

## 2-Irreducible and Strongly 2-Irreducible Submodules of a Module

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ABSTRACT. Let  $R$  be a commutative ring with identity and  $M$  be an  $R$ -module. In this paper, we will introduce the concept of 2-irreducible (resp., strongly 2-irreducible) submodules of  $M$  as a generalization of irreducible (resp., strongly irreducible) submodules of  $M$  and investigated some properties of these classes of modules.

**Keywords:** Irreducible ideal, Strongly 2-irreducible ideal, 2-irreducible submodule, Strongly 2-irreducible submodule.

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### 1. INTRODUCTION

Throughout this paper,  $R$  will denote a commutative ring with identity and  $\mathbb{Z}$  will denote the ring of integers.

An ideal  $I$  of  $R$  is said to be *irreducible* if  $I = J_1 \cap J_2$  for ideals  $J_1$  and  $J_2$  of  $R$  implies that either  $I = J_1$  or  $I = J_2$ . A proper ideal  $I$  of  $R$  is said to be *strongly irreducible* if for ideals  $J_1, J_2$  of  $R$ ,  $J_1 \cap J_2 \subseteq I$  implies that  $J_1 \subseteq I$  or  $J_2 \subseteq I$  [12]. An ideal  $I$  of  $R$  is said to be *2-irreducible* if whenever  $I = J_1 \cap J_2 \cap J_3$  for

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ideals  $J_1, J_2$  and  $J_3$  of  $R$ , then either  $I = J_1 \cap J_2$  or  $I = J_1 \cap J_3$  or  $I = J_2 \cap J_3$ . Clearly, any irreducible ideal is a 2-irreducible ideal [21].

A proper submodule  $N$  of an  $R$ -module  $M$  is said to be *irreducible* (resp., *strongly irreducible*) if for submodules  $H_1$  and  $H_2$  of  $M$ ,  $N = H_1 \cap H_2$  (resp.,  $H_1 \cap H_2 \subseteq N$ ) implies that  $N = H_1$  or  $N = H_2$ . (resp.,  $H_1 \subseteq N$  or  $H_2 \subseteq N$ ).

The main purpose of this paper is to introduce the concept of 2-irreducible and strongly 2-irreducible submodules of an  $R$ -module  $M$  as a generalization of irreducible and strongly irreducible submodules of  $M$  and obtain some related results.

A submodule  $N$  of an  $R$ -module  $M$  is said to be a *2-irreducible submodule* if whenever  $N = H_1 \cap H_2 \cap H_3$  for submodules  $H_1, H_2$  and  $H_3$  of  $M$ , then either  $N = H_1 \cap H_2$  or  $N = H_2 \cap H_3$  or  $N = H_1 \cap H_3$  (Definition 2.1).

A proper submodule  $N$  of an  $R$ -module  $M$  is said to be a *strongly 2-irreducible submodule* if whenever  $H_1 \cap H_2 \cap H_3 \subseteq N$  for submodules  $H_1, H_2$  and  $H_3$  of  $M$ , then either  $H_1 \cap H_2 \subseteq N$  or  $H_2 \cap H_3 \subseteq N$  or  $H_1 \cap H_3 \subseteq N$  (Definition 2.6).

In Section 2 of this paper, for an  $R$ -module  $M$ , among other results, we prove that if  $M$  is a Noetherian  $R$ -module and  $N$  is a 2-irreducible submodule of  $M$ , then either  $N$  is irreducible or  $N$  is an intersection of exactly two irreducible submodules of  $M$  (Theorem 2.22). In Theorem 2.9, we provide a characterization for strongly 2-irreducible submodules of  $M$ . Also, it is shown that if  $M$  is a strong comultiplication  $R$ -module, then every non-zero proper submodule of  $R$  is a strongly sum 2-irreducible  $R$ -module if and only if every non-zero proper submodule of  $M$  is a strongly 2-irreducible submodule of  $M$  (Theorem 2.11). Further, it is proved that if  $N$  is a submodule of a finitely generated multiplication  $R$ -module  $M$ , then  $N$  is a strongly 2-irreducible submodule of  $M$  if and only if  $(N :_R M)$  is a strongly 2-irreducible ideal of  $R$  (Theorem 2.12). In Theorem 2.19 and 2.21, we provide some useful characterizations for strongly 2-irreducible submodules of some special classes of modules. Example 2.14 shows that the concepts of strongly irreducible submodules and strongly 2-irreducible submodules are different in general. Finally, let  $R = R_1 \times R_2 \times \cdots \times R_n$  ( $2 \leq n < \infty$ ) be a decomposable ring and  $M = M_1 \times M_2 \cdots \times M_n$  be an  $R$ -module, where for every  $1 \leq i \leq n$ ,  $M_i$  is an  $R_i$ -module, respectively, it is proved that a proper submodule  $N$  of  $M$  is a strongly 2-irreducible submodule of  $M$  if and only if either  $N = \times_{i=1}^n N_i$  such that for some  $k \in \{1, 2, \dots, n\}$ ,  $N_k$  is a strongly 2-irreducible submodule of  $M_k$ , and  $N_i = M_i$  for every  $i \in \{1, 2, \dots, n\} \setminus \{k\}$  or  $N = \times_{i=1}^n N_i$  such that for some  $k, m \in \{1, 2, \dots, n\}$ ,  $N_k$  is a strongly irreducible submodule of  $M_k$ ,  $N_m$  is a strongly irreducible submodule of  $M_m$ , and  $N_i = M_i$  for every  $i \in \{1, 2, \dots, n\} \setminus \{k, m\}$  (Theorem 2.28).

