Deformation of Outer Representations of Galois Group

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ABSTRACT. To a hyperbolic smooth curve defined over a number-field one naturally associates an "anabelian" representation of the absolute Galois group of the base field landing in outer automorphism group of the algebraic fundamental group. In this paper, we introduce several deformation problems for Lie-algebra versions of the above representation and show that, this way we get a richer structure than those coming from deformations of "abelian" Galois representations induced by the Tate module of associated Jacobian variety. We develop an arithmetic deformation theory of graded Lie algebras with finite dimensional graded components to serve our purpose.

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INTRODUCTION

One canonically associates to a smooth curve X defined over a number field K a continuous group homomorphism

$$\rho_X : Gal(\bar{K}/K) \longrightarrow Out(\pi_1(X))$$

where $Out(\pi_1(X))$ denotes the quotient of the automorphism group $Aut(\pi_1(X))$ by inner automorphisms of the algebraic fundamental group. By a conjecture of Voevodski and Matsumoto the outer Galois representation is injective when

topological fundamental group of X is nonabelian. Special cases of this conjecture are proved by Belyi for $\mathbb{P}^1 - \{0, 1, \infty\}$ [Bel], by Voevodski in cases of genus zero and one [25], and by Matsumoto for affine X using Galois action on profinite braid groups [15]. The importance of the representation ρ_X is due to the fact that, by a result of Mochizuki, for X and X' hyperbolic curves, the natural map

$$Isom_K(X, X') \longrightarrow Out_{Gal(\bar{K}/K)}(Out(\pi_1(X)), Out(\pi_1(X')))$$

is a one-to-one correspondence [17]. Here $Out_{Gal(\bar{K}/K)}$ denotes the set of Galois equivariant isomorphisms between the two profinite groups. In particular, ρ_X determines X completely.

The induced pro-l representation

$$\rho_X^l : Gal(\bar{K}/K) \longrightarrow Out(\pi_1^l(X))$$

after abelianization of the pro-l fundamental group induces the standard Galois representation associated to Tate module of the Jacobian variety of X. Curves with abelian fundamental group are not interesting here, because the outer representation does not give any new information. After dividing $\pi_1^l(X)$ by its Frattini subgroup, or by mod-l reduction of the abelianized representation, one obtains a mod-l representation

$$\bar{\rho}_X^l : Gal(\bar{K}/K) \longrightarrow GSp(2g, \mathbb{F}_l).$$

We are interested in the space of deformations of the representation ρ_X^l fixing the mod-l reduction $\bar{\rho}_X^l$. In order to make sense of deforming a representation landing in $Out(\pi_1^l(X))$ we will translate the outer representation of the Galois group to the language of graded Lie-algebras.

The classical Schlessinger criteria for deformations of functors on Artin local rings is used for deformation of the Galois action on the abelianization of the pro-l fundamental group which is the same as etale cohomology. Using Schlessinger criteria, we will construct universal deformation rings parameterizing all liftings of the mod-l representation $\bar{\rho}_X^l$ to actions of the Galois group on graded Lie-algebras over $\mathbb{Z}_l$ with finite-dimensional graded components. We also show that this deformation theory is equivalent to deformation of abelian representations of the Galois group.

To use the full power of outer representations, we deform the corresponding Galois-Lie algebra representation to the graded Lie algebra associated to weight filtration on outer automorphism group of the pro-l fundamental group. We construct a deformation ring parameterizing all deformations fixing the mod-l Lie-algebra representation. Alongside, we develop an arithmetic theory of deformations of Lie-algebras. The main point we are trying to raise in this paper is that for a hyperbolic curve X the l-adic Lie-algebra representation we associate to a hyperbolic curve X contains more information than the associated abelian l-adic representation.

The main obstacle in generalizing this method is the fact that fundamental groups of curves are one relator groups and therefore very similar to free groups. This makes it possible to mimic many structures which work for free groups in the case of such fundamental groups. This is heavily used in the course of computations in this paper.

1. Background material

The study of outer representations of the Galois group has two origins. One root is the theme of anabelian geometry introduced by Grothendieck [7] which lead to results of Nakamura, Tamagawa and Mochizuki who solved the problem in dimension one [17]. The second theme which is originated by Deligne and Ihara independently deals with Lie-algebras associated to the pro-l outer representation [4] [9]. This lead to a partial proof of a conjecture by Deligne [8]. In the first part, we will review the weight filtration introduced by Oda (after Deligne and Ihara) and a circle of related results.

1.1. **Weight filtration on $\widetilde{Out}(\pi_1^l(X))$.** By a fundamental result of Grothendieck [7] the pro-l geometric fundamental group $\pi_1^l(X_{\bar{K}})$ of a smooth algebraic curve X over $\bar{K}$ is isomorphic to the pro-l completion of its topological fundamental group, after extending the base field to the field of complex numbers. The topological fundamental group of a Riemann surface of genus g with n punctured points has the following standard presentation:

$$\Pi_{g,n} \cong\; < a_1, ..., a_g, b_1, ..., b_g, c_1, ..., c_n | \prod_{i=1}^{g}[a_i, b_i] \prod_{j=1}^{n} c_j = 1 > .$$

Let $Aut(\pi_1^l(X))$ denote the group of continuous automorphisms of the pro-l fundamental group of X and $Out(\pi_1^l(X))$ denote its quotient by the subgroup of inner automorphisms. We will induce filtrations on particular subgroups of these two groups.

Let $\bar{X}$ denote the compactification of X obtained by adding finitely many points. $\bar{X}$ is still defined over K. Let $\lambda : Aut(\pi_1^l(\bar{X})) \to GL(2g, \mathbb{Z}_l)$ denote the map induced by abelianization. The natural actions of $Aut(\pi_1^l(\bar{X}))$ on cohomology groups $H^i(\pi_1^l(\bar{X}), \mathbb{Z}_l)$ are compatible with the non-degenerate alternating form defined by the cup product:

$$H^1(\pi_1^l(\bar{X}), \mathbb{Z}_l) \times H^1(\pi_1^l(\bar{X}), \mathbb{Z}_l) \to H^2(\pi_1^l(\bar{X}), \mathbb{Z}_l) \cong \mathbb{Z}_l.$$

This shows that the image of λ is contained in $GSp(2g, \mathbb{Z}_l)$. One can prove that λ is surjective and if $\tilde{\lambda}$ denote the natural map

$$\tilde{\lambda} : Out(\pi_1^l(\bar{X})) \longrightarrow GSp(2g, \mathbb{Z}_l)$$

there are explicit examples showing that the Galois representation $\rho_X^l \circ \tilde{\lambda}$ does not fully determine the original anabelian Galois representation [1].

