Iranian Journal of Mathematical Sciences and Informatics
Vol. 19, No. 2 (2024), pp 195-206
DOI: 10.61186/ijmsi.19.2.195

Pointwise Inner and Center Actors of a Lie Crossed Module

M. Jamshidi, F. Saeedi*

Department of Mathematics, Mashhad Branch, Islamic Azad University, Mashhad, Iran

E-mail: mehdijamshidi44@yahoo.com
E-mail:saeedi@mashdiau.ac.ir

Abstract. Let $\mathcal{L}$ be a Lie crossed module and $\mathrm{Act}_{pi}(\mathcal{L})$ and $\mathrm{Act}_z(\mathcal{L})$ be the pointwise inner actor and center actor of $\mathcal{L}$, respectively. We will give a necessary and sufficient condition under which $\mathrm{Act}_{pi}(\mathcal{L})$ and $\mathrm{Act}_z(\mathcal{L})$ are equal.

Keywords: Pointwise Inner, Crossed Module, Center Actor.

2020 Mathematics subject classification: 17B40, 17B99.

1. Introduction

Crossed modules of groups are introduced by Whitehead [11] to study homotopy relation among groups. Lie crossed modules are also introduced and used by Lavendhomme and Rosin [8] as a sufficient coefficient of a nonabelian cohomology of T-algebras.

A crossed module $\mathcal{L}$ in Lie algebras is a homomorphism $d : L_1 \longrightarrow L_0$ with an action of L_0 on L_1 satisfying special conditions (see Casas [3], Casas and Ladra [4, 5] for details).

In [9], Norrie extended the definition of actor to the 2-dimensional case by giving a description of the corresponding object in the category of crossed modules of groups. The analogoue construction for the category of crossed modules of Lie algebras is given in [5].

*Corresponding Author

Received 5 February 2020; Accepted 25 November 2021

Actor of crossed module of Leibniz algebras also introduced by Casas et al. in [6].

Allahyari and Saeedi in [1] and [2] introduced a chain of subcrossed modules of $\mathrm{Act}(\mathcal{L})$, and showed that for two Lie crossed module $\mathcal{L}$ and $\mathcal{M}$, $\mathrm{ID}^*\mathrm{Act}(\mathcal{L}) \cong \mathrm{ID}^*\mathrm{Act}(\mathcal{M})$ if $\mathcal{L}$ and $\mathcal{M}$ are isoclinic. Sheikh-Mohseni et al. [10] gives a necessary and sufficient condition for $\mathrm{Der}_c(L)$ and $\mathrm{Der}_z(L)$ of a Lie algebra L to be equal.

In this paper, we shall introduce a new subcrossed module of $\mathrm{Act}(\mathcal{L})$, denoted by $\mathrm{Act}_z(\mathcal{L})$, and study its relationships with subcrossed modules of $\mathrm{Act}(\mathcal{L})$, say $\mathrm{InnAct}(\mathcal{L})$ and $\mathrm{Act}_{pi}(\mathcal{L})$. In section 2, definitions and primary notations used for Lie crossed module and $\mathrm{Act}(\mathcal{L})$ are presented. In section 3, $\mathrm{Act}_z(\mathcal{L})$ is defined and some of its elementary properties are proved. In section 4, we prove the main theorem, which gives a necessary and sufficient condition for the equality of $\mathrm{Act}_{pi}(\mathcal{L})$ and $\mathrm{Act}_z(\mathcal{L})$.

2. Preliminaries on crossed modules

Definition 2.1. A Lie crossed module is a Lie homomorphism $d : L_1 \longrightarrow L_0$ together with an action of L_0 on L_1, denoted as $(l_0, l_1) \mapsto^{l_0} l_1$ for all $l_0 \in L_0$ and $l_1 \in L_1$, such that

(1) $d({}^{l_0}l_1) = [l_0, d(l_1)]$;
(2) ${}^{d(l_1)}l'_1 = [l_1, l'_1]$,

for all $l_0 \in L_0$ and $l_1, l'_1 \in L_1$. The crossed module $\mathcal{L}$ is denoted by $\mathcal{L} : (L_1, L_0, d)$.

The crossed module $\mathcal{L}' : (L'_1, L'_0, d')$ is a subcrossed module of $\mathcal{L} : (L_1, L_0, d)$, and denoted by $\mathcal{L}' \leqslant \mathcal{L}$, if L'_0 and L'_1 are subalgebras of L_0 and L_1, respectively, and d' is the restriction of d on L'_1, and the action of L'_0 on L'_1 is induced from the action of L_0 on L_1.

The subcrossed module $\mathcal{L}' : (L'_1, L'_0, d')$ of $\mathcal{L} : (L_1, L_0, d)$ is an ideal of $\mathcal{L}$, denoted by $\mathcal{L}' \triangleleft \mathcal{L}$, if L'_0 and L'_1 are ideals of L_0 and L_1, respectively, and that we have ${}^{l_0}l'_1 \in L'_1$ and ${}^{l'_0}l_1 \in L'_1$ for all $l_0 \in L_0$, $l'_0 \in L'_0$, $l_1 \in L_1$, and $l'_1 \in L'_1$.