## 2. MAIN RESULTS

**Definition 2.1.** We say that a submodule  $N$  of an  $R$ -module  $M$  is a *2-irreducible submodule* if whenever  $N = H_1 \cap H_2 \cap H_3$  for submodules  $H_1, H_2$  and  $H_3$  of  $M$ , then either  $N = H_1 \cap H_2$  or  $N = H_2 \cap H_3$  or  $N = H_1 \cap H_3$ .

EXAMPLE 2.2. Let  $R = K[X, Y]$  be a polynomial ring in variables  $X$  and  $Y$  over a field  $K$ . Let  $I$  be the ideal  $\langle X^2, XY \rangle$ . Then  $\langle X^2, XY \rangle = \langle X \rangle \cap \langle X^2, Y \rangle$  implies that  $I$  is not an irreducible ideal of  $R$ . But since  $\langle X \rangle \cap \langle X^2, Y \rangle$  is a primary decomposition for  $I$ , one can see that  $I$  is a 2-irreducible ideal of  $R$  by using [17, 9.31].

EXAMPLE 2.3. Let  $R = K[X, Y]$  be a polynomial ring in variables  $X$  and  $Y$  over a field  $K$  and let  $I = \langle X \rangle \cap \langle Y \rangle$ . Then  $I$  is not an irreducible ideal of  $R$ . But since  $\langle X \rangle$  and  $\langle Y \rangle$  are prime and so strongly irreducible ideals of  $R$ , we have  $I$  is a 2-irreducible ideal of  $R$  by [21, Proposition 3].

**Theorem 2.4.** *Let  $M$  be a Noetherian  $R$ -module. If  $N$  is a 2-irreducible submodule of  $M$ , then either  $N$  is irreducible or  $N$  is an intersection of exactly two irreducible submodules of  $M$ .*

*Proof.* Let  $N$  be a 2-irreducible submodule of  $M$ . By [17, Exercise 9.31],  $N$  can be written as a finite irredundant irreducible decomposition  $N = N_1 \cap N_2 \cap \dots \cap N_k$ . We show that either  $k = 1$  or  $k = 2$ . If  $k > 3$ , then since  $N$  is 2-irreducible,  $N = N_i \cap N_j$  for some  $1 \leq i, j \leq k$ , say  $i = 1$  and  $j = 2$ . Therefore  $N_1 \cap N_2 \subseteq N_3$ , which is a contradiction.  $\square$

**Corollary 2.5.** *Let  $M$  be a Noetherian multiplication  $R$ -module. If  $N$  is a 2-irreducible submodule of  $M$ , then  $N$  is a 2-absorbing primary submodule of  $M$ .*

*Proof.* Let  $N$  be a 2-irreducible submodule of  $M$ . By the fact that every irreducible submodule of a Noetherian  $R$ -module is primary and regarding Theorem 2.22, we have either  $N$  is a primary submodule or is a sum of two primary submodules. It is clear that every primary submodule is 2-absorbing primary, also the sum of two primary submodules is a 2-absorbing primary submodule, by [15, Theorem 2.20].  $\square$

**Definition 2.6.** We say that a proper submodule  $N$  of an  $R$ -module  $M$  is a *strongly 2-irreducible submodule* if whenever  $H_1 \cap H_2 \cap H_3 \subseteq N$  for submodules  $H_1, H_2$  and  $H_3$  of  $M$ , then either  $H_1 \cap H_2 \subseteq N$  or  $H_2 \cap H_3 \subseteq N$  or  $H_1 \cap H_3 \subseteq N$ .

EXAMPLE 2.7. [21, Corollary 2] Consider the  $\mathbb{Z}$ -module  $\mathbb{Z}$ . Then  $n\mathbb{Z}$  is a strongly 2-irreducible submodule of  $\mathbb{Z}$  if  $n = 0, p^t$  or  $p^r q^s$ , where  $p, q$  are prime integers and  $t, r, s$  are natural numbers.

**Proposition 2.8.** *The strongly 2-irreducible submodules of a distributive  $R$ -module are precisely the 2-irreducible submodules.*

*Proof.* This is straightforward.  $\square$

**Theorem 2.9.** *Let  $N$  be a proper submodule of an  $R$ -module  $M$ . Then the following conditions are equivalent:*

- (a)  $N$  is a strongly 2-irreducible submodule;
- (b) For all elements  $x, y, z$  of  $M$ , we have  $(Rx + Ry) \cap (Rx + Rz) \cap (Ry + Rz) \subseteq N$  implies that either  $(Rx + Ry) \cap (Rx + Rz) \subseteq N$  or  $(Rx + Ry) \cap (Ry + Rz) \subseteq N$  or  $(Rx + Rz) \cap (Ry + Rz) \subseteq N$ .

*Proof.* (a)  $\Rightarrow$  (b) This is clear.

