.11.2 of [7].

Remark 2.21. If we choose $\theta = k$ and $\lambda = \frac{1}{3}$ in (2.1), then we get the following result.

Corollary 2.22. *Let the assumptions of Theorem 2.1 be valid. Then*

$$m_{15} \leq \rho(k) \int_{\frac{j+5k}{6}}^k \omega(t) dt - \rho(j) \int_{\frac{5j+k}{6}}^j \omega(t) dt + \rho(k) \int_{\frac{5j+k}{6}}^{\frac{j+k}{2}} \omega(t) dt \\ + \rho(j) \int_{\frac{j+k}{2}}^{\frac{j+5k}{6}} \omega(t) dt - \int_j^k \rho(t) \omega(t) dt \leq M_{15}, \quad (2.12)$$

where

$$m_{15} = \int_j^k \left(\left(\int_{\frac{5j+k}{6}}^t \omega(t) dt - \left| \int_{\frac{5j+k}{6}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ + \left. \left(\int_{\frac{5j+k}{6}}^t \omega(t) dt + \left| \int_{\frac{5j+k}{6}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\ + \int_k^j \left(\left(\int_{\frac{j+k}{2}}^t \omega(t) dt - \left| \int_{\frac{j+k}{2}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ + \left. \left(\int_{\frac{j+k}{2}}^t \omega(t) dt + \left| \int_{\frac{j+k}{2}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\ + \int_j^k \left(\left(\int_{\frac{j+5k}{6}}^t \omega(t) dt - \left| \int_{\frac{j+5k}{6}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ + \left. \left(\int_{\frac{j+5k}{6}}^t \omega(t) dt + \left| \int_{\frac{j+5k}{6}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt$$

and

$$M_{15} = \int_j^k \left(\left(\int_{\frac{5j+k}{6}}^t \omega(t) dt + \left| \int_{\frac{5j+k}{6}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ + \left. \left(\int_{\frac{5j+k}{6}}^t \omega(t) dt - \left| \int_{\frac{5j+k}{6}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\ + \int_k^j \left(\left(\int_{\frac{j+k}{2}}^t \omega(t) dt + \left| \int_{\frac{j+k}{2}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ + \left. \left(\int_{\frac{j+k}{2}}^t \omega(t) dt - \left| \int_{\frac{j+k}{2}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\ + \int_j^k \left(\left(\int_{\frac{j+5k}{6}}^t \omega(t) dt + \left| \int_{\frac{j+5k}{6}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ + \left. \left(\int_{\frac{j+5k}{6}}^t \omega(t) dt - \left| \int_{\frac{j+5k}{6}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt.$$

Special Case 2.22.(a) If we take, $\gamma(t) = \gamma_0 \neq 0$, $\Gamma(t) = \Gamma_0 \neq 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.13), then

$$\frac{17(k-j)}{72}(\gamma_0 - \Gamma_0) \leq \left[\frac{\rho(j) + \rho(k)}{2} \right] - \frac{1}{k-j} \int_j^k \rho(t) dt \leq \frac{17(k-j)}{72}(\Gamma_0 - \gamma_0),$$

which is Special Case 2.12.1 of [7].

Special Case 2.22.(b) If we take, $\gamma(t) = \gamma_1 t + \gamma_0 \neq 0$, $\Gamma(t) = \Gamma_1 t + \Gamma_0 \neq 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.13), then

$$m_{16} \leq \left[\frac{\rho(j) + \rho(k)}{2} \right] - \frac{1}{k-j} \int_j^k \rho(t) dt \leq M_{16},$$

where

$$m_{16} = \frac{(k-j)}{72} \left[11(k-j)(\gamma_1 + \Gamma_1) + \frac{3j}{2}(33\gamma_1 - \Gamma_1) + \frac{3k}{2}(\gamma_1 - 33\Gamma_1) \right. \\ \left. + 17(\gamma_0 - \Gamma_0) \right]$$

and

$$M_{16} = \frac{(k-j)}{72} \left[(k-j)(\gamma_1 + \Gamma_1) + \frac{3j}{2}(33\Gamma_1 - \gamma_1) + \frac{3k}{2}(\Gamma_1 - 33\gamma_1) \right. \\ \left. + 17(\Gamma_0 - \gamma_0) \right],$$

which is Special Case 2.12.2 of [7].

Remark 2.23. If we choose $\theta = k$, and $\lambda = \frac{1}{4}$ in (2.1), we obtain a bound for trapezoidal rule in the following result.

Corollary 2.24. *Let all the assumptions of Theorem 2.1 be valid. Then*

$$m_{17} \leq \rho(k) \int_{\frac{j+k}{8}}^k \omega(t) dt - \rho(j) \int_{\frac{j+k}{8}}^j \omega(t) dt + \rho(k) \int_{\frac{j+k}{8}}^{\frac{j+k}{2}} \omega(t) dt \\ + \rho(j) \int_{\frac{j+k}{2}}^{\beta} \omega(t) dt - \int_j^k \rho(t) \omega(t) dt \leq M_{17}, \quad (2.13)$$

where

$$\begin{aligned}
 m_{17} = & \int_j^k \left(\left(\int_{\frac{7j+k}{8}}^t \omega(t) dt - \left| \int_{\frac{7j+k}{8}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\
 & + \left. \left(\int_{\frac{7j+k}{8}}^t \omega(t) dt + \left| \int_{\frac{7j+k}{8}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\
 & + \int_k^j \left(\left(\int_{\frac{j+k}{2}}^t \omega(t) dt - \left| \int_{\frac{j+k}{2}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\
 & + \left. \left(\int_{\frac{j+k}{2}}^t \omega(t) dt + \left| \int_{\frac{j+k}{2}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\
 & + \int_j^k \left(\left(\int_{\beta}^t \omega(t) dt - \left| \int_{\beta}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\
 & + \left. \left(\int_{\beta}^t \omega(t) dt + \left| \int_{\beta}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt.
 \end{aligned}$$

and

$$\begin{aligned}
 M_{17} = & \int_j^k \left(\left(\int_{\frac{7j+k}{8}}^t \omega(t) dt + \left| \int_{\frac{7j+k}{8}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\
 & + \left. \left(\int_{\frac{7j+k}{8}}^t \omega(t) dt - \left| \int_{\frac{7j+k}{8}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\
 & + \int_k^j \left(\left(\int_{\frac{j+k}{2}}^t \omega(t) dt + \left| \int_{\frac{j+k}{2}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\
 & + \left. \left(\int_{\frac{j+k}{2}}^t \omega(t) dt - \left| \int_{\frac{j+k}{2}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\
 & + \int_j^k \left(\left(\int_{\beta}^t \omega(t) dt + \left| \int_{\beta}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\
 & + \left. \left(\int_{\beta}^t \omega(t) dt - \left| \int_{\beta}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt.
 \end{aligned}$$

Special Case 2.24.(a) If we take, $\gamma(t) = \gamma_0 \neq 0$, $\Gamma(t) = \Gamma_0 \neq 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.13), then

$$\frac{17(k-j)}{64}(\gamma_0 - \Gamma_0) \leq \left[\frac{\rho(j) + \rho(k)}{2} \right] - \frac{1}{k-j} \int_j^k \rho(t) dt \leq \frac{17(k-j)}{64}(\Gamma_0 - \gamma_0),$$

which is Special Case 2.13.1 of [7].