Definition 2.2. Let $\mathcal{L} : (L_1, L_0, d)$ be a Lie crossed module. The center $Z(\mathcal{L})$ of $\mathcal{L}$, that is an ideal of $\mathcal{L}$, is defined as

$$Z(\mathcal{L}) : ({}^{L_0}L_1, \mathrm{st}_{L_0}(L_1) \cap Z(L_0), d_|),$$

where

$${}^{L_0}L_1 = \left\{ l_1 \in L_1 \mid^{l_0} l_1 = 0, \ \forall\ l_0 \in L_0 \right\}$$

and

$$\mathrm{st}_{L_0}(L_1) = \left\{ l_0 \in L_0 \mid^{l_0} l_1 = 0, \ \forall\ l_1 \in L_1 \right\}.$$

and $d_|$ is restriction of d to ${}^{L_0}L_1$.
The crossed module $\mathcal{L}$ is abelian if it coincides with its center, i.e.

$$L_1 =^{L_0} L_1 \quad \text{and} \quad L_0 = \mathrm{st}_{L_0}(L_1) \cap Z(L_0).$$

The derived subcrossed module of $\mathcal{L}$, denoted as $\mathcal{L}^2$, is defined as follows:

$$\mathcal{L}^2 : (D_{L_0}(L_1), L_0^2, d_|),$$

where

$$D_{L_0}(L_1) = \langle {}^{l_0}l_1 \mid l_0 \in L_0, l_1 \in L_1 \rangle.$$

and $d_|$ is restriction of d to ${}^{L_0}L_1$.
A homomorphism between two Lie crossed modules $\mathcal{L} : (L_1, L_0, d)$ and $\mathcal{L}' : (L_1', L_0', d')$ is a pair (f, g) of Lie algebra homomorphisms $f : L_1 \longrightarrow L_1'$ and $g : L_0 \longrightarrow L_0'$ satisfying

(1) $d'f = gd$;
(2) $f({}^{l_0}l_1) =^{g(l_0)} f(l_1)$,

for all $l_0 \in L_0$ and $l_1 \in L_1$.

Definition 2.3. Assume $\mathcal{L} : (L_1, L_0, d)$ is a crossed module. A derivation of $\mathcal{L}$ is a pair $(\psi, \phi) : \mathcal{L} \to \mathcal{L}$ satisfying the following conditions:

(1) $\psi \in \mathrm{Der}(L_1)$,
(2) $\phi \in \mathrm{Der}(L_0)$,
(3) $d\psi = \phi d$,
(4) $\psi({}^{l_0}l_1) =^{l_0} \psi(l_1) +^{\phi(l_0)} (l_1)$,

for all $l_0 \in L_0$ and $l_1 \in L_1$.

The set of all derivations of $\mathcal{L}$ is denoted by $\mathrm{Der}(\mathcal{L})$, which is a Lie algebra with bracket as in the following:

$$[(\psi, \phi), (\psi', \phi')] = ([\psi, \psi'], [\phi, \phi']) = (\psi\psi' - \psi'\psi, \phi\phi' - \phi'\phi).$$

Definition 2.4. Assume $\mathcal{L} : (L_1, L_0, d)$ is a Lie algebra crossed module. The a map $\delta : L_0 \to L_1$ is called crossed derivation if

$$\delta([l_0, l_0']) =^{l_0} \delta(l_0') -^{l_0'} \delta(l_0)$$

for all $l_0, l_0' \in L_0$. The set of all crossed derivations from L_0 to L_1 is denoted by $\mathrm{Der}(L_0, L_1)$, which turns into a Lie algebra via the following bracket:

$$[\delta_1, \delta_2] = \delta_1 d\delta_2 - \delta_2 d\delta_1$$

for all $\delta_1, \delta_2 \in \mathrm{Der}(L_0, L_1)$.

Definition 2.5. To each Lie crossed module $\mathcal{L} : (L_1, L_0, d)$, there corresponds a crossed module $\mathrm{Act}(\mathcal{L}) : (\mathrm{Der}(L_0, L_1), \mathrm{Der}(\mathcal{L}), \Delta)$ such that

$$\hom \Delta \mathrm{Der}(L_0, L_1) \mathrm{Der}(\mathcal{L}) \delta (\delta d, d\delta)$$

and the action of $\mathrm{Der}(\mathcal{L})$ on $\mathrm{Der}(L_0, L_1)$ is defined as

$$^{(\alpha,\beta)}\delta = \alpha\delta - \delta\beta$$

for all $(\alpha, \beta) \in \mathrm{Der}(\mathcal{L})$ and $\delta \in \mathrm{Der}(L_0, L_1)$, and it is called the actor of $\mathcal{L}$ (see Casas and Ladra, [5]).

Proposition 2.6. *There exists a canonical homomorphism of crossed modules as*

$$(\varepsilon, \eta) : \mathcal{L} \longrightarrow \mathrm{Act}(\mathcal{L}),$$

where

$$\hom \varepsilon L_1 \mathrm{Der}(L_0, L_1) l_1 \delta_{l_1} \quad \textit{and} \quad \hom \eta L_0 \mathrm{Der}(\mathcal{L}) l_0 (\alpha_{l_0}, \beta_{l_0}),$$

in which $\delta_{l_1}(l_0) =^{l_0} l_1$*,* $\alpha_{l_0}(l_1) =^{l_0} l_1$*, and* $\beta_{l_0}(l_0') = [l_0, l_0']$ *for all* $l_0 \in L_0$*,* $l_0' \in L_0$*, and* $l_1 \in L_1$*.*

The image of (ε, η) *is an ideal of* $\mathrm{Act}(\mathcal{L})$ *and it is denoted as* $\mathrm{InnAct}(\mathcal{L})$*. We have*

$$\mathrm{InnAct}(\mathcal{L}) : (\varepsilon(L_1), \eta(L_0), \Delta_|).$$

On can easily see that $\ker(\varepsilon, \eta) = Z(\mathcal{L})$*. (See allahyary and saeedi* [1]*)*