(b)  $\Rightarrow$  (a) Let  $H_1 \cap H_2 \cap H_3 \subseteq N$  for submodules  $H_1, H_2$  and  $H_3$  of  $M$ . If  $H_1 \cap H_2 \not\subseteq N$ ,  $H_1 \cap H_3 \not\subseteq N$ , and  $H_2 \cap H_3 \not\subseteq N$ , then there exist elements  $x, y, z$  of  $M$  such that  $x \in H_2 \cap H_3$ ,  $y \in H_1 \cap H_3$ , and  $z \in H_1 \cap H_2$  but  $x \notin N$ ,  $y \notin N$ , and  $z \notin N$ . Therefore,

$$(Ry + Rz) \cap (Rx + Rz) \cap (Rx + Ry) \subseteq H_1 \cap H_2 \cap H_3 \subseteq N.$$

Hence by the part (a), either  $(Ry + Rz) \cap (Rx + Rz) \subseteq N$  or  $(Ry + Rz) \cap (Rx + Ry) \subseteq N$  or  $(Rx + Rz) \cap (Rx + Ry) \subseteq N$ . Thus either  $z \in N$  or  $y \in N$  or  $x \in N$ . This contradiction completes the proof.  $\square$

Recall that a *waist submodule* of an  $R$ -module  $M$  is a submodule that is comparable to any other submodules of  $M$ .

**Proposition 2.10.** *Let  $N$  be a proper submodule of an  $R$ -module  $M$ . Then we have the following.*

- (a) If  $N$  is a strongly 2-irreducible submodule of  $M$ , then it is also a 2-irreducible submodule of  $M$ .
- (b) If  $N$  is a strongly 2-irreducible submodule of  $M$ , then  $N$  is a strongly 2-irreducible submodule of  $T$  and  $N/K$  is a strongly 2-irreducible submodule of  $M/K$  for any  $K \subseteq N \subseteq T$ .
- (c) If for all elements  $x, y, z$  of  $M$  we have  $Rx \cap Ry \cap Rz \subseteq N$  implies that either  $Rx \cap Ry \subseteq N$  or  $Rx \cap Rz \subseteq N$  or  $Ry \cap Rz \subseteq N$ , then  $N$  is a strongly 2-irreducible submodule of  $M$ .
- (d) If  $N$  is a waist submodule of  $M$ , then  $N$  is strongly 2-irreducible submodule of  $M$  if and only if  $N$  is 2-irreducible module.
- (e) If  $N$  satisfies  $(N + T) \cap (N + K) = N + (T \cap K)$ , whenever  $T \cap K \subseteq N$ , then  $N$  is strongly 2-irreducible submodule of  $M$  if and only if  $N$  is a 2-irreducible module.

*Proof.* (a) Let  $N$  be a strongly 2-irreducible submodule of  $M$  and let  $N = H_1 \cap H_2 \cap H_3$  for submodules  $H_1, H_2$  and  $H_3$  of  $M$ . Then by assumption, either  $H_1 \cap H_2 \subseteq N$  or  $H_1 \cap H_3 \subseteq N$  or  $H_2 \cap H_3 \subseteq N$ . Now the result follows from the fact that the reverse of inclusions are clear.

The parts (b), (d), and (e) are straightforward.

(c) Let  $H_1 \cap H_2 \cap H_3 \subseteq N$  for submodules  $H_1, H_2$  and  $H_3$  of  $M$ . If  $H_1 \cap H_2 \not\subseteq N$ ,  $H_1 \cap H_3 \not\subseteq N$ , and  $H_2 \cap H_3 \not\subseteq N$ , then there exist elements  $x, y, z$  of  $M$  such that  $x \in H_2 \cap H_3$ ,  $y \in H_1 \cap H_3$ , and  $z \in H_1 \cap H_2$  but  $x \notin N$ ,  $y \notin N$ , and  $z \notin N$ . Now the result follows by assumption.  $\square$

An  $R$ -module  $M$  is said to be a *comultiplication module* if for every submodule  $N$  of  $M$  there exists an ideal  $I$  of  $R$  such that  $N = (0 :_M I)$ , equivalently, for each submodule  $N$  of  $M$ , we have  $N = (0 :_M \text{Ann}_R(N))$  [2].

An  $R$ -module  $M$  satisfies the *double annihilator conditions* (DAC for short) if for each ideal  $I$  of  $R$  we have  $I = \text{Ann}_R(0 :_M I)$  [9].

An  $R$ -module  $M$  is said to be a *strong comultiplication module* if  $M$  is a comultiplication  $R$ -module and satisfies the DAC conditions [4].

A submodule  $N$  of an  $R$ -module  $M$  is said to be a *strongly sum 2-irreducible submodule* if whenever  $N \subseteq H_1 + H_2 + H_3$  for submodules  $H_1, H_2$  and  $H_3$  of  $M$ , then either  $N \subseteq H_1 + H_2$  or  $N \subseteq H_2 + H_3$  or  $N \subseteq H_1 + H_3$ . Also,  $M$  is said to be a *strongly sum 2-irreducible module* if and only if  $M$  is a strongly sum 2-irreducible submodule of itself [10].

**Theorem 2.11.** *Let  $M$  be a strong comultiplication  $R$ -module. Then every non-zero proper submodule of  $R$  is a strongly sum 2-irreducible  $R$ -module if and only if every non-zero proper submodule of  $M$  is a strongly 2-irreducible submodule of  $M$ .*

*Proof.* "  $\Rightarrow$  " Let  $N$  be a non-zero proper submodule of  $M$  and let  $H_1 \cap H_2 \cap H_3 \subseteq N$  for submodules  $H_1, H_2$  and  $H_3$  of  $M$ . Then by using [11, 2.5],

$$\text{Ann}_R(N) \subseteq \text{Ann}_R(H_1) + \text{Ann}_R(H_2) + \text{Ann}_R(H_3).$$

This implies that either  $\text{Ann}_R(N) \subseteq \text{Ann}_R(H_1) + \text{Ann}_R(H_2)$  or  $\text{Ann}_R(N) \subseteq \text{Ann}_R(H_1) + \text{Ann}_R(H_3)$  or  $\text{Ann}_R(N) \subseteq \text{Ann}_R(H_2) + \text{Ann}_R(H_3)$  since by assumption,  $\text{Ann}_R(N)$  is a strongly sum 2-irreducible  $R$ -module. Therefore, either  $H_1 \cap H_2 \subseteq N$  or  $H_1 \cap H_3 \subseteq N$  or  $H_2 \cap H_3 \subseteq N$  since  $M$  is a comultiplication  $R$ -module.