Special Case 2.24.(b) If we take, $\gamma(t) = \gamma_1 t + \gamma_0 \neq 0$, $\Gamma(t) = \Gamma_1 t + \Gamma_0 \neq 0$

and $\omega(t) = \frac{1}{k-j}$ in (2.13), then

$$m_{18} \leq \left[\frac{\rho(j) + \rho(k)}{2} \right] - \frac{1}{k-j} \int_j^k \rho(t) dt \leq M_{18},$$

where

$$m_{18} = \frac{(k-j)}{64} \left[\frac{35}{3}(k-j)(\gamma_1 + \Gamma_1) + \frac{j}{2}(35\gamma_1 + \Gamma_1) - \frac{k}{2}(\gamma_1 + 35\Gamma_1) + 17(\gamma_0 - \Gamma_0) \right]$$

and

$$M_{18} = \frac{(k-j)}{64} \left[\frac{35}{3}(k-j)(\gamma_1 + \Gamma_1) + \frac{j}{2}(35\Gamma_1 + \gamma_1) - \frac{k}{2}(\Gamma_1 + 35\gamma_1) + 17(\Gamma_0 - \gamma_0) \right],$$

which is Special Case 2.13.2 of [7].

Remark 2.25. if we choose $\theta = \frac{2j+k}{3}$ and $\lambda = \frac{1}{4}$ in (2.1), then we get the following result.

Corollary 2.26. *Let all the assumptions of Theorem 2.1 be valid. Then*

$$\begin{aligned} m_{19} &\leq \rho(k) \int_{\frac{j+7k}{8}}^k \omega(t) dt - \rho(j) \int_{\frac{7j+k}{8}}^j \omega(t) dt + \rho\left(\frac{2j+k}{3}\right) \int_{\frac{7j+k}{8}}^{\frac{j+k}{2}} \omega(t) dt \\ &+ \rho\left(\frac{j+2k}{3}\right) \int_{\frac{j+k}{2}}^{\frac{j+7k}{8}} \omega(t) dt - \int_j^k \rho(t) \omega(t) dt \leq M_{19}, \end{aligned} \quad (2.14)$$

where

$$\begin{aligned} m_{19} &= \int_j^{\frac{2j+k}{3}} \left(\left(\int_{\frac{7j+k}{8}}^t \omega(t) dt - \left| \int_{\frac{7j+k}{8}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &+ \left. \left(\int_{\frac{7j+k}{8}}^t \omega(t) dt + \left| \int_{\frac{7j+k}{8}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\ &+ \int_{\frac{2j+k}{3}}^{\frac{j+2k}{3}} \left(\left(\int_{\frac{j+k}{2}}^t \omega(t) dt - \left| \int_{\frac{j+k}{2}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &+ \left. \left(\int_{\frac{j+k}{2}}^t \omega(t) dt + \left| \int_{\frac{j+k}{2}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\ &+ \int_{\frac{j+2k}{3}}^k \left(\left(\int_{\frac{j+7k}{8}}^t \omega(t) dt - \left| \int_{\frac{j+7k}{8}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &+ \left. \left(\int_{\frac{j+7k}{8}}^t \omega(t) dt + \left| \int_{\frac{j+7k}{8}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt. \end{aligned}$$

and

$$\begin{aligned}
 M_{19} &= \int_j^{\frac{2j+k}{3}} \left(\left(\int_{\frac{7j+k}{8}}^t \omega(t) dt + \left| \int_{\frac{7j+k}{8}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\
 &\quad + \left. \left(\int_{\frac{7j+k}{8}}^t \omega(t) dt - \left| \int_{\frac{7j+k}{8}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\
 &\quad + \int_{\frac{2j+k}{3}}^{\frac{j+2k}{3}} \left(\left(\int_{\frac{j+k}{2}}^t \omega(t) dt + \left| \int_{\frac{j+k}{2}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\
 &\quad + \left. \left(\int_{\frac{j+k}{2}}^t \omega(t) dt - \left| \int_{\frac{j+k}{2}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\
 &\quad + \int_{\frac{j+2k}{3}}^k \left(\left(\int_{\frac{j+7k}{8}}^t \omega(t) dt + \left| \int_{\frac{j+7k}{8}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\
 &\quad + \left. \left(\int_{\frac{j+7k}{8}}^t \omega(t) dt - \left| \int_{\frac{j+7k}{8}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt.
 \end{aligned}$$

Special Case 2.26.(a) If we take, $\gamma(t) = \gamma_0 \neq 0$, $\Gamma(t) = \Gamma_0 \neq 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.15), then

$$\begin{aligned}
 \frac{25(k-j)}{576}(\Gamma_0 - \gamma_0) &\leq \frac{3}{8} \left[\frac{\rho(j) + \rho(k)}{3} + \rho\left(\frac{2j+k}{3}\right) + \rho\left(\frac{j+2k}{3}\right) \right] \\
 &\quad - \frac{1}{k-j} \int_j^k \rho(t) dt \leq \frac{25(k-j)}{576}(\gamma_0 - \Gamma_0),
 \end{aligned}$$

which is Special Case 2.15.1 of [7].

Special Case 2.26.(b) If we take, $\gamma(t) = \gamma_1 t + \gamma_0 \neq 0$, $\Gamma(t) = \Gamma_1 t + \Gamma_0 \neq 0$ and $\omega(t) = \frac{1}{k-j}$, in (2.15) then

$$m_{20} \leq \frac{3}{8} \left[\frac{\rho(j) + \rho(k)}{3} + \rho\left(\frac{2j+k}{3}\right) + \rho\left(\frac{j+2k}{3}\right) \right] - \frac{1}{k-j} \int_j^k \rho(t) dt \leq M_{20},$$

where

$$m_{20} = \frac{(k-j)}{192} \left[(k-j)(\gamma_1 + \Gamma_1) + \frac{31}{6}(j\gamma_1 - k\Gamma_1) + \frac{19}{6}(k\gamma_1 - j\Gamma_1) + \frac{25}{3}(\gamma_0 - \Gamma_0) \right]$$

and

$$M_{20} = \frac{(k-j)}{192} \left[(k-j)(\gamma_1 + \Gamma_1) + \frac{31}{6}(k\Gamma_1 - j\gamma_1) + \frac{19}{6}(j\Gamma_1 - k\gamma_1) + \frac{25}{3}(\Gamma_0 - \gamma_0) \right],$$

which is Special Case 2.15.2 of [7].

Now we state two results with their consequences for function ρ whose first derivative is bounded below only and bounded above only respectively.

Theorem 2.27. Let $\rho : I \rightarrow \mathbb{R}$, be a differentiable function on I^0 of I , and let $[j, k] \subset I^0$. If ρ' is bounded below then $\gamma(\theta) \leq \rho'(\theta)$ for any $\gamma \in C[j, k]$, $\theta \in [j, k]$, then for all $\lambda \in [0, 1]$ we have

$$\begin{aligned} m_{21}(\theta, \lambda) &\leq \rho(k) \int_{\beta}^k \omega(t) dt - \rho(j) \int_{\alpha}^j \omega(t) dt + \rho(\theta) \int_{\alpha}^{\frac{\alpha+\beta}{2}} \omega(t) dt \\ &\quad + \rho(j+k-\theta) \int_{\frac{\alpha+\beta}{2}}^{\beta} \omega(t) dt - \int_j^k \rho(t) \omega(t) dt \leq M_{21}(\theta, \lambda), \quad (2.15) \end{aligned}$$