Definition 2.7. Let $\mathcal{L}$ be a Lie crossed module. Then the pointwise inner actor of $\mathcal{L}$ is defined as follows:

$$\mathrm{Act}_{pi}(\mathcal{L}) : (\mathrm{Der}_{pi}(L_0, L_1), \mathrm{Der}_{pi}(\mathcal{L}), \Delta_|),$$

where

$$\mathrm{Der}_{pi}(L_0, L_1) = \left\{\delta \in \mathrm{Der}(L_0, L_1) \mid \forall\, l_0 \in L_0,\ \exists\, l_1 \in L_1 : \delta(l_0) =^{l_0} l_1\right\}$$

and

$$\mathrm{Der}_{pi}(\mathcal{L}) = \left\{(\alpha, \beta) \in \mathrm{Der}(\mathcal{L}) \mid \begin{array}{l} \forall\, l_1 \in L_1,\ \exists\, l_0 \in L_0 : \alpha(l_1) =^{l_0} l_1, \\ \forall\, l_0 \in L_0,\ \exists\, l_0' \in L_0 : \beta(l_0) = [l_0', l_0] \end{array}\right\}.$$

One can easily verify that $\mathrm{Act}_{pi}(\mathcal{L})$ is a subcrossed module of $\mathrm{Act}(\mathcal{L})$ and contains $\mathrm{InnAct}(\mathcal{L})$ (see Allahyari and Saeedi [1]).

Definition 2.8. Let $\mathcal{L} : (L_1, L_0, d)$ be a Lie crossed module. Then $\mathrm{ID}^*\mathrm{Act}(\mathcal{L})$ is defined as

$$\mathrm{ID}^*\mathrm{Act}(\mathcal{L}) : (\mathrm{ID}^*(L_0, L_1), \mathrm{ID}^*(\mathcal{L}), \Delta_|),$$

where

$$\mathrm{ID}^*(L_0, L_1) = \left\{\delta \in \mathrm{Der}(L_0, L_1) \mid \begin{array}{l} \delta(x_0) \in D_{L_0}(L_1),\ \forall\, x_0 \in L_0, \\ \delta(x_0) = 0,\ \forall\, x_0 \in \mathrm{st}_{L_0}(L_1) \cap Z(L_0), \end{array}\right\}$$

and

$$\mathrm{ID}^*(\mathcal{L}) = \left\{(\alpha, \beta) \in \mathrm{Der}(\mathcal{L}) \mid \begin{array}{l} \alpha(x_1) \in D_{L_0}(L_1),\ \forall\, x_1 \in L_1, \\ \alpha(x_1) = 0,\ \forall\, x_1 \in^{L_0} L_1, \\ \beta(x_0) \in L_0^2,\ \forall\, x_0 \in L_0, \\ \beta(x_0) = 0,\ \forall\, x_0 \in \mathrm{st}_{L_0}(L_1) \cap Z(L_0) \end{array}\right\}.$$

On can easily show that $\mathrm{ID}^*\mathrm{Act}(\mathcal{L})$ is a subcrossed module of $\mathrm{Act}(\mathcal{L})$ and contains $\mathrm{Act}_{pi}(\mathcal{L})$ (see Allahyari and Saeedi [1]).

3. Center actor of Lie crossed modules

In this section we define subcrossed module of $\mathrm{Act}(\mathcal{L})$ namely $\mathrm{Act}_z(\mathcal{L})$ and we prove some of its elementary properties.

Definition 3.1. Let $\mathcal{L} : (L_1, L_0, d)$ be a Lie crossed module. The $\mathrm{Act}_z(\mathcal{L})$ is defined as follows:

$$\mathrm{Act}_z(\mathcal{L}) : (\mathrm{Der}_z(L_0, L_1), \mathrm{Der}_z(\mathcal{L}), \Delta_|),$$

where

$$\mathrm{Der}_z(L_0, L_1) = \left\{\delta \in \mathrm{Der}(L_0, L_1) \mid \delta(l_0) \in^{L_0} L_1,\ \forall\ l_0 \in L_0\right\}$$

and

$$\mathrm{Der}_z(\mathcal{L}) = \left\{(\alpha, \beta) \in \mathrm{Der}(\mathcal{L}) \mid \begin{array}{l} \alpha(l_1) \in^{L_0} L_1,\ \forall\ l_1 \in L_1, \\ \beta(l_0) \in \mathrm{st}_{L_0}(L_1) \cap Z(L_0),\ \forall\ l_0 \in L_0. \end{array}\right\}$$

Note that $\Delta_|$ is the restriction of Δ to $\mathrm{Der}_z(L_0, L_1)$.

Proposition 3.2. $\mathrm{Act}_z(\mathcal{L})$ *is a subcrossed module of* $\mathrm{Act}(\mathcal{L})$.

Proof. We have to show that

(1) $\mathrm{Der}_z(L_0, L_1) \leqslant \mathrm{Der}(L_0, L_1)$;
(2) $\mathrm{Der}_z(\mathcal{L}) \leqslant \mathrm{Der}(\mathcal{L})$;
(3) $\Delta_{|\mathrm{Der}_z(L_0,L_1)} \subseteq \mathrm{Der}_z(\mathcal{L})$.

(1) Assume δ, δ' are two arbitrary elements of $\mathrm{Der}_z(L_0, L_1)$. Then

$$\delta(x_0) \in^{L_0} L_1 \quad \text{and} \quad \delta'(x_0) \in^{L_0} L_1$$

for all $x_0 \in L_0$. Now since $[\delta, \delta'](x_0) = \delta d\delta'(x_0) - \delta' d\delta(x_0)$, one can easily verify that

$$[\delta, \delta'](x_0) \in^{L_0} L_1$$

for all $x_0 \in \mathcal{L}$. Hence $\mathrm{Der}_z(L_0, L_1) \leqslant \mathrm{Der}(L_0, L_1)$.