"  $\Leftarrow$  " Let  $I$  be a non-zero proper submodule of  $R$  and let  $I \subseteq I_1 + I_2 + I_3$ . Then

$$(0 :_M I_1) \cap (0 :_M I_2) \cap (0 :_M I_3) \subseteq (0 :_M I).$$

Thus by assumption, either  $(0 :_M I_1) \cap (0 :_M I_2) \subseteq (0 :_M I)$  or  $(0 :_M I_1) \cap (0 :_M I_3) \subseteq (0 :_M I)$  or  $(0 :_M I_2) \cap (0 :_M I_3) \subseteq (0 :_M I)$ . This implies that either  $(0 :_M I_1 + I_2) \subseteq (0 :_M I)$  or  $(0 :_M I_1 + I_3) \subseteq (0 :_M I)$  or  $(0 :_M I_2 + I_3) \subseteq (0 :_M I)$ . Thus either  $I \subseteq I_1 + I_2$  or  $I \subseteq I_1 + I_3$  or  $I \subseteq I_2 + I_3$  since  $M$  is a strong comultiplication  $R$ -module.  $\square$

An  $R$ -module  $M$  is said to be a *multiplication module* if for every submodule  $N$  of  $M$  there exists an ideal  $I$  of  $R$  such that  $N = IM$  [6].

**Theorem 2.12.** *Let  $N$  be a submodule of a finitely generated multiplication  $R$ -module  $M$ . Then  $N$  is a strongly 2-irreducible submodule of  $M$  if and only if  $(N :_R M)$  is a strongly 2-irreducible ideal of  $R$ .*

*Proof.* "  $\Rightarrow$  " Let  $N$  be a strongly 2-irreducible submodule of  $M$  and let  $J_1 \cap J_2 \cap J_3 \subseteq (N :_R M)$  for some ideals  $J_1, J_2$ , and  $J_3$  of  $R$ . Then

$$J_1M \cap J_2M \cap J_3M \subseteq (N :_R M)M = N$$

by [8, Corollary 1.7]. Thus by assumption, either  $J_1M \cap J_2M \subseteq N$  or  $J_1M \cap J_3M \subseteq N$  or  $J_2M \cap J_3M \subseteq N$ . Hence, either  $(J_1 \cap J_2)M \subseteq (N :_R M)M$  or  $(J_1 \cap J_3)M \subseteq (N :_R M)M$  or  $(J_2 \cap J_3)M \subseteq (N :_R M)M$ . Therefore, either  $J_1 \cap J_2 \subseteq (N :_R M)$  or  $J_1 \cap J_3 \subseteq (N :_R M)$  or  $J_2 \cap J_3 \subseteq (N :_R M)$  by [18, Corollary of Theorem 9].

"  $\Leftarrow$  " Let  $(N :_R M)$  is a strongly 2-irreducible ideal of  $R$  and let  $H_1 \cap H_2 \cap H_3 \subseteq N$  for some submodules  $H_1, H_2$  and  $H_3$  of  $M$ . Then we have

$$(H_1 \cap H_2 \cap H_3 :_R M)M = ((H_1 :_R M) \cap (H_2 :_R M) \cap (H_3 :_R M))M \subseteq (N :_R M)M.$$

Thus  $(H_1 :_R M) \cap (H_2 :_R M) \cap (H_3 :_R M) \subseteq (N :_R M)$  by [18, Corollary of Theorem 9]. Hence, either  $(H_1 :_R M) \cap (H_2 :_R M) \subseteq (N :_R M)$  or  $(H_1 :_R M) \cap (H_3 :_R M) \subseteq (N :_R M)$  or  $(H_2 :_R M) \cap (H_3 :_R M) \subseteq (N :_R M)$  since  $(N :_R M)$  is a strongly 2-irreducible ideal of  $R$ . Therefore, either  $H_1 \cap H_2 \subseteq N$  or  $H_1 \cap H_3 \subseteq N$  or  $H_2 \cap H_3 \subseteq N$  by [8, Corollary 1.7].  $\square$

**EXAMPLE 2.13.** Consider the  $\mathbb{Z}$ -module  $\mathbb{Z}_{p^t q^n r^m}$ , where  $p, q, r$  are prime integers and  $t, n, m$  are natural numbers.

- By using Theorem 2.12 and Example 2.7, one can see that  $p^{\bar{t}}\mathbb{Z}_{p^t q^n r^m}$  and  $q^{\bar{n}}r^{\bar{m}}\mathbb{Z}_{p^t q^n r^m}$  are strongly 2-irreducible submodules of  $\mathbb{Z}_{p^t q^n r^m}$ .
- $p\bar{q}r\mathbb{Z}_{p^3 q r} = \bar{p}q\mathbb{Z}_{p^3 q r} \cap \bar{p}r\mathbb{Z}_{p^3 q r} \cap \bar{q}r\mathbb{Z}_{p^3 q r}$  implies that  $p\bar{q}r\mathbb{Z}_{p^3 q r}$  is not a 2-irreducible submodule of  $\mathbb{Z}_{p^3 q r}$ .