where

$$\begin{aligned} m_{21}(\theta, \lambda) &= \int_j^k \left(\int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt \right) \gamma(t) dt + \int_j^{\theta} \left(\int_{\alpha}^{\frac{\alpha+\beta}{2}} \omega(t) dt \right) \gamma(t) dt \\ &\quad - \int_{j+k-\theta}^k \left(\int_{\frac{\alpha+\beta}{2}}^{\beta} \omega(t) dt \right) \gamma(t) dt \\ &\quad - \max \left\{ \int_{\alpha}^{\theta} \omega(t) dt, \int_{\frac{\alpha+\beta}{2}}^{j+k-\theta} \omega(t) dt, \int_{\beta}^k \omega(t) dt \right\} \\ &\quad \times \left(\rho(k) - \rho(j) - \int_j^k \gamma(t) dt \right) \end{aligned}$$

and

$$\begin{aligned} M_{21}(\theta, \lambda) &= \int_j^k \left(\int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt \right) \gamma(t) dt + \int_j^{\theta} \left(\int_{\alpha}^{\frac{\alpha+\beta}{2}} \omega(t) dt \right) \gamma(t) dt \\ &\quad - \int_{j+k-\theta}^k \left(\int_{\frac{\alpha+\beta}{2}}^{\beta} \omega(t) dt \right) \gamma(t) dt \\ &\quad + \max \left\{ \int_{\alpha}^{\theta} \omega(t) dt, \int_{\frac{\alpha+\beta}{2}}^{j+k-\theta} \omega(t) dt, \int_{\beta}^k \omega(t) dt \right\} \\ &\quad \times \left(\rho(k) - \rho(j) - \int_j^k \gamma(t) dt \right). \end{aligned}$$

Proof. Since

$$\begin{aligned} &\int_j^k K(\theta, t) (\rho'(t) - \gamma(t)) dt \\ &= \rho(k) \int_{\beta}^k \omega(t) dt - \rho(j) \int_{\alpha}^j \omega(t) dt + \rho(\theta) \int_{\alpha}^{\frac{\alpha+\beta}{2}} \omega(t) dt \\ &\quad + \rho(j+k-\theta) \int_{\frac{\alpha+\beta}{2}}^{\beta} \omega(t) dt - \int_j^k \rho(t) \omega(t) dt - \int_j^k \omega(t) \rho(t) dt \\ &\quad - \int_j^k K(\theta, t) \gamma(t) dt \end{aligned}$$

$$\begin{aligned}
&= \rho(k) \int_{\beta}^k \omega(t) dt - \rho(j) \int_{\alpha}^j \omega(t) dt + \rho(\theta) \int_{\alpha}^{\frac{\alpha+\beta}{2}} \omega(t) dt \\
&+ \rho(j+k-\theta) \int_{\frac{\alpha+\beta}{2}}^{\beta} \omega(t) dt - \int_j^k \rho(t) \omega(t) dt \\
&- \left[\int_j^{\theta} \left(\int_{\alpha}^t \omega(t) dt \right) \gamma(t) dt + \int_{\theta}^{j+k-\theta} \left(\int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt \right) \gamma(t) dt \right. \\
&\left. + \int_{j+k-\theta}^k \left(\int_{\beta}^t \omega(t) dt \right) \gamma(t) dt \right].
\end{aligned}$$

Using modulus property, we have

$$\begin{aligned}
&\left| \rho(k) \int_{\beta}^k \omega(t) dt - \rho(j) \int_{\alpha}^j \omega(t) dt + \rho(\theta) \int_{\alpha}^{\frac{\alpha+\beta}{2}} \omega(t) dt \right. \\
&+ \rho(j+k-\theta) \int_{\frac{\alpha+\beta}{2}}^{\beta} \omega(t) dt - \int_j^k \rho(t) \omega(t) dt \\
&- \left[\int_j^{\theta} \left(\int_{\alpha}^t \omega(t) dt \right) \gamma(t) dt + \int_{\theta}^{j+k-\theta} \left(\int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt \right) \gamma(t) dt \right. \\
&\left. + \int_{j+k-\theta}^k \left(\int_{\beta}^t \omega(t) dt \right) \gamma(t) dt \right] \Big| \\
&= \left| \int_j^k K(\theta, t) (\rho'(t) - \gamma(t)) dt \right| \leq \int_j^k |K(\theta, t)| (\rho'(t) - \gamma(t)) dt \\
&\leq \max_{t \in [j, k]} |K(\theta, t)| \int_j^k (\rho'(t) - \gamma(t)) dt \\
&= \max \left\{ \int_{\alpha}^{\theta} \omega(t) dt, \int_{\frac{\alpha+\beta}{2}}^{j+k-\theta} \omega(t) dt, \int_{\beta}^k \omega(t) dt \right\} \\
&\times \left(\rho(k) - \rho(j) - \int_j^k \gamma(t) dt \right). \tag{2.16}
\end{aligned}$$

After rearrangement of (2.16) we get required result. $\square$

Remark 2.28. If we put $\omega(t) = \frac{1}{k-j}$ in inequality (2.16), then we will get the following result.

Corollary 2.29. *Let all the assumptions of Theorem 2.27 are valid. Then*

$$m_{22}(\theta, \lambda) \leq \lambda \frac{\rho(j) + \rho(k)}{2} + (1 - \lambda) \frac{\rho(\theta) + \rho(j+k-\theta)}{2} - \int_j^k \rho(t) d(t) \leq M_{22}(\theta, \lambda) \tag{2.17}$$

where

$$m_{22}(\lambda) = \frac{1}{k-j} \left[\int_j^k \left(t - \frac{j+k}{2} \right) \gamma(t) dt + \frac{k-j}{2} \left(\int_j^\theta (1-\lambda) \gamma(t) dt - \int_{j+k-\theta}^k (1-\lambda) \gamma(t) dt \right) - \max \left\{ \lambda \frac{k-j}{2}, \left(\theta - \frac{(2-\lambda)j + \lambda k}{2} \right), \left(\frac{j+k}{2} - \theta \right) \right\} \times \left(\rho(k) - \rho(j) - \int_j^k \gamma(t) dt \right) \right]$$

and

$$M_{22}(\lambda) = \frac{1}{k-j} \left[\int_j^k \left(t - \frac{j+k}{2} \right) \gamma(t) dt + \frac{k-j}{2} \left(\int_j^{\frac{j+k}{2}} (1-\lambda) \gamma(t) dt - \int_{\frac{j+k}{2}}^k (1-\lambda) \gamma(t) dt \right) + \max \left\{ \lambda \frac{k-j}{2}, \left(\theta - \frac{(2-\lambda)j + \lambda k}{2} \right), \left(\frac{j+k}{2} - \theta \right) \right\} \times \left(\rho(k) - \rho(j) - \int_j^k \gamma(t) dt \right) \right],$$

which is Theorem 2.8 of [7].

Remark 2.30. If we choose $\theta = \frac{j+k}{2}$ in (2.15), we get the following result.