(2) Let (α, β) and (α', β') be two elements of $\mathrm{Der}_z(\mathcal{L})$. Then

$$\alpha(x_1) \in^{L_0} L_1 \quad \text{and} \quad \alpha'(x_1) \in^{L_0} L_1,$$
$$\beta(x_0) \in \mathrm{st}_{L_0}(L_1) \cap Z(L_0) \quad \text{and} \quad \beta'(x_0) \in \mathrm{st}_{L_0}(L_1) \cap Z(L_0)$$

for all $x_0 \in L_0$ and $x_1 \in L_1$. Since

$$[(\alpha, \beta), (\alpha', \beta')] = ([\alpha, \alpha'], [\beta, \beta']) = (\alpha\alpha' - \alpha'\alpha, \beta\beta' - \beta'\beta),$$

one can see that

$$(\alpha\alpha' - \alpha'\alpha)(x_1) = \alpha\alpha'(x_1) - \alpha'\alpha(x_1) \in^{L_0} L_1,$$
$$(\beta\beta' - \beta'\beta)(x_0) = \beta\beta'(x_0) - \beta'\beta(x_0) \in \mathrm{st}_{L_0}(L_1) \cap Z(L_0)$$

for all $x_0 \in L_0$ and $x_1 \in L_1$. Therefore $[(\alpha,\beta),(\alpha',\beta')] \in \mathrm{Der}_z(\mathcal{L})$ so that $\mathrm{Der}_z(\mathcal{L}) \leqslant \mathrm{Der}(\mathcal{L})$.

(3) Assume $\delta \in \mathrm{Der}_z(L_0, L_1)$. From the definition of Δ, we have

$$\Delta(\delta) = (\delta d, d\delta).$$

One can easily check that

$$\delta d(x_1) \in^{L_0} L_1,$$
$$d\delta(x_0) \in \mathrm{st}_{L_0}(L_1) \cap Z(L_0)$$

for all $x_0 \in L_0$ and $x_1 \in L_1$. Thus $\Delta(\delta) = (\delta d, d\delta) \in \mathrm{Der}_z(\mathcal{L})$, and so $\Delta_{|\mathrm{Der}_z(L_0,L_1)} \subseteq \mathrm{Der}_z(\mathcal{L})$. Therefore $\mathrm{Act}_z(\mathcal{L}) \leqslant \mathrm{Act}(\mathcal{L})$, and the proof is complete. $\square$

Definition 3.3. Let $\mathcal{L} : (L_1, L_0, d)$ be a Lie crossed module and $\mathcal{M} : (M_1, M_0, d)$ be an ideal of $\mathcal{L}$. Then the centralizer of $\mathcal{M}$ in $\mathcal{L}$, denoted as $\mathcal{C}_{\mathcal{L}}(\mathcal{M})$, is defined as

$$\mathcal{C}_{\mathcal{L}}(\mathcal{M}) : ({}^{M_0}L_1, C_{L_0}(M_0) \cap \mathrm{st}_{L_0}(M_1), d_|),$$

where

$$^{M_0}L_1 = \{x_1 \in L_1 \mid^{x_0} x_1 = 0,\ \forall\ x_0 \in M_0\},$$
$$C_{L_0}(M_0) = \{x_0 \in L_0 \mid [x_0, y_0] = 0,\ \forall\ y_0 \in M_0\},$$
$$\mathrm{st}_{L_0}(M_1) = \{x_0 \in L_0 \mid^{x_0} x_1 = 0,\ \forall\ x_1 \in M_1\}.$$

Let $\mathcal{M} : (M_1, M_0, d_1)$ and $\mathcal{N} : (N_1, N_0, d_1)$ be two ideals of the crossed module $\mathcal{L} : (L_1, L_0, d)$. Then the ideal $\mathcal{M} \cap \mathcal{N}$ of $\mathcal{L}$ is defined as

$$\mathcal{M} \cap \mathcal{N} : (M_1 \cap N_1, M_0 \cap N_0, d_|).$$

Lemma 3.4. *Let $\mathcal{L} : (L_1, L_0, d)$ be a Lie crossed module and $\mathcal{M} : (M_1, M_0, d)$ be an ideal of $\mathcal{L}$. Then $\mathcal{M} \cap \mathcal{C}_{\mathcal{L}}(\mathcal{M}) = Z(\mathcal{M})$.*

Proof. It is obvious. $\square$

Lemma 3.5. *Let $\mathcal{L} : (L_1, L_0, d)$ be a Lie crossed module and* $\mathrm{InnAct}(\mathcal{L}) \leqslant \mathcal{H} \leqslant \mathrm{ID}^*\mathrm{Act}(\mathcal{L})$. *Then*

$$C_{\mathrm{Act}(\mathcal{L})}(\mathcal{H}) = \mathrm{Act}_z(\mathcal{L}).$$

Proof. Assume $\mathcal{H} : (H_1, H_0, \Delta)$. We need to show that

(1) ${}^{H_0}\mathrm{Der}(L_0, L_1) = \mathrm{Der}_z(L_0, L_1)$;

(2) $C_{\mathrm{Der}(\mathcal{L})}(H_0) \cap \mathrm{st}_{\mathrm{Der}(\mathcal{L})}(H_1) = \mathrm{Der}_z(\mathcal{L})$.

(1) Let $\delta \in \mathrm{Der}_z(L_0, L_1)$. Then $\delta(l_0) \in^{L_0} L_1$ for all $l_0 \in L_0$. Now if $(\alpha,\beta) \in H_0$, then we observe that

$$^{(\alpha,\beta)}\delta(l_0) = (\alpha\delta - \delta\beta)(l_0) = \alpha(\delta(l_0)) - \delta(\beta(l_0)) = -\delta(\beta(l_0)).$$

Since $\beta(l_0) \in L_0^2$, there exist $x_0, y_0 \in L_0$ such that $\beta(l_0) = [x_0, y_0]$. Then

$$^{(\alpha,\beta)}\delta(l_0) = \delta([x_0, y_0]) =^{y_0} \delta(x_0) -^{x_0} \delta(y_0) = 0.$$

Thus $\delta \in^{H_0} \mathrm{Der}(L_0, L_1)$ and consequently $\mathrm{Der}_z(L_0, L_1) \subseteq^{H_0} \mathrm{Der}(L_0, L_1)$.