Let $\widetilde{Aut}(\pi_1^l(X))$ denote the Braid subgroup of $Aut(\pi_1^l(X))$ which consists of those elements taking each c_i to a conjugate of a power c_i^σ for some σ in $\mathbb{Z}_l^*$. There is a natural surjective map

$$\pi_X : \widetilde{Aut}(\pi_1^l(X)) \longrightarrow Aut(\pi_1^l(\bar{X}))$$

Oda uses $\tilde{\lambda}$ to define and study natural filtrations on $\widetilde{Aut}(\pi_1^l(X))$ and $\widetilde{Out}(\pi_1^l(X))$. In the special case of $X = \mathbb{P}^1 - \{0, 1, \infty\}$ this is the same filtration as the filtration introduced by Deligne and Ihara. This filtration is also used by Nakamura in bounding Galois centralizers [18]. Consider the central series of the pro-l fundamental group

$$\pi_1^l(X) = I^1 \pi_1^l(X) \supset I^2 \pi_1^l(X) \supset ... \supset I^m \pi_1^l(X) \supset ...$$

and let $I^1 Aut(\pi_1^l(X))$ denote the kernel of $\pi_X \circ \lambda$. The central series filtration is not the most appropriate for non-compact X. In general, we consider the weight filtration, namely the fastest decreasing central filtration such that

$$I^2 \pi_1^l(X) = < [\pi_1^l(X), \pi_1^l(X)], c_1, ..., c_n >_{\text{norm}}$$

where $< \, . \, >_{\text{norm}}$ means the closed normal subgroup generated by these elements. For $m \geq 3$ we define

$$I^m \pi_1^l(X) = < [I^i \pi_1^l(X), I^j \pi_1^l(X)] | i + j = m >_{\text{norm}} .$$

The weight filtration induces a filtration on the automorphism group of braid type by normal subgroups

$$\widetilde{Aut}(\pi_1^l(X)) = I^0 \widetilde{Aut}(\pi_1^l(X)) \supset I^1 \widetilde{Aut}(\pi_1^l(X)) \supset ... \supset I^m \widetilde{Aut}(\pi_1^l(X)) \supset ...$$

$$I^m \widetilde{Aut}(\pi_1^l(X)) = \{\sigma \in \widetilde{Aut}(\pi_1^l(X)) | x^\sigma x^{-1} \in I^{m+1} \pi_1^l(X) \text{ for all } x \in \pi_1^l(X)\}$$

Proposition 1.1. *The weight filtration on* $\widetilde{Aut}(\pi_1^l(X))$ *satisfies*

$$[I^m \widetilde{Aut}(\pi_1^l(X)), I^n \widetilde{Aut}(\pi_1^l(X))] \subset I^{m+n} \widetilde{Aut}(\pi_1^l(X))$$

for all m and n, and induces a Lie-algebra structure on the associated graded object $Gr_I^\bullet \widetilde{Aut}(\pi_1^l(X)) = \bigoplus gr^m \widetilde{Aut}(\pi_1^l(X))$. *The graded pieces*

$$gr^m \widetilde{Aut}(\pi_1^l(X)) = I^m \widetilde{Aut}(\pi_1^l(X))/I^{m+1} \widetilde{Aut}(\pi_1^l(X))$$

are free $\mathbb{Z}_l$-modules of finite rank for all positive m.

The weight filtration on the automorphism group of pro-l fundamental group induces a filtration on the outer automorphism group of braid type

$$\widetilde{Out}(\pi_1^l(X)) = I^0 \widetilde{Out}(\pi_1^l(X)) \supset I^1 \widetilde{Out}(\pi_1^l(X)) \supset ... \supset I^m \widetilde{Out}(\pi_1^l(X)) \supset ...$$

Proposition 1.2. *The induced filtration on* $\widetilde{Out}(\pi_1^l(X))$ *satisfies*

$$[I^m \widetilde{Out}(\pi_1^l(X)), I^n \widetilde{Out}(\pi_1^l(X))] \subset I^{m+n} \widetilde{Out}(\pi_1^l(X))$$

for all m and n, and induces a Lie-algebra structure on the associated graded object $Gr_I^\bullet\widetilde{Out}(\pi_1^l(X)) = \bigoplus gr^m\widetilde{Out}(\pi_1^l(X))$. *The graded pieces*

$$gr^m\widetilde{Out}(\pi_1^l(X)) = I^m\widetilde{Out}(\pi_1^l(X))/I^{m+1}\widetilde{Out}(\pi_1^l(X))$$

are finitely generated $\mathbb{Z}_l$-*module for all positive m.*

These two propositions are proved in [kon]. One can induce filtrations on $Aut(\pi_1^l(X))$ and $Out(\pi_1^l(X))$ using the natural surjection

$$Aut(\pi_1^l(X)) \longrightarrow GSp(2g, \mathbb{Z}_l)$$

which is induced by the action of $Aut(\pi_1^l(X))$ on $gr^1\pi_1^l(X) \cong \mathbb{Z}_l^{2g}$.

1.2. Graded pieces of $\widetilde{Out}(\pi_1^l(X))$.

The explicit presentation of the fundamental group of a Riemann surface given in the previous section implies that $\pi_1^l(X)$ is a one-relator pro-l group and therefore very close to a free pro-l group. The groups $Aut(\pi_1^l(X))$ and $Out(\pi_1^l(X))$ also look very similar to automorphism group and outer automorphism group of a free pro-l group [20]. This can be shown more precisely in the particular case of $\widetilde{Out}(\pi_1^l(X))$.

The graded pieces of $Gr_I^\bullet\widetilde{Out}(\pi_1^l(X))$ can be completely determined in terms of the graded pieces of $Gr_I^\bullet\pi_1^l(X)$ which are free $\mathbb{Z}_l$-modules. In fact, $Gr_I^\bullet\pi_1^l(X)$ is a free Lie-algebra over $\mathbb{Z}_l$ generated by images of a_i's and b_i's in $gr^2\pi_1^l(X)$ for $1 \leq i \leq g$ and c_j's in $gr^3\pi_1^l(X)$ for $1 \leq j \leq n$. We denote these generators by $\bar{a}_i, \bar{b}_i$ and $\bar{c}_j$ respectively.

Let g_m denote the following injective $\mathbb{Z}_l$-linear homomorphism

$$g_m : gr^m\pi_1^l(X) \longrightarrow (gr^{m+1}\pi_1^l(X))^{2g} \times (gr^m\pi_1^l(X))^n$$

$$g \mapsto ([g, \bar{a}_i])_{1 \leq i \leq g} \times ([g, \bar{b}_i])_{1 \leq i \leq g} \times (g)_{1 \leq j \leq n}$$

and f_m denote the following surjective $\mathbb{Z}_l$-linear homomorphism

$$f_m : (gr^{m+1}\pi_1^l(X))^{2g} \times (gr^m\pi_1^l(X))^n \longrightarrow gr^{m+2}\pi_1^l(X)$$

$$(r_i)_{1 \leq i \leq g} \times (s_i)_{1 \leq i \leq g} \times (t_j)_{1 \leq j \leq n} \mapsto \sum_{i=1}^{g}([\bar{a}_i, s_i] + [r_i, \bar{b}_i]) + \sum_{j=1}^{n}[t_j, \bar{c}_j].$$

Proposition 1.3. *The graded pieces of* $Gr_I^\bullet\widetilde{Out}(\pi_1^l(X))$ *fit into the following short exact sequence of* $\mathbb{Z}_l$-*modules*

$$gr^m\widetilde{Out}(\pi_1^l(X)) \hookrightarrow (gr^{m+1}\pi_1^l(X))^{2g} \times (gr^m\pi_1^l(X))^n / gr^m\pi_1^l(X) \twoheadrightarrow gr^{m+2}\pi_1^l(X)$$

where embedding of $gr^m\pi_1^l(X)$ *inside* $(gr^{m+1}\pi_1^l(X))^{2g} \times (gr^m\pi_1^l(X))^n$ *is defined by* g_m *and the final surjection is induced by* f_m.