Corollary 2.31. *Let all the assumptions of Theorem 2.1 be valid. Then*

$$m_{23}(\lambda) \leq \rho(k) \int_\beta^k \omega(t) dt - \rho(j) \int_\alpha^j \omega(t) dt + \rho \left(\frac{j+k}{2} \right) \int_\alpha^\beta \omega(t) dt - \int_j^k \rho(t) \omega(t) dt \leq M_{23}(\lambda), \quad (2.18)$$

where

$$m_{23}(\lambda) = \int_j^{\frac{j+k}{2}} \left(\int_\alpha^t \omega(t) dt \right) \gamma(t) dt + \int_{\frac{j+k}{2}}^k \left(\int_\beta^t \omega(t) dt \right) \gamma(t) dt - \max \left\{ \int_\alpha^{\frac{j+k}{2}} \omega(t) dt, \int_{\frac{\alpha+\beta}{2}}^{\frac{j+k}{2}} \omega(t) dt, \int_\beta^k \omega(t) dt \right\} \times \left(\rho(k) - \rho(j) - \int_j^k \gamma(t) dt \right)$$

and

$$M_{23}(\lambda) = \int_j^{\frac{j+k}{2}} \int_\alpha^t \omega(t) dt \gamma(t) dt + \int_{\frac{j+k}{2}}^k \int_\beta^t \omega(t) dt \gamma(t) dt \\ + \max \left\{ \int_\alpha^{\frac{j+k}{2}} \omega(t) dt, \int_{\frac{\alpha+\beta}{2}}^{\frac{j+k}{2}} \omega(t) dt, \int_\beta^k \omega(t) dt \right\} \\ \times \left(\rho(k) - \rho(j) - \int_j^k \gamma(t) dt \right).$$

Special Case 2.31. If we choose $\omega(t) = \frac{1}{k-j}$ in (2.18), we obtain the following result.

$$m_{24}(\lambda) \leq \left[\lambda \frac{\rho(j) + \rho(k)}{2} + (1-\lambda) \rho \left(\frac{j+k}{2} \right) \right] - \frac{1}{k-j} \int_j^k \rho(t) dt \\ \leq M_{24}(\lambda), \quad (2.19)$$

where

$$m_{24}(\lambda) = \frac{1}{k-j} \left[\int_j^k \left(t - \frac{j+k}{2} \right) \gamma(t) dt + \frac{k-j}{2} \left(\int_j^{\frac{j+k}{2}} (1-\lambda) \gamma(t) dt \right. \right. \\ \left. \left. - \int_{\frac{j+k}{2}}^k (1-\lambda) \gamma(t) dt \right) - \max \left\{ \lambda \frac{k-j}{2}, (1-\lambda) \frac{k-j}{2} \right\} \right. \\ \left. \times \left(\rho(k) - \rho(j) - \int_j^k \gamma(t) dt \right) \right]$$

and

$$M_{24}(\lambda) = \frac{1}{k-j} \left[\int_j^k \left(t - \frac{j+k}{2} \right) \gamma(t) dt + \frac{k-j}{2} \left(\int_j^{\frac{j+k}{2}} (1-\lambda) \gamma(t) dt \right. \right. \\ \left. \left. - \int_{\frac{j+k}{2}}^k (1-\lambda) \gamma(t) dt \right) + \max \left\{ \lambda \frac{k-j}{2}, (1-\lambda) \frac{k-j}{2} \right\} \right. \\ \left. \times \left(\rho(k) - \rho(j) - \int_j^k \gamma(t) dt \right) \right],$$

which is Corollary 2.9 of [7].

Theorem 2.32. Let $\rho : I \rightarrow \mathbb{R}$, be a differentiable function on I^0 of I , and let $[j, k] \subset I^0$. If ρ' is bounded above then $\rho'(\theta) \leq \Gamma(\theta)$ for any $\Gamma \in C[j, k]$,

$\theta \in [j, k]$, then for all $\lambda \in [0, 1]$ we have

$$\begin{aligned} m_{25}(\theta, \lambda) &\leq \rho(k) \int_{\beta}^k \omega(t) dt - \rho(j) \int_{\alpha}^j \omega(t) dt + \rho(\theta) \int_{\alpha}^{\frac{\alpha+\beta}{2}} \omega(t) dt \\ &\quad + \rho(j+k-\theta) \int_{\frac{\alpha+\beta}{2}}^{\beta} \omega(t) dt - \int_j^k \rho(t) \omega(t) dt \leq M_{25}(\theta, \lambda), \quad (2.20) \end{aligned}$$

where

$$\begin{aligned} m_{25}(\theta, \lambda) &= \int_j^k \left(\int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt \right) \Gamma(t) dt + \int_j^{\theta} \left(\int_{\alpha}^{\frac{\alpha+\beta}{2}} \omega(t) dt \right) \Gamma(t) dt \\ &\quad - \int_{j+k-\theta}^k \left(\int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt \right) \Gamma(t) dt \\ &\quad - \max \left\{ \int_{\alpha}^{\theta} \omega(t) dt, \int_{\frac{\alpha+\beta}{2}}^{j+k-\theta} \omega(t) dt, \int_{\beta}^k \omega(t) dt \right\} \\ &\quad \times \left(\int_j^k \Gamma(t) dt - \rho(k) + \rho(j) \right) \end{aligned}$$

and

$$\begin{aligned} M_{25}(\theta, \lambda) &= \int_j^k \left(\int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt \right) \Gamma(t) dt + \int_j^{\theta} \left(\int_{\alpha}^{\frac{\alpha+\beta}{2}} \omega(t) dt \right) \Gamma(t) dt \\ &\quad - \int_{j+k-\theta}^k \left(\int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt \right) \Gamma(t) dt \\ &\quad + \max \left\{ \int_{\alpha}^{\theta} \omega(t) dt, \int_{\frac{\alpha+\beta}{2}}^{j+k-\theta} \omega(t) dt, \int_{\beta}^k \omega(t) dt \right\} \\ &\quad \times \left(\int_j^k \Gamma(t) dt - \rho(k) + \rho(j) \right). \end{aligned}$$

Proof. Since

$$\begin{aligned}
 & \int_j^k K(\theta, t) (\rho'(t) - \Gamma(t)) dt \\
 &= \rho(k) \int_\beta^k \omega(t) dt - \rho(j) \int_\alpha^j \omega(t) dt + \rho(\theta) \int_\alpha^{\frac{\alpha+\beta}{2}} \omega(t) dt \\
 &+ \rho(j+k-\theta) \int_{\frac{\alpha+\beta}{2}}^\beta \omega(t) dt - \int_j^k \rho(t) \omega(t) dt - \int_j^k K(\theta, t) \Gamma(t) dt \\
 &= \rho(k) \int_\beta^k \omega(t) dt - \rho(j) \int_\alpha^j \omega(t) dt + \rho(\theta) \int_\alpha^{\frac{\alpha+\beta}{2}} \omega(t) dt \\
 &+ \rho(j+k-\theta) \int_{\frac{\alpha+\beta}{2}}^\beta \omega(t) dt - \int_j^k \rho(t) \omega(t) dt \\
 &- \left[\int_j^\theta \int_\alpha^t \omega(t) dt \Gamma(t) dt + \int_\theta^{j+k-\theta} \int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt \Gamma(t) dt \right. \\
 &\left. + \int_{j+k-\theta}^k \int_\beta^t \omega(t) dt \Gamma(t) dt \right],
 \end{aligned}$$

so we have

$$\begin{aligned}
 & \left| \rho(k) \int_\beta^k \omega(t) dt - \rho(j) \int_\alpha^j \omega(t) dt + \rho(\theta) \int_\alpha^{\frac{\alpha+\beta}{2}} \omega(t) dt \right. \\
 &+ \rho(j+k-\theta) \int_{\frac{\alpha+\beta}{2}}^\beta \omega(t) dt - \int_j^k \rho(t) \omega(t) dt \\
 &- \left[\int_j^\theta \int_\alpha^t \omega(t) dt \Gamma(t) dt + \int_\theta^{j+k-\theta} \int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt \Gamma(t) dt \right. \\
 &\left. + \int_{j+k-\theta}^k \int_\beta^t \omega(t) dt \Gamma(t) dt \right] \Big| \\
 &= \left| \int_j^k K(\theta, t) (\rho'(t) - \Gamma(t)) dt \right| \\
 &\leq \int_j^k |K(\theta, t)| (\Gamma(t) - \rho'(t)) dt \\
 &\leq \max_{t \in [j, k]} |K(\theta, t)| \int_j^k (\Gamma(t) - \rho'(t)) dt \\
 &= \max \left\{ \int_\alpha^\theta \omega(t) dt, \int_{\frac{\alpha+\beta}{2}}^{j+k-\theta} \omega(t) dt, \int_\beta^k \omega(t) dt \right\} \\
 &\times \left(\int_j^k \Gamma(t) dt - \rho(k) + \rho(j) \right), \tag{2.21}
 \end{aligned}$$

we get required result after some rearrangement. $\square$

Remark 2.33. If we put $\omega(t) = \frac{1}{k-j}$ in inequality (2.20) then we get Theorem 3 of [18] and Theorem 3 of [19].