The following example shows that the concepts of strongly irreducible submodules and strongly 2-irreducible submodules are different in general.

**EXAMPLE 2.14.** Consider the  $\mathbb{Z}$ -module  $\mathbb{Z}_6$ . Then  $0 = \bar{3}\mathbb{Z}_6 \cap \bar{2}\mathbb{Z}_6$  implies that the 0 submodule of  $\mathbb{Z}_6$  is not strongly irreducible. But  $(0 :_{\mathbb{Z}} \mathbb{Z}_6) = 6\mathbb{Z}$  is a strongly 2-irreducible ideal of  $\mathbb{Z}$  by Example 2.7. Since the  $\mathbb{Z}$ -module  $\mathbb{Z}_6$  is a finitely generated multiplication  $\mathbb{Z}$ -module, 0 is a strongly 2-irreducible submodule of  $\mathbb{Z}_6$  by Theorem 2.12.

**Lemma 2.15.** *Let  $M$  be an  $R$ -module. If  $N_1$  and  $N_2$  are strongly irreducible submodules of  $M$ , then  $N_1 \cap N_2$  is a strongly 2-irreducible submodule of  $M$ .*

*Proof.* This is straightforward.  $\square$

A proper submodule  $P$  of an  $R$ -module  $M$  is said to be *prime* if for any  $r \in R$  and  $m \in M$  with  $rm \in P$ , we have  $m \in P$  or  $r \in (P :_R M)$  [7].

**Proposition 2.16.** *Let  $M$  be a multiplication  $R$ -module and let  $N_1, N_2,$  and  $N_3$  be prime submodules of  $M$  such that  $N_1 + N_2 = N_1 + N_3 = N_2 + N_3 = M$ . Then  $N_1 \cap N_2 \cap N_3$  is not a strongly 2-irreducible submodule of  $M$ .*

*Proof.* Assume on the contrary that  $N_1 \cap N_2 \cap N_3$  is a strongly 2-irreducible submodule of  $M$ . Then  $N_1 \cap N_2 \cap N_3 \subseteq N_1 \cap N_2 \cap N_3$  implies that either  $N_1 \cap N_2 \subseteq N_1 \cap N_2 \cap N_3$  or  $N_1 \cap N_3 \subseteq N_1 \cap N_2 \cap N_3$  or  $N_2 \cap N_3 \subseteq N_1 \cap N_2 \cap N_3$ . We can assume without loss of generality that  $N_1 \cap N_2 \subseteq N_1 \cap N_2 \cap N_3$ . Then  $N_1 \cap N_2 \subseteq N_3$ . It follows that  $(N_1 :_R M)N_2 \subseteq N_3$ . As  $N_3$  is a prime submodule of  $M$ , we have  $N_2 \subseteq N_3$  or  $(N_2 :_R M) \subseteq (N_3 :_R M)$ . Thus  $N_2 \subseteq N_3$  or  $N_1 \subseteq N_3$  since  $M$  is a multiplication  $R$ -module. Therefore,  $N_3 = M$ , which is a contradiction.  $\square$

**Corollary 2.17.** *Let  $M$  be a multiplication  $R$ -module such that every proper submodule of  $M$  is strongly 2-irreducible. Then  $M$  has at most two maximal submodules.*

*Proof.* This follows from Proposition 2.16  $\square$

Let  $N$  be a submodule of an  $R$ -module  $M$ . The intersection of all prime submodules of  $M$  containing  $N$  is said to be the (*prime*) *radical* of  $N$  and denote by  $rad_M N$  (or simply by  $rad(N)$ ). In case  $N$  does not contained in any prime submodule, the radical of  $N$  is defined to be  $M$ . Also,  $N \neq M$  is said to be a *radical submodule* of  $M$  if  $rad(N) = N$  [14]

**Lemma 2.18.** *Let  $I$  be an ideal of  $R$  and  $N$  be a submodule of an  $R$ -module  $M$ . Then  $rad(IN) = rad(N) \cap rad(IM)$ .*

*Proof.* By [13, Corollary of Theorem 6], we have  $rad(N \cap IM) = rad(N) \cap rad(IM)$ . Since  $IN \subseteq IM \cap N$ ,  $rad(IN) \subseteq rad(IM \cap N)$ . Thus  $rad(IN) \subseteq rad(N) \cap rad(IM)$ . Now let  $P$  be a prime submodule of  $M$  such that  $IN \subseteq P$ . As  $P$  is prime,  $N \subseteq P$  or  $I \subseteq (P :_R M)$ . Hence  $N \cap IM \subseteq P$ . This in turn implies that  $rad(N) \cap rad(IM) \subseteq rad(IN)$ , as desired.  $\square$

A proper ideal  $I$  of  $R$  is said to be a *2-absorbing ideal* of  $R$  if whenever  $a, b, c \in R$  and  $abc \in I$ , then  $ab \in I$  or  $ac \in I$  or  $bc \in I$  [5].

A proper submodule  $N$  of an  $R$ -module  $M$  is said to be a *2-absorbing primary submodule* of  $M$  if whenever  $a, b \in R, m \in M$ , and  $abm \in N$ , then  $am \in rad(N)$  or  $bm \in rad(N)$  or  $ab \in (N :_R M)$  [15].

A proper submodule  $N$  of an  $R$ -module  $M$  is called a *2-absorbing submodule* of  $M$  if whenever  $abm \in N$  for some  $a, b \in R$  and  $m \in M$ , then  $am \in N$  or  $bm \in N$  or  $ab \in (N :_R M)$  [19] and [16].