Conversely, assume $\delta \in^{H_0} \mathrm{Der}(L_0, L_1)$. Then ${}^{(\alpha,\beta)}\delta(x_0) = 0$ for all $x_0 \in L_0$ and $(\alpha, \beta) \in H_0$. Now since $\mathcal{H}$ contains $\mathrm{InnAct}(\mathcal{L})$, we can write $(\alpha, \beta) = (\alpha_{l_0}, \beta_{l_0})$ for some $l_0 \in L_0$. Then

$$\begin{aligned}
{}^{(\alpha_{l_0},\beta_{l_0})}\delta(x_0) = 0 &\Rightarrow (\alpha_{l_0}\delta - \delta\beta_{l_0})(x_0) = 0,\\
&\Rightarrow \alpha_{l_0}(\delta(x_0)) - \delta(\beta_{l_0}(x_0)) = 0,\\
&\Rightarrow^{l_0} \delta(x_0) - \delta([l_0, x_0]) = 0,\\
&\Rightarrow^{l_0} \delta(x_0) -^{l_0} \delta(x_0) +^{x_0} \delta(l_0) = 0,\\
&\Rightarrow^{x_0} \delta(l_0) = 0
\end{aligned}$$

for all $x_0, l_0 \in L_0$. Therefore $\delta \in \mathrm{Der}_z(L_0, L_1)$ so that ${}^{H_0}\mathrm{Der}(L_0, L_1) \subseteq \mathrm{Der}_z(L_0, L_1)$.

(2) Let $(\alpha, \beta) \in \mathrm{Der}_z(\mathcal{L})$. Then

$$\alpha(l_1) \in^{L_0} L_1 \quad \text{and} \quad \beta(l_0) \in \mathrm{st}_{L_0}(L_1) \cap Z(L_0)$$

for all $l_0 \in L_0$ and $l_1 \in L_1$. Now assume $(\alpha', \beta') \in H_0$ is any element. Then

$$\begin{aligned}
&[(\alpha, \beta), (\alpha', \beta')] = ([\alpha, \alpha'], [\beta, \beta']),\\
&\qquad [\alpha, \alpha'](l_1) = (\alpha\alpha' - \alpha'\alpha)(l_1) = \alpha(\alpha'(l_1)) - \alpha'(\alpha(l_1)) = \alpha(\alpha'(l_1)).
\end{aligned}$$

Since $\alpha'(l_1) \in D_{L_0}(L_1)$, there exist $x_0 \in L_0$ and $x_1 \in L_1$ such that

$$[\alpha, \alpha'](l_1) = \alpha(\alpha'(l_1)) = \alpha({}^{x_0}x_1) =^{x_0} \alpha(x_1) +^{\beta(x_0)} x_1 = 0.$$

Similarly, we can show that

$$\begin{aligned}
[\beta, \beta'](l_0) &= (\beta\beta' - \beta'\beta)(l_0) = \beta(\beta'(l_0)) - \beta'(\beta(l_0))\\
&= \beta([x_0, y_0]) = [\beta(x_0), y_0] + [x_0, \beta(y_0)] = 0
\end{aligned}$$

for some $x_0, y_0 \in L_0$. Hence, we conclude that $[(\alpha, \beta), (\alpha', \beta')] = 0$ and so

$$\mathrm{Der}_z(\mathcal{L}) \subseteq C_{\mathrm{Der}(\mathcal{L})}(H_0). \tag{3.1}$$

Now suppose that $\delta \in H_1$. Then

$${}^{(\alpha,\beta)}\delta(x_0) = \alpha(\delta(x_0)) - \delta(\beta(x_0)) = \alpha(\delta(x_0)).$$

Since $H_1 \subseteq \mathrm{ID}^*(L_0, L_1)$, there exist elements $y_0 \in L_0$ and $y_1 \in L_1$ such that $\delta(x_0) =^{y_0} y_1$. Then we have

$${}^{(\alpha,\beta)}\delta(x_0) = \alpha(\delta(x_0)) = \alpha({}^{y_0}y_1) =^{y_0} \alpha(y_1) +^{\beta(y_0)} y_1 = 0.$$

Thus

$$\mathrm{Der}_z(\mathcal{L}) \subseteq \mathrm{st}_{\mathrm{Der}(\mathcal{L})}(H_1). \tag{3.2}$$

From (3.1) and (3.2) it follows that

$$\mathrm{Der}_z(\mathcal{L}) \subseteq C_{\mathrm{Der}(\mathcal{L})}(H_0) \cap \mathrm{st}_{\mathrm{Der}(\mathcal{L})}(H_1).$$

Conversely, assume $(\alpha,\beta) \in C_{\mathrm{Der}(\mathcal{L})}(H_0) \cap \mathrm{st}_{\mathrm{Der}(\mathcal{L})}(H_1)$. Then

$$^{(\alpha,\beta)}\delta = 0 \quad \text{and} \quad [(\alpha,\beta),(\alpha',\beta')] = 0$$

for all $\delta \in H_1$ and $(\alpha',\beta') \in H_0$. Now since $\mathrm{InnAct}(\mathcal{L}) \subseteq \mathcal{H}$, we can write $\delta = \delta_{l_1}$ for some $l_1 \in L_1$. Then

$$\begin{aligned} ^{(\alpha,\beta)}\delta_{l_1}(x_0) = 0 &\Rightarrow \alpha(\delta_{l_1}(x_0)) - \delta_{l_1}(\beta(x_0)) = 0, \\ &\Rightarrow \alpha(^{x_0}l_1) -^{\beta(x_0)} l_1 = 0, \\ &\Rightarrow^{x_0} \alpha(l_1) +^{\beta(x_0)} l_1 -^{\beta(x_0)} l_1 = 0, \\ &\Rightarrow^{x_0} \alpha(l_1) = 0 \end{aligned}$$

for all $x_0 \in L_0$ and $l_1 \in L_1$. This shows that

$$\alpha(l_1) \in^{L_0} L_1 \tag{3.3}$$

for all $l_1 \in L_1$.