This computational tool helps to work with the graded pieces of $Gr_I^\bullet\widetilde{Out}(\pi_1^l(X))$ as fluent as the graded pieces of $Gr_I^\bullet\pi_1^l(X)$. In particular, it enabled Koneko to prove the following profinite version of the Dehn-Nielson theorem [Kon]:

Theorem 1.4. *(Koneko) Let X be a smooth curve defined over a number-field K and let Y denote an embedded curve in X obtained by omitting finitely many K-rational points. Then the natural map*

$$\widetilde{Out}(\pi_1^l(Y)) \longrightarrow \widetilde{Out}(\pi_1^l(X))$$

is a surjection.

This can be easily proved by diagram chasing between the corresponding short exact sequences for Y and X. The above exact sequence first appeared in the work of Ihara [10] and then generalized by Asada and Koneko [1].

1.3. **Filtrations on the Galois group.** If all of the points in the complement $\bar{X} - X$ are K-rational, then the pro-l outer representation of the Galois group lands in the braid type outer automorphism group

$$\tilde{\rho}_X^l : Gal(\bar{K}/K) \longrightarrow \widetilde{Out}(\pi_1^l(X))$$

and the weight filtration on the pro-l outer automorphism group induce a filtration on the absolute Galois group mapping to $\widetilde{Out}(\pi_1^l(X))$ and also an injection between associated Lie algebras over $\mathbb{Z}_l$ defined by each of these filtrations

$$Gr_{X,l}^\bullet Gal(\bar{K}/K) \hookrightarrow Gr_I^\bullet \widetilde{Out}(\pi_1^l(X)).$$

Proposition 1.5. *Let X and Y denote smooth curves over K and let $\phi : X \to Y$ denote a morphism also defined over K. Then ϕ induces a commutative diagram of Lie algebras*

$$\begin{array}{ccc} Gr_{X,l}^\bullet Gal(\bar{K}/K) & \hookrightarrow & Gr_I^\bullet \widetilde{Out}(\pi_1^l(X)) \\ \downarrow \phi_{**} & & \downarrow \phi_* \\ Gr_{Y,l}^\bullet Gal(\bar{K}/K) & \hookrightarrow & Gr_I^\bullet \widetilde{Out}(\pi_1^l(Y)) \end{array}$$

If ϕ induces a surjection on topological fundamental groups, then ϕ_* and ϕ_{**} will also be surjective.

Proof. The claim is true because ϕ_* is Galois equivariant and graded pieces of $Gr_I^\bullet \widetilde{Out}(\pi_1^l(X))$ can be represented in terms of exact sequences on graded pieces of $Gr_I^\bullet \pi_1^l(X)$ [Kon]. $\qquad\square$

Proposition 1.6. *Let X be an affine smooth curve X over K whose complement has a K-rational point. Then there is a morphism*

$$Gr_{X,l}^\bullet Gal(\bar{K}/K) \to Gr_{\mathbb{P}^1 - \{0,1,\infty\},l}^\bullet Gal(\bar{K}/K).$$

Proof. This is a consequence of theorem 3.1 in [15]. $\qquad\square$

Proposition 1.7. *There exists a finite set of primes S such that we have an isomorphism*

$$Gr_{X,l}^\bullet Gal(\bar{K}/K) \cong Gr_{X,l}^\bullet Gal(K_S^{un}/K)$$

where $Gal(K_S^{un}/K)$ denotes the Galois group of the maximal algebraic extension unramified outside S

Proof. Indeed, Grothendieck proved that the representation $\tilde{\rho}_X^l$ factors through $Gal(K_S^{un}/K)$ for a finite set of primes S. S can be taken to be primes of bad reduction of X and primes over l [7]. This is also proved independently by Ihara in the special case of $X = \mathbb{P}^1 - \{0, 1, \infty\}$ [10]. □

The importance of this result of Grothensieck is the fact that $Gal(K_S^{un}/K)$ is a finitely generated profinite group [Neu-Sch-Wei] and therefore, the moduli of its representations is a scheme of finite type.

For a Lie algebra L over $\mathbb{Z}_l$ let $Der(L)$ denote the set of derivations, which are defined to be $\mathbb{Z}_l$-linear homomorphisms $D : L \to L$ with

$$D([u, v]) = [D(u), v] + [u, D(v)]$$

for all u and v in L, and let $Inn(L)$ denote the set of inner derivations, which are defined to be derivations with $D(u) = [u, v]$ for some fixed $v \in L$. Then we have the following Lie algebra version of the outer representation of the Galois group

$$Gr_{X,l}^\bullet Gal(\bar{K}/K) \to Der(Gr_I^\bullet \pi_1^l(X))/Inn(Gr_I^\bullet \pi_1^l(X))$$

$$\sigma \in gr^m Gal(\bar{K}/K) \mapsto (u \mapsto \tilde{\sigma}(\tilde{u}).\tilde{u}^{-1}) \bmod I^{m+n+1}\pi_1^l(X)$$

where $u \in gr^n \pi_1^l(X)$ with $\tilde{u} \in I^n \pi_1^l(X)$ and $\tilde{\sigma} \in I^m Gal(\bar{K}/K)$ is a lift of σ. In fact, for a free graded algebra, one can naturally associate a grading on the the algebra of derivations. Let $L = \oplus_i L^i$ be a free graded Lie algebra and let D denote the derivation algebra of L. Then define

$$D^i = \{d \in D | D(L^j) \subset L^{i+j}\}.$$

Then every element $d \in D$ is uniquely represented in the form $d = \sum_i d^i$ with $d^i \in D^i$ such that for any $f \in L$ the component $d^i f$ vanishes for almost all i. One can prove that

$$[D^i, D^j] \subset D^{i+j} \text{ and } [D_1, D_2]^k = \sum_{i+j=k} [D_1^i, D_2^j].$$

One can mimic the same construction on the graded algebra associated to $\pi_1^l(X)$ to get a graded algebra of derivations [24].

One shall notice that in case $X = \mathbb{P}^1 - \{0, 1, \infty\}$ the group $\pi_1^l(X)$ is the pro-l completion of a free group with two generators. Ihara proves that the associated Lie algebra $Gr_I^\bullet(\pi_1^l(X))$ is also free over two generators say x and y [10]. Now for f in the m-th piece of the grading of the Lie algebra, there is a unique derivation $D_f \in Der(Gr_I^\bullet(\pi_1^l(X)))$ which satisfies $D_f(x) = 0$ and $D_f(y) = [y, f]$. One can show that $D_f(y)$ is non-zero for non-zero m and that for any $\sigma \in Gr_I^\bullet(\pi_1^l(X))$ there exists a unique $f \in Gr_I^\bullet(\pi_1^l(X))$ with image of σ being equal to D_f [16]. Now it is enough to let $\sigma = \sigma_m$ the Soule elements, to get a non-zero image $D_{f_{\sigma_m}}$.