Corollary 2.34. Let all the assumptions of Theorem 2.32 be valid and if we choose $\theta = \frac{j+k}{2}$ in (2.20), we get

$$m_{26}(\lambda) \leq \rho(k) \int_{\beta}^k \omega(t) dt - \rho(j) \int_{\alpha}^j \omega(t) dt + \rho\left(\frac{j+k}{2}\right) \int_{\alpha}^{\beta} \omega(t) dt - \int_j^k \rho(t) \omega(t) dt \leq M_{26}(\lambda), \quad (2.22)$$

where

$$m_{26}(\lambda) = \int_j^{\frac{j+k}{2}} \int_{\alpha}^t \omega(t) dt \Gamma(t) dt + \int_{\frac{j+k}{2}}^k \int_{\beta}^t \omega(t) dt \Gamma(t) dt - \max \left\{ \int_{\alpha}^{\frac{j+k}{2}} \omega(t) dt, \int_{\frac{\alpha+\beta}{2}}^{\frac{j+k}{2}} \omega(t) dt, \int_{\beta}^k \omega(t) dt \right\} \left(\int_j^k \Gamma(t) dt - \rho(k) + \rho(j) \right)$$

and

$$M_{26}(\lambda) = \int_j^{\frac{j+k}{2}} \int_{\alpha}^t \omega(t) dt \Gamma(t) dt + \int_{\frac{j+k}{2}}^k \int_{\beta}^t \omega(t) dt \Gamma(t) dt + \max \left\{ \int_{\alpha}^{\frac{j+k}{2}} \omega(t) dt, \int_{\frac{\alpha+\beta}{2}}^{\frac{j+k}{2}} \omega(t) dt, \int_{\beta}^k \omega(t) dt \right\} \left(\int_j^k \Gamma(t) dt - \rho(k) + \rho(j) \right).$$

Special Case 2.34. If we choose $\omega(t) = \frac{1}{k-j}$ in (2.22), we obtain the following result.

$$m_{27}(\lambda) \leq \left[\lambda \frac{\rho(j) + \rho(k)}{2} + (1-\lambda) \rho\left(\frac{j+k}{2}\right) \right] - \frac{1}{k-j} \int_j^k \rho(t) dt \leq M_{27}(\lambda),$$

where

$$m_{27}(\lambda) = \frac{1}{k-j} \left[\int_j^k \left(t - \frac{j+k}{2} \right) \Gamma(t) dt + \frac{k-j}{2} \left(\int_j^{\frac{j+k}{2}} (1-\lambda) \Gamma(t) dt - \int_{\frac{j+k}{2}}^k (1-\lambda) \Gamma(t) dt \right) - \max \left\{ \lambda \frac{k-j}{2}, (1-\lambda) \frac{k-j}{2} \right\} \left(\int_j^k \Gamma(t) dt - \rho(k) + \rho(j) \right) \right]$$

and

$$\begin{aligned} M_{27}(\lambda) = & \frac{1}{k-j} \left[\int_j^k \left(t - \frac{j+k}{2} \right) \Gamma(t) dt \right. \\ & + \frac{k-j}{2} \left(\int_j^{\frac{j+k}{2}} (1-\lambda) \Gamma(t) dt - \int_{\frac{j+k}{2}}^k (1-\lambda) \Gamma(t) dt \right) \\ & \left. + \max \left\{ \lambda \frac{k-j}{2}, (1-\lambda) \frac{k-j}{2} \right\} \left(\int_j^k \Gamma(t) dt - \rho(k) + \rho(j) \right) \right], \end{aligned}$$

which is Corollary 2.11 of [7].

Remark 2.35. If we put $w(t) = \frac{1}{k-j}$ in inequality (2.20) is Special Case of Theorem 3 of [18] and Theorem 3 of [19].

Corollary 2.36. *Let all the assumptions of Theorem 2.32 be valid. Then*

$$\begin{aligned} m_{28}(\theta, \lambda) \leq & \left[\rho(k) \int_{\beta}^k \omega(t) dt - \rho(j) \int_{\alpha}^j \omega(t) dt + f \left(\frac{j+k}{2} \right) \int_{\alpha}^{\beta} \omega(t) dt \right. \\ & \left. - \int_j^k \rho(t) \omega(t) dt \right] \leq M_{28}(\theta, \lambda), \end{aligned} \quad (2.23)$$

where

$$\begin{aligned} m_{28}(\theta, \lambda) = & \int_j^{\theta} \left(\int_{\alpha}^t \omega(t) dt \right) \gamma(t) dt + \int_{\theta}^{j+k-\theta} \left(\int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt \right) \gamma(t) dt \\ & + \int_{j+k-\theta}^k \left(\int_{\beta}^t \omega(t) dt \right) \gamma(t) dt - \max \left\{ \int_{\alpha}^{\frac{j+k}{2}} \omega(t) dt, \int_{\frac{\alpha+\beta}{2}}^{\frac{j+k}{2}} \omega(t) dt, \int_{\beta}^k \omega(t) dt \right\} \\ & \times \left(\rho(k) - \rho(j) - \int_j^k \gamma(t) dt \right) \end{aligned}$$

and

$$\begin{aligned} M_{28}(\theta, \lambda) = & \int_j^{\theta} \left(\int_{\alpha}^t \omega(t) dt \right) \gamma(t) dt + \int_{\theta}^{j+k-\theta} \left(\int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt \right) \gamma(t) dt \\ & + \int_{j+k-\theta}^k \left(\int_{\beta}^t \omega(t) dt \right) \gamma(t) dt + \max \left\{ \int_{\alpha}^{\frac{j+k}{2}} \omega(t) dt, \int_{\frac{\alpha+\beta}{2}}^{\frac{j+k}{2}} \omega(t) dt, \int_{\beta}^k \omega(t) dt \right\} \\ & \times \left(\rho(k) - \rho(j) - \int_j^k \gamma(t) dt \right). \end{aligned}$$

Remark 2.37. If $\gamma(\theta) \leq \rho'(\theta) \leq \Gamma(\theta)$ for any $\theta \in [j, k]$ and $\gamma, \Gamma \in C[j, k]$, and if one put $\lambda = 0$, then error bounds of non-standard quadrature $A_5(\rho)$ are given

as:

$$m_{29} \leq \frac{1}{2} \left[-\rho(j) + 2\rho\left(\frac{j+k}{2}\right) + \rho(k) \right] - \frac{1}{k-j} \int_j^k \rho(t) dt \leq M_{29}, \quad (2.24)$$

where

$$m_{29} = \frac{1}{k-j} \left[\int_j^{\frac{j+k}{2}} (t-j) \gamma(t) dt + \int_{\frac{j+k}{2}}^k (t-k) \gamma(t) dt + \frac{(k-j)}{2} \int_j^k \gamma(t) dt \right]$$

and

$$M_{29} = \frac{1}{k-j} \left[\int_j^{\frac{j+k}{2}} (t-j) \Gamma(t) dt + \int_{\frac{j+k}{2}}^k (t-k) \Gamma(t) dt + \frac{(k-j)}{2} \int_j^k \Gamma(t) dt \right],$$

which is Corollary 3 of [18], Corollary 4 of [19] and Corollary 2.11 of [7].