On the other hand, for all $l_0 \in L_0$, we have

$$\begin{aligned} [(\alpha,\beta),(\alpha_{l_0},\beta_{l_0})] = 0 &\Rightarrow [\alpha,\alpha_{l_0}](x_1) = 0, \\ &\Rightarrow \alpha(\alpha_{l_0}(x_1) - \alpha_{l_0}(\alpha(x_1)) = 0, \\ &\Rightarrow \alpha(^{l_0}x_1) -^{l_0} \alpha(x_1) =^{l_0} \alpha(x_1) +^{\beta(l_0)} x_1 -^{l_0} \alpha(x_1) =^{\beta(l_0)} x_1 = 0 \end{aligned}$$

for all $x_1 \in L_1$, which implies that $\beta(l_0) \in \mathrm{st}_{L_0}(L_1)$. Also

$$\begin{aligned} [\beta,\beta_{l_0}] = 0 &\Rightarrow [\beta,\beta_{l_0}](x_0) = 0, \\ &\Rightarrow \beta(\beta_{l_0}(x_0)) - \beta_{l_0}(\beta(x_0)) = 0, \\ &\Rightarrow \beta([l_0,x_0]) - [l_0,\beta(x_0)] = 0, \\ &\Rightarrow [\beta(l_0),x_0] + [l_0,\beta(x_0)] - [l_0,\beta(x_0)] = [\beta(l_0),x_0] = 0 \end{aligned}$$

for all $x_0 \in L_0$, which implies that $\beta(l_0) \in Z(L_0)$. Hence

$$\beta(l_0) \in \mathrm{st}_{L_0}(L_1) \cap Z(L_0). \tag{3.4}$$

From (3.3) and (3.4), we get $(\alpha,\beta) \in \mathrm{Der}_z(\mathcal{L})$. $\square$

Corollary 3.6. *Let $\mathcal{L} : (L_1, L_0, d)$ be a Lie crossed module and* $\mathrm{InnAct}(\mathcal{L}) \leqslant \mathcal{H} \leqslant \mathrm{ID}^*\mathrm{Act}(\mathcal{L})$. *Then*

$$\mathcal{H} \cap \mathrm{Act}_z(\mathcal{L}) = Z(\mathcal{H}).$$

Proof. The result follows by Lemmas 3.4 and 3.5. $\square$

4. Main theorem

We are now ready to prove our main theorem, which gives a necessary and sufficient condition for $\mathrm{Act}_{pi}(\mathcal{L})$ and $\mathrm{Act}_z(\mathcal{L})$ to be equal. To this end, we need some preliminary lemmas.

Lemma 4.1. *Let $\mathcal{L} : (L_1, L_0, d)$ be a Lie crossed module and* $\mathrm{Act}_{pi}(\mathcal{L}) = \mathrm{Act}_z(\mathcal{L})$. *Then* $\mathrm{InnAct}(\mathcal{L})$ *is abelian.*

Proof. The result follows from the fact that $\mathrm{InnAct}(\mathcal{L}) \subseteq \mathrm{Act}_{pi}(\mathcal{L})$ and $\mathrm{Act}_{pi}(\mathcal{L}) = \mathrm{Act}_z(\mathcal{L})$. □

Definition 4.2. Let $\mathcal{L} : (L_1, L_0, d_{\mathcal{L}})$ and $\mathcal{M} : (M_1, M_0, d_{\mathcal{M}})$ be two Lie crossed modules. The set of all linear transformations from $\mathcal{L}$ to $\mathcal{M}$ is denoted by $T(\mathcal{L}, \mathcal{M})$ and it is defined as

$$T(\mathcal{L}, \mathcal{M}) : (T(L_0, M_1), (T(L_1, M_1), T(L_0, M_0))),$$

where for example $T(L_0, M_1)$ is the vector space of linear transformations from L_0 to M_1.

Definition 4.3. Let $\mathcal{L} : (L_1, L_0, d)$ be a Lie crossed module. The dimension of $\mathcal{L}$ is defined as

$$\dim \mathcal{L} = (\dim L_1, \dim L_0).$$

Lemma 4.4. *Let $\mathcal{L} : (L_1, L_0, d)$ be a Lie crossed module. Then we have the following vector space isomorphisms:*

(1) $\mathrm{Der}_z(L_0, L_1) \cong T(L_0/L_0^2, {}^{L_0} L_1)$;
(2) $\mathrm{Der}_z(\mathcal{L}) \cong (T(L_1/D_{L_0}(L_1), {}^{L_0} L_1), T(L_0/L_0^2, \mathrm{st}_{L_0}(L_1) \cap Z(L_0))$.