Conjecture 1.8. *(Deligne) The graded Lie algebra* $(Gr^{\bullet}_{\mathbb{P}^1-\{0,1,\infty\},l}Gal(\bar{\mathbb{Q}}/\mathbb{Q}))\otimes$ $\mathbb{Q}_l$ *is a free graded Lie algebra over* $\mathbb{Q}_l$ *which is generated by Soule elements and the Lie algebra structure is induced from a Lie algebra over* $\mathbb{Z}$ *independent of* l.

Remark 1.9. *It is reasonable to expect freeness to hold for* $(Gr^{\bullet}_{X,l}Gal(\bar{\mathbb{Q}}/\mathbb{Q}))\otimes$ $\mathbb{Q}_l$.

This implies that the above graded Lie algebra representation is also injective. Ihara showed that Soule elements do generate $Gr^{\bullet}_{X,l}(Gal(\bar{\mathbb{Q}}/\mathbb{Q})) \otimes \mathbb{Q}_l$ if one assumes freeness of this Lie algebra [10]. Hain and Matsumoto proved the same result without assuming any part of Deligne's conjecture [8].

2. Deformation theory

We search for deformations of Galois outer representations which are equipped with extra deformation data, for example with fixed mod-l representation. One can deform such representations both in geometric and algebraic senses. A geometric example would be given by p-adic deformation of a smooth curve X defined over a local field $K_\wp$. M. Kisin has considered such deformation problems and has proved rigidity results in case $l \neq p$ [12]. We are interested in algebraic deformations which also work over global fields. In this part, we will introduce several deformation problems for representations landing in graded Lie algebras over $\mathbb{Z}_l$. We also develop an arithmetic deformation theory for graded Lie algebras and morphisms between them. In some cases universal deformations exist and in some others, we are only able to construct a hull which parameterizes all possible deformations.

2.1. **Several deformation problems.** Let k be a finite field of characteristic p and let Λ be any complete Noetherian local ring. For example Λ can be $W(k)$, the ring of Witt vectors of k, or O, the ring of integers of any local field with quotient field $K_\wp$ and residue field k. Let C denote the category of Artinian local Λ-algebras with residue field k. A covariant functor from C to $Sets$ is called pro-representable if it has the form

$$F(A) \cong Hom_\Lambda(R_{univ}, A) \qquad A \in C$$

where R_{univ} is a complete local Λ-algebra with maximal ideal m_{univ} such that R_{univ}/m^n_{univ} is in C for all n. There are a number of deformation functors related to our problem.

The first deformation problem coming to mind is deforming actions of Galois group on graded Lie algebras. Indeed, Galois group maps to $\widetilde{Out}(\pi^l_1(X))$ which acts on itself and therefore on $Gr^{\bullet}_I\widetilde{Out}(\pi^l_1(X))$ by conjugation. By results of Koneko, this action is completely determined by the associated abelian Galois representation [Kon]. This means that, the corresponding deformation problem

is pro-representable and we get exactly the same deformation ring as in the abelian case.

In order to get a more delicate deformation theory, we could deform the Lie algebra representation

$$Gr^\bullet_{X,l} Gal(\bar{K}/K) \to Gr^\bullet_I \widetilde{Out}(\pi^l_1(X))$$

of the Galois graded Lie algebra, induced by the outer representation of the Galois group. We could define $D_{\bar\rho}(A)$ for any Artinian local algebra A to be the set of isomorphism classes of deformations of the mod-l reduction of the above representation $\bar\rho$ to $Gr^\bullet_I \widetilde{Out}(\pi^l_1(X)) \otimes A$. One is interested in some variant of $D_{\bar\rho}(A)$ being pro-representable.

The following Lie algebra version of an outer representation of the Galois group

$$Gr^\bullet_{X,l} Gal(\bar{K}/K) \to Der(Gr^\bullet_I \pi^l_1(X))/Inn(Gr^\bullet_I \pi^p_1(X))$$

which was defined in the first part is another candidate which could be deformed.

2.2. Galois actions on graded Lie algebras.

The method of proving modularity results by finding isomorphisms between Hecke algebras and universal deformation rings as originated by Wiles [Wi], can be reformulated in the language of Lie algebras. One can define a Hecke-Lie algebra and a canonical graded representation of the Galois-Lie algebra to Hecke-Lie algebra which contains all the information of modular Galois representations.

Let us first reformulate the theory of Galois representations in the language Lie algebas. We start with elliptic curves. To each elliptic curve E defined over $\mathbb{Q}$ which has a rational point, one associates a Galois outer representation

$$Gal(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow Out(\pi^l_1(E - \{0\})).$$

By analogy to Shimura-Taniyama-Weil conjecture, we expect this representation be encoded in the representations

$$Gal(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow Out(\pi^l_1(Y_0(N)))$$

associated to modular curves $Y_0(N)$ which have a model over $\mathbb{Q}$. By $Y_0(N)$ we mean the non-compactified modular curve of level N which is given as the quotient of the upper half-plane by the congruence subgroup $\Gamma_0(N)$ of $SL_2(\mathbb{Z})$ consisting of matrices which are upper triangular modulo N.

For any smooth curve X defined over $\mathbb{Q}$ the outer automorphism group of braid type acts on $Gr^\bullet_I \widetilde{Out}(\pi^l_1(X))$ by conjugation and therefore for each m we get a Galois representation

$$Gal(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow Aut(gr^m \widetilde{Out}(\pi^l_1(X))).$$

In grade zero, we recover the usual abelian Galois representation and in higher grades on can canonically construct this representation by the grade-zero standard representation. Indeed, for each $m \geq 1$ the isomorphism in proposition 1.3 is $\widetilde{Out}(\pi_1^l(X))$-equivariant. From this we can determine the representation from the inner action of $\widetilde{Out}(\pi_1^l(X))$ on $Gr_I^\bullet \pi_1^l(X)$. This action is fully determined by the grade-zero action [Kon]. Therefore, the Galois representations

$$Gal(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow Aut(gr^m \widetilde{Out}(\pi_1^l(X)))$$

are all determined by the abelian Galois representation associated to X over $\mathbb{Q}$. Together with Shimura-Tanyama-Weil conjecture proved by Wiles and his collaborators [26], [22] and [3] we get the following

Theorem 2.1. *Let E be an elliptic curve over $\mathbb{Q}$ together with a rational point $0 \in E$. For each m the Galois representation*

$$Gal(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow Aut(gr^m \widetilde{Out}(\pi_1^l(E - \{0\})))$$

appear as direct summand of the Galois representation

$$Gal(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow Aut(gr^m \widetilde{Out}(\pi_1^l(Y_0(N))))$$

for some integer N.