Proof. To prove (2.22), we use Corollary 2.31 and 2.34. First by putting $\lambda = 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.18), we get

$$\begin{aligned} & \frac{1}{k-j} \left[\int_j^k (t-j) \gamma(t) dt + \frac{k-j}{2} \left(\int_j^{\frac{j+k}{2}} \gamma(t) dt - \int_{\frac{j+k}{2}}^k \gamma(t) dt \right) \right] \\ & \leq \frac{1}{2} \left[-\rho(j) + 2\rho\left(\frac{j+k}{2}\right) + \rho(k) \right] - \frac{1}{k-j} \int_j^k \rho(t) dt \end{aligned} \quad (2.25)$$

provided that $\gamma(t) \leq \rho'(t) \forall t \in [j, k]$.

On the other hand, by assuming $\lambda = 0$ in (2.21), we obtain

$$\begin{aligned} & \frac{1}{2} \left[-\rho(j) + 2\rho\left(\frac{j+k}{2}\right) + \rho(k) \right] - \frac{1}{k-j} \int_j^k \rho(t) dt \\ & \leq \frac{1}{k-j} \left[\int_j^k (t-j) \Gamma(t) dt + \frac{k-j}{2} \left(\int_j^{\frac{j+k}{2}} \Gamma(t) dt - \int_{\frac{j+k}{2}}^k \Gamma(t) dt \right) \right] \end{aligned} \quad (2.26)$$

provided that $\rho'(t) \leq \Gamma(t) \forall t \in [j, k]$. Combining the above two inequalities obtain the required result. $\square$

Remark 2.38. If $\gamma(\theta) \leq \rho'(\theta) \leq \Gamma(\theta)$ for any $\theta \in [j, k]$ and $\gamma, \Gamma \in C[j, k]$, and if we choose $\lambda = 0$, then error bound of non-standard quadrature $A_6(\rho)$ would be

$$m_{30} \leq \frac{1}{2} \left[\rho(j) + 2\rho\left(\frac{j+k}{2}\right) - \rho(k) \right] - \frac{1}{k-j} \int_j^k \rho(t) dt \leq M_{30} \quad (2.27)$$

where

$$m_{30} = \frac{1}{k-j} \left[\int_j^{\frac{j+k}{2}} (t-j) \Gamma(t) dt + \int_{\frac{j+k}{2}}^k (t-k) \Gamma(t) dt - \frac{(k-j)}{2} \int_j^k \Gamma(t) dt \right]$$

and

$$M_{30} = \frac{1}{k-j} \left[\int_j^{\frac{j+k}{2}} (t-j) \gamma(t) dt + \int_{\frac{j+k}{2}}^k (t-k) \gamma(t) dt - \frac{(k-j)}{2} \int_j^k \gamma(t) dt \right],$$

which is Corollary 4 of [18], Corollary 5 of [19] and Corollary 2.11 of [7].

Proof. Proof of (2.25) is similar to that of Remark 2.37, if one replaces $\lambda = 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.18) and (2.21), respectively, and combines them together. $\square$

Remark 2.39. If $\gamma(\theta) \leq \rho'(\theta) \leq \Gamma(\theta)$ for any $\theta \in [j, k]$ and $\gamma, \Gamma \in C[j, k]$ then by replacing $\theta = k$, $\lambda = 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.18) and (2.21), respectively, then error of non-standard quadrature $A_7(\rho)$ is given as

$$m_{31} \leq \rho(j) - \frac{1}{(k-j)} \int_j^k \rho(t) dt \leq M_{31},$$

where

$$m_{31} = \frac{1}{(k-j)} \int_j^k (t-k)\Gamma(t) dt$$

and

$$M_{31} = \frac{1}{(k-j)} \int_j^k (t-k)\gamma(t) dt,$$

which is Corollary 5 of [18], Corollary 2 of [19] and Corollary 2.11.

Remark 2.40. If $\gamma(\theta) \leq \rho'(\theta) \leq \Gamma(\theta)$ for any $\theta \in [j, k]$ and $\gamma, \Gamma \in C[j, k]$ then by replacing $\theta = j$, $\lambda = 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.18) and (2.21), respectively, one gets error bounds of non-standard quadrature $A_8(\rho)$ as follows:

$$m_{32} \leq \rho(k) - \frac{1}{(k-j)} \int_j^k \rho(t) dt \leq M_{32},$$

where

$$m_{32} = \frac{1}{(k-j)} \int_j^k (t-j)\gamma(t) dt \quad (2.28)$$

and

$$M_{32} = \frac{1}{(k-j)} \int_j^k (t-j)\Gamma(t) dt.$$

The inequality presented above is same as the Corollary 6 of [18], Corollary 3 of [19] and Corollary 2.11 of [7].

3. APPLICATIONS TO NUMERICAL QUADRATURE RULES

Let $I_n : j = z_0 < z_1 < \dots < z_n = k$ be a partition of interval $[j, k]$ and let $h_i = z_{i+1} - z_i, i \in \{0, 1, 2, \dots, n-1\}$. Then

$$\int_j^k \omega(t) \rho(t) dt = Q_n(\rho, \omega, \lambda) + R_n(\rho, \omega, \lambda). \quad (3.1)$$

Consider a general quadrature rule

$$Q_n(\rho, \omega, \lambda) := \sum_{i=0}^{n-1} \left[\rho(z_{i+1}) \int_{\beta_i}^{z_{i+1}} \omega(t) dt - \rho(z_i) \int_{\alpha_i}^{z_i} \omega(t) dt + \rho(\theta_i) \int_{\alpha_i}^{\frac{\alpha_i + \beta_i}{2}} \omega(t) dt \right. \\ \left. + \rho(z_i + z_{i+1} - \theta_i) \int_{\frac{\alpha_i + \beta_i}{2}}^{\beta_i} \omega(t) dt \right], \quad (3.2)$$

where $\lambda \in [0, 1]$ and $\theta_i \in [z_i, z_{i+1}]$. Then we get following result:

Theorem 3.1. *Let all the assumptions of Theorem 2.1 be valid. Then (3.1) holds where $Q_n(\rho, \omega, \lambda)$ is given by formula (3.2) and remainder $R_n(I_n, \rho, \omega)$ satisfies estimates*

$$|R_n(\rho, \omega, \lambda)| \leq \sum_{i=0}^{n-1} \sup \{|R_1|, |R_2|\}, \quad (3.3)$$

where

$$R_1 = \int_{z_i}^{\theta_i} \left(\left(\int_{\alpha_i}^t \omega(t) dt - \left| \int_{\alpha_i}^t \omega(t) dt \right| \right) \frac{\Gamma_i(t)}{2} \right. \\ \left. + \left(\int_{\alpha_i}^t \omega(t) dt + \left| \int_{\alpha_i}^t \omega(t) dt \right| \right) \frac{\gamma_i(t)}{2} \right) dt \\ + \int_{\theta_i}^{z_i + z_{i+1} - \theta_i} \left(\left(\int_{\frac{\alpha_i + \beta_i}{2}}^t \omega(t) dt - \left| \int_{\frac{\alpha_i + \beta_i}{2}}^t \omega(t) dt \right| \right) \frac{\Gamma_i(t)}{2} \right. \\ \left. + \left(\int_{\frac{\alpha_i + \beta_i}{2}}^t \omega(t) dt + \left| \int_{\frac{\alpha_i + \beta_i}{2}}^t \omega(t) dt \right| \right) \frac{\gamma_i(t)}{2} \right) dt \\ + \int_{z_i + z_{i+1} - \theta_i}^{z_{i+1}} \left(\left(\int_{\beta_i}^t \omega(t) dt - \left| \int_{\beta_i}^t \omega(t) dt \right| \right) \frac{\Gamma_i(t)}{2} \right. \\ \left. + \left(\int_{\beta_i}^t \omega(t) dt + \left| \int_{\beta_i}^t \omega(t) dt \right| \right) \frac{\gamma_i(t)}{2} \right) dt$$

and

$$\begin{aligned}
 R_2 &= \int_{z_i}^{\theta_i} \left(\left(\int_{\alpha_i}^t \omega(t) dt + \left| \int_{\alpha_i}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\
 &+ \left. \left(\int_{\alpha_i}^t \omega(t) dt - \left| \int_{\alpha_i}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\
 &+ \int_{\theta_i}^{z_i+z_{i+1}-\theta_i} \left(\left(\int_{\frac{\alpha_i+\beta_i}{2}}^t \omega(t) dt + \left| \int_{\frac{\alpha_i+\beta_i}{2}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\
 &+ \left. \left(\int_{\frac{\alpha_i+\beta_i}{2}}^t \omega(t) dt - \left| \int_{\frac{\alpha_i+\beta_i}{2}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\
 &+ \int_{z_i+z_{i+1}-\theta_i}^{z_{i+1}} \left(\left(\int_{\beta_i}^t \omega(t) dt + \left| \int_{\beta_i}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\
 &+ \left. \left(\int_{\beta_i}^t \omega(t) dt - \left| \int_{\beta_i}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt,
 \end{aligned}$$

$$\forall \theta_i \in [z_i, z_{i+1}].$$

Proof. Applying inequality (2.1) on the intervals $[z_i, z_{i+1}]$ for $r \in \{1, 2, \dots, n-1\}$, and using (3.3), we get

$$\begin{aligned}
 R_i(\rho, \omega, \lambda) &= \int_{z_i}^{z_{i+1}} \omega(t) \rho(t) dt - \left[\rho(z_{i+1}) \int_{\beta_i}^{z_{i+1}} \omega(t) dt - \rho(z_i) \int_{\alpha_i}^{z_i} \omega(t) dt \right. \\
 &+ \left. \rho(\theta_i) \int_{\alpha_i}^{\frac{\alpha_i+\beta_i}{2}} \omega(t) dt + \rho(z_i + z_{i+1} - \theta_i) \int_{\frac{\alpha_i+\beta_i}{2}}^{\beta_i} \omega(t) dt \right].
 \end{aligned}$$

Summing it over i from 0 to $n-1$ we get

$$\begin{aligned}
 R_n(\rho, \omega, \lambda) &= \int_j^k \rho(t) \omega(t) dt - \sum_{i=0}^{n-1} \left[\rho(z_{i+1}) \int_{\beta_i}^{z_{i+1}} \omega(t) dt - \rho(z_i) \int_{\alpha_i}^{z_i} \omega(t) dt \right. \\
 &\times \left. \omega(t) dt + \rho(\theta_i) \int_{\alpha_i}^{\frac{\alpha_i+\beta_i}{2}} \omega(t) dt + \rho(z_i + z_{i+1} - \theta_i) \int_{\frac{\alpha_i+\beta_i}{2}}^{\beta_i} \omega(t) dt \right].
 \end{aligned}$$

It follows from (2.1) that

$$\begin{aligned}
 |R_n(\rho, \omega, \lambda)| &= \left| \int_j^k \rho(t) \omega(t) dt \right. \\
 &- \sum_{i=0}^{n-1} \left[\rho(z_{i+1}) \int_{\beta_i}^{z_{i+1}} \omega(t) dt - \rho(z_i) \int_{\alpha_i}^{z_i} \omega(t) dt \right. \\
 &+ \left. \left. \rho(\theta_i) \int_{\alpha_i}^{\frac{\alpha_i+\beta_i}{2}} \omega(t) dt + \rho(z_i + z_{i+1} - \theta_i) \int_{\frac{\alpha_i+\beta_i}{2}}^{\beta_i} \omega(t) dt \right] \right|
 \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{i=0}^{n-1} \sup \left\{ \left| \left[\int_{z_i}^{\theta_i} \left(\left(\int_{\alpha_i}^t \omega(t) dt - \left| \int_{\alpha_i}^t \omega(t) dt \right| \right) \frac{\Gamma_i(t)}{2} \right. \right. \right. \right. \\
&\quad + \left. \left(\int_{\alpha_i}^t \omega(t) dt + \left| \int_{\alpha_i}^t \omega(t) dt \right| \right) \frac{\gamma_i(t)}{2} \right) \right] dt \\
&\quad + \int_{\theta_i}^{z_i+z_{i+1}-\theta_i} \left(\left(\int_{\frac{\alpha_i+\beta_i}{2}}^t \omega(t) dt - \left| \int_{\frac{\alpha_i+\beta_i}{2}}^t \omega(t) dt \right| \right) \frac{\Gamma_i(t)}{2} \right. \\
&\quad + \left. \left(\int_{\frac{\alpha_i+\beta_i}{2}}^t \omega(t) dt + \left| \int_{\frac{\alpha_i+\beta_i}{2}}^t \omega(t) dt \right| \right) \frac{\gamma_i(t)}{2} \right) \right] dt \\
&\quad + \int_{z_i+z_{i+1}-\theta_i}^{z_{i+1}} \left(\left(\int_{\beta_i}^t \omega(t) dt - \left| \int_{\beta_i}^t \omega(t) dt \right| \right) \frac{\Gamma_i(t)}{2} \right. \\
&\quad + \left. \left(\int_{\beta_i}^t \omega(t) dt + \left| \int_{\beta_i}^t \omega(t) dt \right| \right) \frac{\gamma_i(t)}{2} \right) \right] dt \Big|, \\
&\quad \left| \int_{z_i}^{\theta_i} \left(\left(\int_{\alpha_i}^t \omega(t) dt + \left| \int_{\alpha_i}^t \omega(t) dt \right| \right) \frac{\Gamma_i(t)}{2} \right. \right. \\
&\quad + \left. \left(\int_{\alpha_i}^t \omega(t) dt - \left| \int_{\alpha_i}^t \omega(t) dt \right| \right) \frac{\gamma_i(t)}{2} \right) \right] dt \\
&\quad + \int_{\theta_i}^{z_i+z_{i+1}-\theta_i} \left(\left(\int_{\frac{\alpha_i+\beta_i}{2}}^t \omega(t) dt + \left| \int_{\frac{\alpha_i+\beta_i}{2}}^t \omega(t) dt \right| \right) \frac{\Gamma_i(t)}{2} \right. \\
&\quad + \left. \left(\int_{\frac{\alpha_i+\beta_i}{2}}^t \omega(t) dt - \left| \int_{\frac{\alpha_i+\beta_i}{2}}^t \omega(t) dt \right| \right) \frac{\gamma_i(t)}{2} \right) \right] dt \\
&\quad + \int_{z_i+z_{i+1}-\theta_i}^{z_{i+1}} \left(\left(\int_{\beta_i}^t \omega(t) dt + \left| \int_{\beta_i}^t \omega(t) dt \right| \right) \frac{\Gamma_i(t)}{2} \right. \\
&\quad + \left. \left(\int_{\beta_i}^t \omega(t) dt - \left| \int_{\beta_i}^t \omega(t) dt \right| \right) \frac{\gamma_i(t)}{2} \right) \right] dt \Big| \Big\}.
\end{aligned}$$

□

Remark 3.2. Similarly, we can state applications of other results and their cases as given in Section 2.