**Theorem 2.19.** *Let  $M$  be a finitely generated multiplication  $R$ -module and  $N$  be a radical submodule of  $M$ . Then the following conditions are equivalent:*

- (a)  $N$  is a strongly 2-irreducible submodule of  $M$ ;

- (b)  $N$  is a 2-absorbing submodule of  $M$ ;
- (c)  $N$  is a 2-absorbing primary submodule of  $M$ ;
- (d)  $N$  is either a prime submodule of  $M$  or is an intersection of exactly two prime submodules of  $M$ .

*Proof.* (a)  $\Rightarrow$  (b) Let  $I, J$  be ideals of  $R$  and  $K$  be a submodule of  $M$  such that  $IJK \subseteq N$ . Then by using Lemma 2.18,

$$K \cap IM \cap JM \subseteq \text{rad}(K) \cap \text{rad}(IM) \cap \text{rad}(JM) = \text{rad}(IJK) \subseteq \text{rad}(N) = N$$

Hence by part (a), either  $K \cap IM \subseteq N$  or  $K \cap JM \subseteq N$  or  $IM \cap JM \subseteq N$ . Thus either  $IK \subseteq N$  or  $JK \subseteq N$  or  $IJM \subseteq N$  as needed.

(b)  $\Rightarrow$  (c) This is clear.

(c)  $\Rightarrow$  (b) This is clear by using [15, Theorem 2.6].

(b)  $\Rightarrow$  (d) Since  $N$  is a 2-absorbing submodule of  $M$ ,  $(N :_R M)$  is a 2-absorbing ideal of  $R$  by [20, Proposition 1]. Hence  $\sqrt{(N :_R M)} = P$  is a prime ideal of  $R$  or  $\sqrt{(N :_R M)} = P \cap Q$ , where  $P$  and  $Q$  are distinct prime ideals of  $R$  that are minimal over  $(N :_R M)$  by [5, Theorem 2.4]. We have  $\sqrt{(N :_R M)}M = \text{rad}(N)$  by [14, Theorem 4]. If  $\sqrt{(N :_R M)} = P$ , then  $PM = \text{rad}(N)$ . Since  $M$  is a multiplication  $R$ -module,  $PM$  is a prime submodule of  $M$  by [8, Corollary 2.11]. Now let  $\sqrt{\text{Ann}_R(N)} = P \cap Q$ , where  $P$  and  $Q$  are distinct prime ideals of  $R$ . Then  $(P \cap Q)M = \text{rad}(N)$ . By [8, Corollary 1.7],  $(P \cap Q)M = PM \cap QM$ . Thus by [8, Corollary 2.11],  $\text{rad}(N)$  is an intersection of two prime submodules of  $M$ . Now, we prove the claim by assumption that  $N$  is a radical submodule of  $M$ .

(d)  $\Rightarrow$  (a) This follows from Lemma 2.15. □

The following example shows that parts (a) and (b) of Theorem 2.19 are not equivalent in general.

**EXAMPLE 2.20.** Consider the submodule  $G_t = \langle 1/p^t + \mathbb{Z} \rangle$  of the  $\mathbb{Z}$ -module  $\mathbb{Z}_{p^\infty}$ . Then  $G_t$  is a strongly 2-irreducible submodule of  $\mathbb{Z}_{p^\infty}$ . But  $G_t$  is not a 2-absorbing submodule of  $\mathbb{Z}_{p^\infty}$ . It should be note that the  $\mathbb{Z}$ -module  $\mathbb{Z}_{p^\infty}$  is not a finitely generated multiplication  $\mathbb{Z}$ -module.

A submodule  $N$  of an  $R$ -module  $M$  is said to be *pure* if  $IN = IM \cap N$  for every ideal  $I$  of  $R$  [1]. Also, an  $R$ -module  $M$  is said to be *fully pure* if every submodule of  $M$  is pure [3].

**Theorem 2.21.** *Let  $M$  be a fully pure multiplication  $R$ -module and  $N$  be a submodule of  $M$ . Then the following conditions are equivalent:*

- (a)  $N$  is a strongly 2-irreducible submodule of  $M$ ;
- (b)  $N$  is a 2-absorbing submodule of  $M$ ;
- (c)  $N$  is a 2-irreducible submodule of  $M$ .



*Proof.* (a)  $\Rightarrow$  (b) Let  $I, J$  be ideals of  $R$  and  $K$  be a submodule of  $M$  such that  $IJK \subseteq N$ . Then since  $M$  is fully pure,

$$K \cap IM \cap JM = IJK \subseteq N.$$

Hence by part (a), either  $K \cap IM \subseteq N$  or  $K \cap JM \subseteq N$  or  $IM \cap JM \subseteq N$ . Thus either  $IK \subseteq N$  or  $JK \subseteq N$  or  $IJM \subseteq N$ .

(b)  $\Rightarrow$  (a) Let  $H_1 \cap H_2 \cap H_3 \subseteq N$  for submodules  $H_1, H_2$  and  $H_3$  of  $M$ . Then

$$(H_1 :_R M) \cap (H_2 :_R M) \cap (H_3 :_R M) = (H_1 \cap H_2 \cap H_3 :_R M) \subseteq (N :_R M).$$

Thus either  $(H_1 :_R M)(H_2 :_R M) \subseteq (N :_R M)$  or  $(H_1 :_R M)(H_3 :_R M) \subseteq (N :_R M)$  or  $(H_2 :_R M)(H_3 :_R M) \subseteq (N :_R M)$  since  $(N :_R M)$  is a 2-absorbing ideal of  $R$  by [20, Proposition 1]. We can assume without loss of generality that  $(H_1 :_R M)(H_2 :_R M) \subseteq (N :_R M)$ . Thus as  $M$  is fully pure, we have

$$(H_1 :_R M)M \cap (H_2 :_R M)M \subseteq (N :_R M)M \subseteq N.$$

Therefore,  $H_1 \cap H_2 \subseteq N$  since  $M$  is a multiplication  $R$ -module.