2.3. **Deformations of local graded Lie algebras.** We are interested in deforming the coefficient ring of graded Lie algebras over $\mathbb{Z}_l$ of the form $L = Gr_I^\bullet \widetilde{Out}(\pi_1^l(X))$ and then deforming representations of Galois graded Lie algebra

$$Gr_{X,l}^\bullet Gal(\bar{K}/K) \to Gr_I^\bullet \widetilde{Out}(\pi_1^l(X)).$$

One can reduce the coefficient ring $\mathbb{Z}_l$ modulo l and get a graded Lie algebra $\bar{L}$ over $\mathbb{F}_l$ and a representation

$$Gr_{X,l}^\bullet Gal(\bar{K}/K) \to \bar{L}.$$

We look for liftings of this representation which is landing in $\bar{L}$ among representations landing in graded Lie algebras over Artin local rings A of the form $L = \oplus_i L^i$ where $L^0 \cong GSp(2g, A)$ and L^i is a finitely generated A-module for positive i. In this section, we are only concerned with deformations of Lie algebras and leave deformation of their representations for the next section.

For a local ring $A \in C$ with maximal ideal m, the set of deformations of $\bar{L}$ to A is denoted by $D_c(\bar{L}, A)$ and is defined to be the set of isomorphism classes of graded Lie algebras L/A of the above form which reduce to $\bar{L}$ modulo m. In this notation c stands for coefficients, since we are only deforming coefficients not the Lie algebra structure. The functor $D(\bar{L})$ as defined above is not a pro-representable functor. As we will see, there exists a "hull" for this functor (Schlessinger's terminology [21]) parameterizing all possible deformations. One can also deform the Lie-algebra structure of graded Lie algebras. The idea of deforming the Lie structure of Lie algebras has been extensively used by

geometers. For example, Fialowski studied this problem in double characteristic zero case [5]. In this paper, we are interested in double characteristic $(0, l)$-version.

We shall first review cohomology of Lie algebras with the adjoint representation as coefficients. Let L be a graded $\mathbb{Z}_l$-algebra and let $C^q(L, L)$ denote the space of all skew-symmetric q-linear forms on a Lie algebra L with values in L. Define the differential

$$\delta : C^q(L, L) \longrightarrow C^{q+1}(L, L)$$

where the action of δ on a skew-symmetric q-linear form γ is a skew-symmetric $(q+1)$-linear form which takes $(l_1, ..., l_{q+1}) \in L^{q+1}$ to

$$\sum (-1)^{s+t-1} \gamma([l_s, l_t], l_1, ..., \hat{l}_s, ..., \hat{l}_t, ..., l_{q+1}) + \sum [l_u, \gamma(l_1, ..., \hat{l}_u, ..., l_{q+1})]$$

where the first sum is over s and t with $1 \le s < t \le q+1$ and the second sum is over u with $1 \le u \le q+1$. Then $\delta^2 = 0$ and we can define $H^q(L, L)$ to be the cohomology of the complex $\{C^q(L, L), \delta\}$. If we put $C^m = C^{m+1}(L, L)$ and $H^m = H^{m+1}(L, L)$, then there exists a natural bracket operation which makes $C = \oplus C^m$ a differential graded algebra and $H = \oplus H^m$ a graded Lie algebra. Look in [6]. If L is $\mathbb{Z}$-graded, $L = \oplus L(m)$, we say $\phi \in C^q(L, L)(m)$ if for $l_i \in L(g_i)$ we have $\phi(l_1, ..., l_q) \in L(g_1 + ... + g_q - m)$. Then, there exists a grading induced on the Lie algebra cohomology $H^q(L, L) = \oplus H^q(L, L)(m)$.

Now we make an assumption for further constructions. Assume $H^2(L, L)(m)$ is finite dimensional for all m and consider the algebra $\mathbb{D}_1 = \mathbb{Z}_l \oplus \bigoplus_m H^2(L, L)(m)'$ where $'$ means the dual over $\mathbb{Z}_l$. Fix a graded homomorphism of degree zero

$$\mu : H^2(L, L) \longrightarrow C^2(L, L)$$

which takes any cohomology class to a cocycle representing this class. Now define a Lie algebra structure on

$$\mathbb{D}_1 \otimes L = L \oplus Hom^0(H^2(L, L), L)$$

where Hom^0 means degree zero graded homomorphisms, by the following bracket

$$[(l_1, \phi_1), (l_2, \phi_2)] := ([l_1, l_2], \psi)$$

where $\psi(\alpha) = \mu(\alpha)(l_1, l_2) + [\phi_1(\alpha), l_2] + [\phi_2(\alpha), l_1]$. The Jacobi identity is implied by $\delta\mu(\alpha) = 0$. It is clear that this in an infinitesimal deformation of L and it can be shown that, up to an isomorphism, this deformation does not depend on the choice of μ. We shall denote this deformation by η_L after Fialowski and Fuchs [6].

Proposition 2.2. *Any infinitesimal deformation of $\bar{L}$ to a finite dimensional local ring A is induced by pushing forward η_L by a unique morphism*

$$\phi : \mathbb{Z}_l \oplus \bigoplus_m H^2(L, L)(m)' \longrightarrow A.$$

Proof. This is the double characteristic version of proposition 1.8 in [6]. □

Note that, in our case $H^2(L, L)$ is not finite dimensional. This is why we restrict our deformations to the space of graded deformations. Since $H^2(L, L)(0)$ is the tangent space of the space of graded deformations, and the grade zero piece $H^2(L, L)(0)$ is finite dimensional, the following version is more appropriate:

Proposition 2.3. *Any infinitesimal graded deformation of $\bar{L}$ to a finite dimensional local ring A is induced by pushing forward η_L^0 by a unique morphism*

$$\phi : \mathbb{Z}_l \oplus H^2(L, L)(0)' \longrightarrow A.$$

where η_L^0 denotes the restriction of η_L to $\mathbb{Z}_l \oplus H^2(L, L)(0)'$.

Let A be a small extension of $\mathbb{F}_l$ and L be a graded deformation of $\bar{L}$ over the base A. The deformation space $D(\bar{L}, A)$ can be identified with $H^2(L, L)(0)$ which is finite dimensional. Therefore, by Schlessinger criteria, in the subcategory C' of C consisting of local algebras with $m^2 = 0$ for the maximal ideal m, the functor $D(\bar{L}, A)$ is pro-representable. This means that there exists a unique map

$$\mathbb{Z}_l \oplus H^2(L, L)(0)' \to R_{univ}$$

inducing the universal infinitesimal graded deformation.

2.4. **Obstructions to deformations.** The computational tool used by geometers to study deformations of Lie algebras is a cohomology theory of K-algebras where K is a field, which is developed by Harrison [9]. This cohomology theory is generalized by Barr to algebras over general rings [2]. Here we use a more modern version of the latter introduced independently by Andre and Quillen which works for general algebras [19].