Proof. (1) For each $\delta \in \mathrm{Der}_z(L_0, L_1)$, we can define the map $\psi_\delta : L_0/L_0^2 \longrightarrow^{L_0} L_1$ by $\psi_\delta(l_0 + \mathcal{L}_0^2) = \delta(l_0)$ for all $l_0 \in L_0$. Clearly, ψ_δ is well-defined. Also, it is easy to see that the map

$$\psi : \mathrm{Der}_z(L_0, L_1) \longrightarrow T\left(\frac{L_0}{L_0^2}, {}^{L_0} L_1\right)$$

define by $\psi(\delta) = \psi_\delta$ is an one-to-one and onto linear transformation. Thus

$$\mathrm{Der}_z(L_0, L_1) \cong T\left(\frac{L_0}{L_0^2}, {}^{L_0} L_1\right).$$

(2) For each $(\alpha, \beta) \in \mathrm{Der}_z(\mathcal{L})$, we may define the maps $\phi_\alpha : L_1/D_{L_0}(L_1) \longrightarrow^{L_0} L_1$ and $\phi_\beta : L_0/L_0^2 \longrightarrow \mathrm{st}_{L_0}(L_1) \cap Z(L_0)$ by $\phi_\alpha(l_1 + D_{L_0}(L_1)) = \alpha(l_1)$ and $\psi_\beta(l_0 + L_0^2) = \beta(l_0)$, respectively. One can easily check that, the maps ϕ_α and ϕ_β are well-defined linear transformations. Now, it is easy to show that the map

$$\hom \phi \mathrm{Der}_z(\mathcal{L}) \left(T\left(\frac{L_1}{D_{L_0}(L_1)}, {}^{L_0} L_1\right), T\left(\frac{L_0}{L_0^2}, \mathrm{st}_{L_0}(L_1) \cap Z(L_0)\right)\right) (\alpha, \beta)(\phi_\alpha, \phi_\beta)$$

is a one-to-one and onto linear transformation. Thus

$$\mathrm{Der}_z(\mathcal{L}) \cong \left(T\left(\frac{L_1}{D_{L_0}(L_1)}, {}^{L_0} L_1\right), T\left(\frac{L_0}{L_0^2}, \mathrm{st}_{L_0}(L_1) \cap Z(L_0)\right)\right),$$

as required □

Corollary 4.5. *We have*

$$\dim \mathrm{Act}_z(\mathcal{L}) = \left(\dim T\left(\frac{L_0}{L_0^2}, {}^{L_0}L_1\right),\right.$$
$$\left.\dim\left(T\left(\frac{L_1}{D_{L_0}(L_1)}, {}^{L_0}L_1\right), T\left(\frac{L_0}{L_0^2}, \mathrm{st}_{L_0}(L_1)\cap Z(L_0)\right)\right)\right).$$

Theorem 4.6. *Let $\mathcal{L} : (L_1, L_0, d)$ be a nonabelian Lie crossed module of finite dimension with $Z(\mathcal{L}) \neq 0$. Then $\mathrm{Act}_z(\mathcal{L}) = \mathrm{Act}_{pi}(\mathcal{L})$ if and only if $Z(\mathcal{L}) = \mathcal{L}^2$ and*

$$\dim \mathrm{Act}_{pi}(\mathcal{L}) = \left(\dim T\left(\frac{L_0}{\mathrm{st}_{L_0}(L_1)\cap Z(L_0)}, D_{L_0}(L_1)\right),\right.$$
$$\left.\dim\left(T\left(\frac{L_1}{{}^{L_0}L_1}, D_{L_0}(L_1)\right), T\left(\frac{L_0}{\mathrm{st}_{L_0}(L_1)\cap Z(L_0)}, L_0^2\right)\right)\right).$$

Proof. First assume that $\mathrm{Act}_z(\mathcal{L}) = \mathrm{Act}_{pi}(\mathcal{L})$. Since $\mathrm{InnAct}(\mathcal{L}) \subseteq \mathrm{Act}_{pi}(\mathcal{L})$, we get $\mathcal{L}^2 \subseteq Z(\mathcal{L})$. For each $\delta \in \mathrm{Der}_{pi}(L_0, L_1)$, we define the well-defined linear transformation $\psi_\delta : L_0/\mathrm{st}_{L_0}(L_1)\cap Z(L_0) \longrightarrow D_{L_0}(L_1)$ by $\psi_\delta(x_0 + \mathrm{st}_{L_0}(L_1)\cap Z(L_0)) = \delta(x_0)$. One can easily check that the map

$$\psi : \mathrm{Der}_{pi}(L_0, L_1) \longrightarrow T\left(\frac{L_0}{\mathrm{st}_{L_0}(L_1)\cap Z(L_0)}, D_{L_0}(L_1)\right)$$

define by $\psi(\delta) = \psi_\delta$ is a one-to-one and onto linear transformation. Thus

$$\dim \mathrm{Der}_{pi}(L_0, L_1) = \dim T\left(\frac{L_0}{\mathrm{st}_{L_0}(L_1)\cap Z(L_0)}, D_{L_0}(L_1)\right). \tag{4.1}$$

Also, for each $(\alpha, \beta) \in \mathrm{Der}_{pi}(\mathcal{L})$, the maps $\phi_\alpha : L_1/{}^{L_0}L_1 \longrightarrow D_{L_0}(L_1)$ and $\phi_\beta : L_0/\mathrm{st}_{L_0}(L_1)\cap Z(L_0) \longrightarrow L_0^2$ defined by $\phi_\alpha(x_1 + {}^{L_0}L_1) = \alpha(x_1)$ and $\phi_\beta(x_0 + \mathrm{st}_{L_0}(L_1)\cap Z(L_0)) = \beta(x_0)$, respectively, are well-defined linear transformations. One can easily see that

$$\phi : \mathrm{Der}_{pi}(\mathcal{L}) \longrightarrow \left(T\left(\frac{L_1}{{}^{L_0}L_1}, D_{L_0}(L_1)\right), T\left(\frac{L_0}{\mathrm{st}_{L_0}(L_1)\cap Z(L_0)}, L_0^2\right)\right)$$

given by $\phi(\alpha, \beta) = (\phi_\alpha, \phi_\beta)$ is a one-to-one and onto linear transformation. Thus

$$\dim \mathrm{Der}_{pi}(\mathcal{L}) = \dim\left(T\left(\frac{L_1}{{}^{L_0}L_1}, D_{L_0}(L_1)\right), T\left(\frac{L_0}{\mathrm{st}_{L_0}(L_1)\cap Z(L_0)}, L_0^2\right)\right). \tag{4.2}$$