(a)  $\Leftrightarrow$  (c) By [3, proof of Theorem 2.19],  $M$  is a distributive  $R$ -module. Now the result follows from Proposition 2.8.  $\square$

**Lemma 2.22.** *Let  $M$  be an  $R$ -module,  $S$  a multiplicatively closed subset of  $R$ , and  $N$  be a finitely generated submodule of  $M$ . If  $S^{-1}N \subseteq S^{-1}K$  for a submodule  $K$  of  $M$ , then there exists  $s \in S$  such that  $sN \subseteq K$ .*

*Proof.* This is straightforward.  $\square$

**Proposition 2.23.** *Let  $M$  be an  $R$ -module,  $S$  be a multiplicatively closed subset of  $R$  and  $N$  be a finitely generated prime strongly 2-irreducible submodule of  $M$  such that  $(N :_R M) \cap S = \emptyset$ . Then  $S^{-1}N$  is a strongly 2-irreducible submodule of  $S^{-1}M$  if  $S^{-1}N \neq S^{-1}M$ .*

*Proof.* Let  $S^{-1}H_1 \cap S^{-1}H_2 \cap S^{-1}H_3 \subseteq S^{-1}N$  for submodules  $S^{-1}H_1, S^{-1}H_2$  and  $S^{-1}H_3$  of  $S^{-1}M$ . Then  $S^{-1}(H_1 \cap H_2 \cap H_3) \subseteq S^{-1}N$ . By Lemma 2.22, there exists  $s \in S$  such that  $s(H_1 \cap H_2 \cap H_3) \subseteq N$ . This implies that  $H_1 \cap H_2 \cap H_3 \subseteq N$  since  $N$  is prime and  $(N :_R M) \cap S = \emptyset$ . Now as  $N$  is a strongly 2-irreducible submodule of  $M$ , we have either  $H_1 \cap H_2 \subseteq N$  or  $H_1 \cap H_3 \subseteq N$  or  $H_2 \cap H_3 \subseteq N$ . Therefore, either  $S^{-1}H_1 \cap S^{-1}H_2 \subseteq S^{-1}N$  or  $S^{-1}H_1 \cap S^{-1}H_3 \subseteq S^{-1}N$  or  $S^{-1}H_2 \cap S^{-1}H_3 \subseteq S^{-1}N$ , as needed.  $\square$

**Proposition 2.24.** *Let  $M$  be an  $R$ -module and  $\{K_i\}_{i \in I}$  be a chain of strongly 2-irreducible submodules of  $M$ . Then  $\bigcap_{i \in I} K_i$  is a strongly 2-irreducible submodule of  $M$ .*

*Proof.* Let  $H_1 \cap H_2 \cap H_3 \subseteq \bigcap_{i \in I} K_i$  for submodules  $H_1, H_2$  and  $H_3$  of  $M$ . Assume that  $H_1 + H_2 \not\subseteq \bigcap_{i \in I} K_i$ ,  $H_1 + H_3 \not\subseteq \bigcap_{i \in I} K_i$ , and  $H_2 + H_3 \not\subseteq \bigcap_{i \in I} K_i$ .

Then there are  $m, n, t \in I$ , where  $H_1 \cap H_2 \not\subseteq K_m$ ,  $H_1 \cap H_3 \not\subseteq K_n$ , and  $H_2 \cap H_3 \not\subseteq K_t$ . Since  $\{K_i\}_{i \in I}$  is a chain we can assume that  $K_m \subseteq K_n \subseteq K_t$ . But as  $H_1 \cap H_2 \cap H_3 \subseteq K_m$  and  $K_m$  is a strongly sum 2-irreducible submodule of  $M$ , we have either  $H_1 \cap H_2 \subseteq K_m$  or  $H_1 \cap H_3 \subseteq K_m$  or  $H_2 \cap H_3 \subseteq K_m$ . In any case, we get a contradiction.  $\square$

**Theorem 2.25.** *Let  $f : M \rightarrow \hat{M}$  be a epimorphism of  $R$ -modules. Then we have the following.*

- (a) *If  $N$  is a strongly 2-irreducible submodule of  $M$  such that  $\ker(f) \subseteq N$ , then  $f(N)$  is a strongly 2-irreducible submodule of  $\hat{M}$ .*
- (b) *If  $\hat{N}$  is a strongly 2-irreducible submodule of  $\hat{M}$ , then  $f^{-1}(\hat{N})$  is a strongly 2-irreducible submodule of  $M$ .*