Let A be a commutative algebra with identity over $\mathbb{Z}_l$ or any ring R and let M be an A-module. By an n-long singular extension of A by M we mean an exact sequence of A-modules

$$0 \to M \to M_{n-1} \to ... \to M_1 \to T \to R \to 0$$

where T is a commutative $\mathbb{Z}_l$-algebra and the final map a morphism of $\mathbb{Z}_l$-algebras whose kernel has square zero. It is trivial how to define morphisms and isomorphisms between n-long singular extensions. Baer defines a group structure on these isomorphism classes [2], which defines $H_{Barr}^n(A, M)$ for $n > 1$ and we put $H_{Barr}^1(A, M) = Der(A, M)$. Barr proves that for a multiplicative subset S of R not containing zero

$$H_{Barr}^n(A, M) \cong H_{Barr}^n(A_S, M)$$

for all n and any A_S-module M. According to this isomorphism, the cohomology of the algebra A over $\mathbb{Z}_l$ is the same after tensoring A with $\mathbb{Q}_l$ if M

is $A \otimes \mathbb{Q}_l$-module. Thus one could assume that we are working with an algebra over a field, and then direct definitions given by Harrison would serve our computations better. Consider the complex

$$0 \to Hom(A, M) \to Hom(S^2 A, M) \to Hom(A \otimes A \otimes A, M)$$

where $\psi \in Hom(A, M)$ goes to

$$d_1 \psi : (a, b) \mapsto a\psi(b) - \psi(ab) + b\psi(a)$$

and $\phi \in Hom(S^2 A, M)$ goes to

$$d_2 \phi(a, b, c) \mapsto a\phi(b, c) - \phi(ab, c) + \phi(a, bc) - c\phi(a, b)$$

The cohomology of this complex defines $H^i_{Harr}(R, M)$ for $i = 1, 2$. If A is a local algebra with maximal ideal m and residue field k, the Harrison cohomology $H^1_{Harr}(A, k) = (m/m^2)'$, which is the space of homomorphisms $A \to k[t]/t^2$ such that m is the kernel of the composition $A \to k[t]/t^2 \to k$.

Andre-Quillen cohomology is the same as Barr cohomology in low dimensions and can be described directly in terms of derivations and extensions. For any morphism of commutative rings $A \to B$ and B-module M we denote the B-module of A-algebra derivations of B with values in M by $Der_A(B, M)$. Let $Ext_A^{inf}(B, M)$ denote the B-module of infinitesimal A-algebra extensions of B by M. The functors Der and Ext^{inf} have transitivity property. Namely, given morphisms of commutative rings $A \to B \to C$ and a C-module M, there is an exact sequence

$$0 \longrightarrow Der_B(C, M) \longrightarrow Der_A(C, M) \longrightarrow Der_A(B, M)$$

$$\longrightarrow Ext_B^{inf}(C, M) \longrightarrow Ext_A^{inf}(C, M) \longrightarrow Ext_A^{inf}(B, M).$$

The two functors Der and Ext^{inf} also satisfy flat base-change property. Namely, given morphisms $A \to B$ and $A \to A'$ if $Tor_1^A(A', B) = 0$, then there are isomorphisms $Der_{A'}(A' \otimes_A B, M) \cong Der_A(B, M)$ and $Ext_{A'}^{inf}(A' \otimes_A B, M) \cong Ext_A^{inf}(B, M)$. Andre-Quillen cohomology associates $Der_A(B, M)$ and $Ext_A^{inf}(B, M)$ to any morphism of commutative rings $A \to B$ and B-module M as the first two cohomologies and extends it to higher dimensional cohomologies such that transitivity and flat base-change extend in the obvious way.

Let A be an object in the category C and $L \in Def(\bar{L}, A)$. The pair (A, L) defines a morphism of functors $\theta : Mor(A, B) \to Def(\bar{L}, B)$. We say that (A, L) is universal if θ is an isomorphism for any choice of B. We say that (A, L) is miniversal if θ is always surjective, and gives an isomorphism for $B = k[\varepsilon]/\varepsilon^2$. We intend to construct a miniversal deformation of $\bar{L}$.

Consider a graded deformation with base in a local algebra A with residue field $k = \mathbb{F}_l$. One can define a map

$$\Phi_A : Ext_{\mathbb{Z}_l}^{inf}(A, k) \longrightarrow H^3(\bar{L}, \bar{L}).$$

Indeed, choose an extension $0 \to k \to B \to A \to 0$ corresponding to an element in $Ext_{\mathbb{Z}_l}^{inf}(A, k)$. Consider the B-linear skew-symmetric operation $\{., .\}$ on $\bar{L} \otimes_k B$ commuting with $[., .]$ on $\bar{L} \otimes A$ defined by $\{l, l_1\} = [l, \bar{l}_1]$ for l in the kernel of $\bar{L} \otimes_k B \to \bar{L} \otimes_k A$ which can be identified by $\bar{L}$. Here $\bar{l}_l$ is the image of l_1 under the projection map $B \to k$ tensored with $\bar{L}$ whose kernel is the inverse image of the maximal ideal of A. The Jacobi expression induces a multilinear skew-symmetric form on $\bar{L}$ which could be regarded as a closed element in $C^3(\bar{L}, \bar{L})$. The image in $H^3(\bar{L}, \bar{L})$ is independent of the choices made.

Theorem 2.4. *(Fialowski) One can deform the Lie algebra structure on $L \otimes_k A$ to $L \otimes_k B$ if and only if the image of the above extension vanishes under the morphism $Ext_{\mathbb{Z}_l}^{inf}(A, k) \to H^3(\bar{L}, \bar{L})$.*

Proof. The proof presented in [5] and [6] works for algebras over fields of finite characteristic.   □

Using the above criteria for extending deformations, one can follow the methods of Fialowski and Fuchs to introduce a miniversal deformation for $\bar{L}$.

Proposition 2.5. *Given a local commutative algebra A over $\mathbb{Z}_l$ there exists a universal extension*

$$0 \longrightarrow Ext_{\mathbb{Z}_l}^{inf}(A, k)' \longrightarrow C \longrightarrow A \longrightarrow 0$$

among all extensions of A with modules M over A with $mM = 0$ where m is the maximal ideal of A.

This is proposition 2.6 in [6]. Consider the canonical split extension

$$0 \longrightarrow H^2(L, L)(0)' \longrightarrow \mathbb{D}_1 \longrightarrow k \longrightarrow 0.$$

We will initiate an inductive construction of $\mathbb{D}_k$ such that

$$0 \longrightarrow Ext_{\mathbb{Z}_l}^{inf}(\mathbb{D}_k, k)' \longrightarrow \bar{\mathbb{D}}_{k+1} \longrightarrow \mathbb{D}_k \longrightarrow 0.$$

together with a deformation η_k of L to the base $\mathbb{D}_k$. For η_1 take η_L, and assume $\mathbb{D}_i$ and η_k is constructed for $i \leq k$. Given a local commutative algebra $\mathbb{D}_k$ with maximal ideal m, there exists a unique universal extension for all extensions of $\mathbb{D}_k$ by $\mathbb{D}_k$-modules M with $mM = 0$ of the following form

$$0 \longrightarrow Ext_{\mathbb{Z}_l}^{inf}(\mathbb{D}_k, k)' \longrightarrow C \longrightarrow \mathbb{D}_k \longrightarrow 0.$$

associated to the cocycle $f_k : S^2(\mathbb{D}_k) \to Ext_{\mathbb{Z}_l}^{inf}(\mathbb{D}_k, k)'$ which is dual to the homomorphism

$$\mu : Ext_{\mathbb{Z}_l}^{inf}(\mathbb{D}_k, k) \longrightarrow S^2(\mathbb{D}_k)'$$

which takes a cohomology class to a cocycle from the same class. The obstruction to extend η_k lives in $Ext_{\mathbb{Z}_l}^{inf}(\mathbb{D}_k, k)' \otimes H^3(\bar{L}, \bar{L})$. Consider the composition of the associated dual map

$$\Phi_k' : H^3(\bar{L}, \bar{L})' \longrightarrow Ext_{\mathbb{Z}_l}^{inf}(\mathbb{D}_k, k)'.$$

with $Ext^{inf}_{\mathbb{Z}_l}(\mathbb{D}_k, k)' \to C$ and define $\mathbb{D}_{k+1}$ to be the cokernel of this map. We get the following exact sequence

$$0 \longrightarrow (ker\Phi_k)' \longrightarrow \mathbb{D}_{k+1} \longrightarrow \mathbb{D}_k \longrightarrow 0.$$