4. ACKNOWLEDGEMENT

I am thankful to my supervisor Dr. Asif Raza Khan who introduced to me such an interesting branch of mathematics and helped me a lot during this paper.

5. CONCLUSION

In this paper, weighted Ostrowski type inequality is discussed for function differentiable with variable bounds. Applications to solve error bounds

of midpoint, trapezoidal, Simpson's and Simpson's quadrature and some non-standard quadrature rules are discussed. We also have many proven results as our special cases. In particular our results would generalize of [7, 18, 19, 27].

REFERENCES

1. M. A. Ali, H. Budak, A. Akkurt, Y. M. Chu, A Quantum Ostrowski-type Inequalities for Twice Quantum Differentiable Functions in Quantum Calculus, *Open Mathematics*, **19**, (2021), 440–449.
2. M. A. Ali, Y. M. Chu, H. Budak, A. Akkurt, H. Yıldırım, M. A. Zahid, Quantum Variant of Montgomery Identity and Ostrowski-type Inequalities for the Mappings of Two Variables, *Advances in Difference Equations*, **2021**, (2021), Article 25.
3. Y. M. Chu, M. U. Awan, S. Talib, M. A. Noor, K. I. Noor, New Post Quantum Analogues of Ostrowski-type Inequalities Using New Definitions of Left-right (P, Q)-Derivatives and Definite Integrals, *Advances in Difference Equations*, **2020**, (2020), Article 634.
4. S. S. Dragomir, On the Ostrowski's Integral Inequality for Mappings With Bounded Variation and Applications, *Mathematical Inequalities and Applications*, **4**(1), (2001), 59–66.
5. S. S. Dragomir, P. Cerone, N. S. Barnett, J. Roumeliotis, An Inequality of the Ostrowski-type for Double Integrals Integrals and Applications for Cubature Formulae, *Tamsui Oxford Journal of Information and Mathematical Sciences*, **16**(1), (2000), 1–16.
6. S. S. Dragomir, P. Cerone, J. Roumeliotis, A New Generalization of Ostrowski's Integral Inequality for Mappings Whose Derivatives are Bounded and Applications in Numerical Integration, *Applied Mathematics Letters*, **13**, (2000), 19–25.
7. N. Irshad, Asif R. Khan, Generalization of Ostrowski Inequality for Differentiable Functions and its Applications to Numerical Quadrature Rule, *Journal of Mathematical Analysis and Applications*, **8**(1), (2017), 79–102.
8. M. Imtiaz, N. Irshad, Asif R. Khan, Generalization of Weighted Ostrowski Integral Inequality for Twice Differentiable Mappings, *Advances in Inequalities and Applications*, **2016**, (2016), Article 20, pp. 17.
9. N. Irshad, Asif R. Khan, H. Musharraf, Generalized Fractional Ostrowski-type Inequality, *Journal of Inequalities and Special Functions*, **11**(4), (2020), 16–26.
10. N. Irshad, M. A. Shaikh, Generalized Weighted Ostrowski-Grüss Type Inequality with Applications, *Journal of Mathematical Analysis and Applications*, **8**(1), (2017), 79–102.
11. N. Irshad, Asif R. Khan, H. Musharraf, Generalization of Ostrowski-type Inequality and its Computational Integration, *Journal of Numerical Analysis and Approximation Theory*, **49**(2), (2020), 155–176.
12. N. Irshad, Asif R. Khan, On Weighted Ostrowski-Grüss Inequality With Applications, *Transylvanian Journal of Mechanics and Mathematics*, **10**(1), (2018), 15–22.
13. N. Irshad, Asif R. Khan, Some Applications of Quadrature Rules for Mappings on L_p Space via Ostrowski-type Inequality, *Journal of Numerical Analysis and Approximation Theory*, **46**(4), (2017), 141–149.
14. N. Irshad, Asif R. Khan, M. A. Shaikh, Generalization of Weighted Ostrowski With Weights for Second Order Differentiable Mappings, *Advances in Inequalities and Applications*, **7**, (2019), 1–14.
15. N. Irshad, Asif R. Khan, M. A. Shaikh, Generalized Weighted Ostrowski-Grüss Type Inequality With Applications, *Global Journal of Pure and Applied Mathematics*, **15**(5), (2019), 675–692.

16. N. Irshad, Asif R. Khan, A. Nazir, Extension of Ostrowski-type Inequality via Moment Generating Function, *Advances in Inequalities and Applications*, **2020**(2), (2020), 1-15.
17. N. Irshad, Asif R. Khan, M. A. Shaikh, Generalized Ostrowski Inequality With Applications in Numerical Integration and Special Means, *Advances in Inequalities and Applications*, **2018**(7), (2018), 1-22.
18. M. M. Jamei, S. S. Dragomir, Generalization of Ostrowski-Grüss Inequality, *World Scientific*, **12**(2), (2014), 117-130.
19. M. M. Jamei, Severs Dragomir, An Analogue of the Ostrowski Inequality and Applications, *Filomat*, **28**(2), (2014), 373-381.
20. A. R. Khan, General Inequalities for Generalized Convex Functions, (Unpublished doctoral dissertation), Abdul Salam School of Mathematical Sciences GC University Lahore, Pakistan, 2014.
21. H. Kalsoom, M. Idrees, A. Kashuri, M. U. Awan, Y. M. Chu, Some New (p_1, p_2, q_1, q_2) -Estimates of Ostrowski-Type Integral Inequalities via N-Polynomials S-Type Convexity, *AIMS Mathematics*, **5**(6), (2020), 7122-7144.
22. H. Kalsoom, M. Idrees, D. Baleanu, Y. M. Chu, New Estimates of q_1, q_2 -Ostrowski-Type Inequalities Within a Class of N-Polynomial Preconvexity of Functions, *Journal of Function Spaces*, **2020**(1), (2020), Article ID 3720798, 13 pages.
23. M. A. Khan, S. Begum, Y. Khurshid, Y. M. Chu, Ostrowski-type Inequalities Involving Conformable Fractional Integrals, *Journal of inequalities and applications*, **70**, (2018), 2018.
24. S. Naz, M. N. Naeem, Y. M. Chu, Ostrowski-Type Inequalities for N-Polynomial P - Convex Function for K-Fractional Hilfer-Katugampola Derivative, *Journal of Inequalities and Applications*, **2021**, (2021), 117.
25. A. M. Ostrowski, Über Die Absolutabweichung Einer Differentiebaren Funktion Von Ihren Integralmittelwert, *Commentarii Mathematici Helvetici*, **10**, (1938), 226-227.
26. S. Rashid, M. A. Noor, K. I. Noor, Y. M. Chu, Ostrowski-type Inequalities in the Sense of Generalized K-Fractional Integral Operator for Exponentially Convex Functions, *AIMS Mathematics*, **5**(3), (2020), 2629-2645.
27. N. Ujević, A Generalization of Ostrowski's Inequality and Applications in Numerical Integration, *Applied Mathematics Letters*, **17**, (2004), 133-137.