From (4.1) and (4.2), it follows that

$$\dim \mathrm{Act}_{pi}(\mathcal{L}) = \left(\dim T\left(\frac{L_0}{\mathrm{st}_{L_0}(L_1)\cap Z(L_0)}, D_{L_0}(L_1)\right),\right.$$
$$\left.\dim\left(T\left(\frac{L_1}{{}^{L_0}L_1}, D_{L_0}(L_1)\right), T\left(\frac{L_0}{\mathrm{st}_{L_0}(L_1)\cap Z(L_0)}, L_0^2\right)\right)\right).$$

Suppose on the contrary that $\mathcal{L}^2 \subset Z(\mathcal{L})$. Then

$$\dim T\left(\frac{\mathcal{L}}{Z(\mathcal{L})}, \mathcal{L}^2\right) < \dim T\left(\frac{\mathcal{L}}{\mathcal{L}^2}, Z(\mathcal{L})\right),$$

which contradicts the equality of $\mathrm{Act}_{pi}(\mathcal{L})$ and $\mathrm{Act}_z(\mathcal{L})$. Therefore $\mathcal{L}^2 = Z(\mathcal{L})$.

Conversely, assume that $\mathcal{L}^2 = Z(\mathcal{L})$ and

$$\begin{aligned}\dim \mathrm{Act}_{pi}(\mathcal{L}) = \Bigg(&\dim T\left(\frac{L_0}{\mathrm{st}_{L_0}(L_1) \cap Z(L_0)}, D_{L_0}(L_1)\right),\\ &\dim\left(T\left(\frac{L_1}{{}^{L_0}L_1}, D_{L_0}(L_1)\right), T\left(\frac{L_0}{\mathrm{st}_{L_0}(L_1) \cap Z(L_0)}, L_0^2\right)\right)\Bigg).\end{aligned}$$

Since $\mathcal{L}^2 \subseteq Z(\mathcal{L})$, we have

$$\mathrm{Act}_{pi}(\mathcal{L}) \leqslant \mathrm{Act}_z(\mathcal{L}). \tag{4.3}$$

On the other hand, we have

$$\begin{aligned}\dim \mathrm{Der}_z(L_0, L_1) &= \dim T\left(\frac{L_0}{L_0^2}, {}^{L_0} L_1\right)\\ &= \dim\left(\frac{L_0}{\mathrm{st}_{L_0}(L_1) \cap Z(L_0)}, D_{L_0}(L_1)\right)\\ &= \dim \mathrm{Der}_{pi}(L_0, L_1)\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}\dim \mathrm{Der}_z(\mathcal{L}) &= \dim T\left(\frac{L_1}{{}^{L_0}L_1}, D_{L_0}(L_1)\right), T\left(\frac{L_0}{L_0^2}, \mathrm{st}_{L_0}(L_1) \cap Z(L_0)\right)\\ &= \dim\left(T\left(\frac{L_1}{{}^{L_0}L_1}, D_{L_0}(L_1)\right), T\left(\frac{L_0}{\mathrm{st}_{L_0}(L_1) \cap Z(L_0)}, L_0^2\right)\right)\\ &= \dim \mathrm{Der}_{pi}(\mathcal{L})\end{aligned} \tag{4.5}$$

From (4.4) and (4.5), we conclude that $\dim \mathrm{Act}_z(\mathcal{L}) = \dim \mathrm{Act}_{pi}(\mathcal{L})$. Since $\mathrm{Act}_{pi}(\mathcal{L}) \leqslant \mathrm{Act}_z(\mathcal{L})$ by (4.3), it follows that $\mathrm{Act}_z(\mathcal{L}) = \mathrm{Act}_{pi}(\mathcal{L})$. The proof is completed. $\square$

Acknowledgments

We thank the two referees for careful readings of the manuscript and for a number of constructive corrections and suggestions.

References

1. A. Allahyari, F. Saeedi, Some Properties of Isoclinism of Lie Algebra Crossed Module, *Asian–Eur. J. Math.*, **13**(3), (2020), 2050055 (14 pages).
2. A. Allahyari, F. Saeedi, On Nilpotency of Outer Pointwise Inner Actor of the Lie Algebra Crossed Modules, *Journal of Mathematical Extension*, **14**(1), (2020), 19–40.
3. J. M. Casas, *Invariants de Modulos Cruzados en Algebras de Lie*, Ph.D. Thesis, Universty of Santiago, 1991.

4. J. M. Casas, M. Ladra, Perfect Crossed Modules in Lie Algebras, *Comm. Algebra*, **23**, (1995), 1625–1644.
5. J. M. Casas, M. Ladra, The Actor of a Crossed Module in Lie Algebra, *Comm. Algebra*, **26**, (1998), 2065–2089.
6. J. M.Casas, R. Fernandez-Casado, X. Garcia-Martinez, E. Khmaladze, Actor of a Crossed Module of Leibniz Algebras, *Theory Appl. Categ.*, **33**, (2018), 23–42.
7. B. Edalatzadeh, Capability of Crossed Modules of Lie Algebras, *Comm. Algebra*, **42**, (2014), 3366–3380.
8. R. Lavendhomme, J. R. Roisin, Cohomologie Non Abelienne de Structures Algebriques, *J. Algebra*, **67**, (1980), 385–414.
9. K. Norrie, Actions and Automorphisms of Crossed Modules, *Bull. Soc. Math. France*, **118**(2), (1990), 129–146.
10. S. Sheikh-Mohseni, F. Saeedi, M. Badrkhani-Asl, On Special Subalgebras of Derivations of Lie Algebras, *Asian-European J. Math.*, **8**(2), (2015), 1550032 (12 pages).
11. J. H. C. Whitehead, Combinatorial Homotopy II, *Bull. Amer. Math. Soc.*, **55**, (1949), 453–496.