We can extend η_k to η_{k+1}. Now, taking a projective limit of $\mathbb{D}_k$ we get a base and a formal deformation of $\bar{L}$.

Theorem 2.6. *Let $\mathbb{D}$ denote the projective limit $\underline{\lim}\mathbb{D}_k$ which is a $\mathbb{Z}_l$-module. One can deform $\bar{L}$ uniquely to a graded Lie algebra with base $\mathbb{D}$ which is miniveral among all deformations of $\bar{L}$ to local algebras over $\mathbb{Z}_l$.*

Proof. This is the double characteristic version of theorem 4.5 in [6]. The same proof works here because theorems 11 and 18 in [9] which are used in the arguments of Fialowski and Fuchs work for algebras over any perfect field. $\square$

Proposition 2.7. *(Fialowski-Fuchs) The base of the minversal deformation of $\bar{L}$ is the zero locus of a formal map $H^2(\bar{L}, \bar{L})(0) \to H^3(\bar{L}, \bar{L})(0)$.*

This is the graded version of proposition 7.2 in [6].

2.5. **Deformations of graded Lie algebra representations.** In the previous section we discussed deformation theory of the mod l reduction of the Lie algebra $Gr_I^\bullet \widetilde{Out}(\pi_1^l(X))$. We shall mention the following

Theorem 2.8. *The cohomology groups $H^i(Gr_I^\bullet \widetilde{Out}(\pi_1^l(X)), Gr_I^\bullet \widetilde{Out}(\pi_1^l(X)))(0)$ are finite dimensional for all non-negative integer i.*

Proof. By a theorem of Labute $Gr_I^\bullet \pi_1^l(X)$ is quotient of a finitely generated free Lie algebra with finitely generated module of relations [13]. Therefore, the cohomology groups $H^i(Gr_I^\bullet \pi_1^l(X), Gr_I^\bullet \pi_1^l(X))(0)$ are finite dimensional. Finite dimensionality of the cohomology of $Gr_I^\bullet \widetilde{Out}(\pi_1^l(X))$ follows from proposition 1.3. $\square$

We are interested in deforming the following graded representation of the Galois graded Lie algebra

$$\rho : Gr_{X,l}^\bullet Gal(\bar{K}/K) \to Gr_I^\bullet \widetilde{Out}(\pi_1^l(X))$$

among all graded representations which modulo l reduce to the graded representation

$$\bar{\rho} : Gr_{X,l}^\bullet Gal(\bar{K}/K) \to \bar{L}$$

where the Lie algebra $\bar{L}$ over $\mathbb{F}_l$ is the mod-l reduction of $Gr_I^\bullet \widetilde{Out}(\pi_1^l(X))$. There are suggestions from the classical deformation theory of Galois representations on how to get a representable deformation functor. Let $D(\bar{\rho}, A)$ denote the set of isomorphism classes of Galois graded Lie algebra representations to graded Lie algebras L/A of the above form which reduce to $\bar{\rho}$ modulo m. The first ingredient we need to prove $D(\bar{\rho})$ is representable is that the tangent

space of the functor is finite-dimensional. The tangent space of the deformation functor $D(\bar{\rho})$ for an object $A \in C$ is canonically isomorphic to

$$H^1(Gr^{\bullet}_{X,l}Gal(\bar{K}/K), Ad \circ \overline{\rho})$$

where the Lie algebra module is given by the composition of $\bar{\rho}$ with the adjoint representation of $Gr^{\bullet}_I \widetilde{Out}(\pi_1^l(X/K))$. To get finite dimensionality, we restrict ourselves to the graded deformations of the graded representation $\bar{\rho}$.

Theorem 2.9. $H^1(Gr^{\bullet}_{X,l}Gal(\bar{K}/K), Ad \circ \overline{\rho})(0)$ *is finite dimensional.*

Proof. The Galois-Lie representation $Gr^{\bullet}_I Gal(\bar{K}/K) \to Gr^{\bullet}_I \widetilde{Out}(\pi_1^l(X))$ is an injection. Derivation inducing cohomology commutes with inclusion of Lie algebras. Therefore $H^1(Gr^{\bullet}_{X,l}Gal(\bar{K}/K), Ad \circ \overline{\rho})(0)$ injects in $H^1(Gr^{\bullet}_I \widetilde{Out}(\pi_1^l(X)), Ad \circ \overline{\rho})(0)$ which is finite dimensional by previous theorem. $\square$

For a surjective mapping $A_1 \to A_0$ of Artinian local rings in C such that the kernel $I \subset A_1$ satisfies $I.m_1 = 0$ and given any deformation ρ_0 of $\bar{\rho}$ to $L \otimes A_0$ one can associate a canonical obstruction class in

$$H^2(Gr^{\bullet}_{X,l}Gal(\bar{K}/K), I \otimes Ad \circ \bar{\rho}) \cong H^2(Gr^{\bullet}_{X,l}Gal(\bar{K}/K), Ad \circ \bar{\rho}) \otimes I$$

which vanishes if and only if ρ_0 can be extended to a deformation with coefficients A_1. Therefore, vanishing results on second cohomology are important.

Theorem 2.10. *Suppose* $Gr^{\bullet}_{X,l}Gal(\bar{K}/K)$ *is a free Lie algera over* $\mathbb{Z}_l$, *then the Galois cohomology* $H^2(Gr^{\bullet}_{X,l}Gal(\bar{K}/K), Ad \circ \bar{\rho})$ *vanishes.*

Proof. The free Lie algebra $G = Gr^{\bullet}_{X,l}Gal(\bar{K}/K)$ is rigid, and therefore has trivial infinitesimal deformations. Thus, we get vanishing of its second cohomology:

$$H^2(Gr^{\bullet}_{X,l}Gal(\bar{K}/K), Gr^{\bullet}_{X,l}Gal(\bar{K}/K)) = 0.$$

The injection of G inside $L = Gr^{\bullet}_I \widetilde{Out}(\pi_1^l(X))$ as Lie-algebras over $\mathbb{Z}_l$ implies that, the cohomology group $H^2(Gr^{\bullet}_{X,l}Gal(\bar{K}/K), Ad \circ \rho)$ vanishes again by freeness of G. Let $\bar{G}$ denote the reduction modulo l of G which is a free Lie algebra over $\mathbb{F}_l$. The cohomology $H^2(G, \bar{G})$ is the mod-l reduction of $H^2(G, G)$, hence it also vanishes. So does the cohomology $H^2(Gr^{\bullet}_{X,l}Gal(\bar{K}/K), Ad \circ \bar{\rho})$ by similar reasoning. $\square$

We have obtained conceptual conditions implying H_4 of [21]. What we have proved can be summarized as follows.

Main Theorem 2.11. *Suppose that* $Gr^{\bullet}_{X,l}Gal(\bar{K}/K)$ *is a free Lie algera over* $\mathbb{Z}_l$. *There exists a universal deformation ring* $R_{univ} = R(X, K, l)$ *and a universal deformation of the representation* $\bar{\rho}$

$$\rho^{univ} : Gr^{\bullet}_{X,l}Gal(\bar{K}/K) \longrightarrow Gr^{\bullet}_I \widetilde{Out}(\pi_1^l(X)) \otimes R_{univ}$$

which is unique in the usual sense. If $Gr_{X,l}^\bullet Gal(\bar{K}/K)$ is not free, then a mini-versal deformation exists which is universal among infinitesimal deformations of $\bar\rho$.

Remark 2.12. *Note that, freeness of $Gr_{X,l}^\bullet Gal(\bar{K}/K)$ in the special case of $K = \mathbb{Q}$ where filtration comes from punctured projective curve $X = \mathbb{P}^1 - \{0, 1, \infty\}$ or a punctured elliptic curve $X = E - \{0\}$ is implied by Deligne's conjecture.*

As in the classical case, the Lie algebra structure on $Ad \circ \bar\rho$ induces a graded Lie algebra structure on the cohomology $H^*(Gr_{X,l}^\bullet Gal(\bar{K}/K), Ad \circ \bar\rho)$ via cup-product, and in particular, a symmetric bilinear pairing

$$H^1(Gr_{X,l}^\bullet Gal(\bar{K}/K), Ad \circ \bar\rho) \times H^1(Gr_{X,l}^\bullet Gal(\bar{K}/K), Ad \circ \bar\rho)$$

$$\longrightarrow H^2(Gr_{X,l}^\bullet Gal(\bar{K}/K), Ad \circ \bar\rho)$$

which gives the quadratic relations satisfied by the minimal set of formal parameters of R_{univ}/lR_{univ} for characteristic l different from 2.

2.6. **Hecke-Teichmuller graded Lie algebras.** There is no general analogue for Hecke operators in the context of Lie algebras constructed in this manner. What we need is an analogue of Hecke algebra which contains all the information of Galois outer representations associated to elliptic curves. In fact, we will provide an algebra containing such information for hyperbolic smooth curves of given topological type. From now on, we assume that $2g - 2 + n > 0$. The stack $M_{g,n}$ is defined as the moduli stack of n-pointed genus g curves.

Definition 2.13. *A family of n-pointed genus g curves over a scheme S is a proper smooth morphism $C \to S$ whose fibers are proper smooth curves of genus g together with n sections $s_i : S \to C$ for $i = 1, ..., n$ whose images do not intersect.*

The moduli stack is an algebraic stack over $Spec(\mathbb{Z})$. One can define the etale fundamental group of the stack $M_{g,n}$ in the same manner one defines etale fundamental group of schemes. Oda showed that the etale homotopy type of the algebraic stack $M_{g,n} \otimes \bar{\mathbb{Q}}$ is the same as the analytic stack $M_{g,n}^{an}$ and its algebraic fundamental group is isomorphic to the completion $\widehat{\Gamma_{g,n}}$ of the Teichmuller modular group, or the mapping class group of n-punctured genus g Riemann surfaces [14]:

$$\pi_1^{alg}(M_{g,n} \otimes \bar{\mathbb{Q}}) \cong \widehat{\Gamma_{g,n}}.$$

Triviality of π_2 implies exactness of the following short sequence for the universal family $C_{g,n} \longrightarrow M_{g,n}$ over the moduli stack

$$0 \longrightarrow \pi_1^{alg}(C, b) \longrightarrow \pi_1^{alg}(C_{g,n}, b) \longrightarrow \pi_1^{alg}(M_{g,n}, a) \longrightarrow 0$$

where C is the fiber on a and b is a point on $C_{g,n}$. Using this exact sequence, one defines the arithmetic universal monodromy representation

$$\rho_{g,n} : \pi_1^{alg}(M_{g,n}, a) \longrightarrow Out(\pi_1^{alg}(C))$$

In fact, this is the completion of the natural map $\Gamma_{g,n} \longrightarrow Out(\Pi_{g,n})$. By composition with the natural projection to outer automorphism group of the l-adic completion $\Pi_{g,n}^l$ we get a representation

$$\rho_{g,n} : \pi_1^{alg}(M_{g,n}, a) \longrightarrow Out(\pi_1^l(C)).$$

This map induces filtrations on $\pi_1^{alg}(M_{g,n})$ and its subgroup $\pi_1^{alg}(M_{g,n} \otimes \bar{\mathbb{Q}})$ and an injection of $\mathbb{Z}_l$-Lie algebras

$$Gr_I^\bullet \pi_1^{alg}(M_{g,n} \otimes \bar{\mathbb{Q}}) \hookrightarrow Gr_I^\bullet \pi_1^{alg}(M_{g,n}).$$

It is conjectured by Oda and proved by a series of papers by Ihara, Matsumoto, Nakamura and Takao that the cokernel of the above map after tensoring with $\mathbb{Q}_l$ is independent of g and n [11] [16] [18]. Note that $M_{0,3} \cong Spec(\mathbb{Q})$.

On can think of $\pi_1^{alg}(M_{g,n} \otimes \bar{\mathbb{Q}})$ as a replacement for the Hecke algebra. This object has the information of all outer representations associated to smooth curves over $\mathbb{Q}$. Indeed, by fixing such a curve C of genus $g \geq 2$ we have introduced a point on the moduli stack $a \in M_{g,0}$ and thus a Galois representation

$$Gal(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \pi_1^{alg}(M_{g,n})$$

which splits the following short exact sequence

$$0 \longrightarrow \pi_1^{alg}(M_{g,n} \otimes \bar{\mathbb{Q}}, a) \longrightarrow \pi_1^{alg}(M_{g,n}) \longrightarrow Gal(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow 0.$$

Combining with the arithmetic universal monodromy representation we get

$$Gal(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow Out(\Pi_{g,n}^l)$$

which recovers the canonical outer representation associated to C.

Definition 2.14. *We define the Hecke-Teichmuller Lie algebra to be the image of the following morphism of graded Lie algebras*

$$Gr_I^\bullet \pi_1^{alg}(M_{g,n}) \longrightarrow Gr_I^\bullet \widetilde{Out}(\Pi_{g,n}^l).$$

We expect Hecke-Teichmuller Lie algebra to serve the role of Hecke algebra in proving modularity results for elliptic curves or other motivic objects.

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