*Proof.* (a) Let  $N$  be a strongly 2-irreducible submodule of  $M$ . If  $f(N) = \hat{M}$ , then we have  $N + \ker(f) = f^{-1}(f(N)) = f^{-1}(\hat{M}) = f^{-1}(f(M)) = M$ . Now as  $\ker(f) \subseteq N$ , we get that  $N = M$ , which is a contradiction. Therefore,  $f(N) \neq \hat{M}$ . Suppose that  $\hat{H}_1 \cap \hat{H}_2 \cap \hat{H}_3 \subseteq f(N)$  for submodules  $\hat{H}_1, \hat{H}_2$  and  $\hat{H}_3$  of  $\hat{M}$ . Then  $f^{-1}(\hat{H}_1) \cap f^{-1}(\hat{H}_2) \cap f^{-1}(\hat{H}_3) \subseteq f^{-1}(f(N)) = N$  since  $\ker(f) \subseteq N$ . Thus by assumption, either  $f^{-1}(\hat{H}_1) \cap f^{-1}(\hat{H}_2) \subseteq N$  or  $f^{-1}(\hat{H}_1) \cap f^{-1}(\hat{H}_3) \subseteq N$  or  $f^{-1}(\hat{H}_2) \cap f^{-1}(\hat{H}_3) \subseteq N$ . Now as  $f$  is epimorphism, we have either  $\hat{H}_1 \cap \hat{H}_2 \subseteq f(N)$  or  $\hat{H}_1 \cap \hat{H}_3 \subseteq f(N)$  or  $\hat{H}_2 \cap \hat{H}_3 \subseteq f(N)$ , as needed.

(b) Let  $\hat{N}$  be a strongly 2-irreducible submodule of  $\hat{M}$ . Since  $\hat{N} \neq \hat{M}$  and  $f$  is a epimorphism, we have  $f^{-1}(\hat{N}) \neq M$ . Now let  $H_1 \cap H_2 \cap H_3 \subseteq f^{-1}(\hat{N})$  for submodules  $H_1, H_2$  and  $H_3$  of  $M$ . Then  $f(H_1) \cap f(H_2) \cap f(H_3) \subseteq f(f^{-1}(\hat{N})) = \hat{N}$ . Thus by assumption, either  $f(H_1) \cap f(H_2) \subseteq \hat{N}$  or  $f(H_1) \cap f(H_3) \subseteq \hat{N}$  or  $f(H_2) \cap f(H_3) \subseteq \hat{N}$ . Now we have either  $H_1 \cap H_2 \subseteq f^{-1}(\hat{N})$  or  $H_1 \cap H_3 \subseteq f^{-1}(\hat{N})$  or  $H_2 \cap H_3 \subseteq f^{-1}(\hat{N})$ , as required.  $\square$

**Theorem 2.26.** *Let  $M$  be a finitely generated multiplication distributive  $R$ -module and let  $N$  be a non-zero proper submodule of  $M$ . Then the following statements are equivalent:*

- (a)  *$N$  is a strongly 2-irreducible submodule of  $M$ ;*
- (b)  *$(N :_R M)$  is a strongly 2-irreducible ideal of  $R$ ;*
- (c)  *$(N :_R M)$  is a 2-irreducible ideal of  $R$ .*

*Proof.* (a)  $\Rightarrow$  (b) This follows from Theorem 2.12.

(b)  $\Rightarrow$  (c) This follows from [21, Proposition 1].

(c)  $\Rightarrow$  (a) Let  $H_1 \cap H_2 \cap H_3 \subseteq N$  for submodules  $H_1, H_2$  and  $H_3$  of  $M$ . Then as  $M$  is a distributive  $R$ -module, we have

$$N = N + (H_1 \cap H_2 \cap H_3) = (N + H_1) \cap (N \cap H_2) \cap (N \cap H_3).$$

This implies that  $(N :_R M) = (N + H_1 :_R M) \cap (N + H_2 :_R M) \cap (N + H_3 :_R M)$ . Thus by assumption, either  $(N :_R M) = (N + H_1 :_R M) \cap (N + H_2 :_R M)$  or  $(N :_R M) = (N + H_1 :_R M) \cap (N + H_3 :_R M)$  or  $(N :_R M) = (N + H_2 :_R M)$

$M) \cap (N + H_3 :_R M)$ . Therefore, by [8, Corollary 1.7], either  $N = N + (H_1 \cap H_2)$  or  $N = N + (H_1 \cap H_3)$  or  $N = N + (H_2 \cap H_3)$ , since  $M$  is a finitely generated multiplication  $R$ -module. Thus either,  $H_1 \cap H_2 \subseteq N$  or  $H_1 \cap H_3 \subseteq N$  or  $H_2 \cap H_3 \subseteq N$  as needed.  $\square$

Let  $R_i$  be a commutative ring with identity and  $M_i$  be an  $R_i$ -module, for  $i = 1, 2$ . Let  $R = R_1 \times R_2$ . Then  $M = M_1 \times M_2$  is an  $R$ -module and each submodule of  $M$  is in the form of  $N = N_1 \times N_2$  for some submodules  $N_1$  of  $M_1$  and  $N_2$  of  $M_2$ .

**Theorem 2.27.** *Let  $R = R_1 \times R_2$  be a decomposable ring and  $M = M_1 \times M_2$  be an  $R$ -module, where  $M_1$  is an  $R_1$ -module and  $M_2$  is an  $R_2$ -module. Suppose that  $N = N_1 \times N_2$  is a proper submodule of  $M$ . Then the following conditions are equivalent:*

- (a)  $N$  is a strongly 2-irreducible submodule of  $M$ ;
- (b) Either  $N_1 = M_1$  and  $N_2$  strongly 2-irreducible submodule of  $M_2$  or  $N_2 = M_2$  and  $N_1$  is a strongly 2-irreducible submodule of  $M_1$  or  $N_1, N_2$  are strongly irreducible submodules of  $M_1, M_2$ , respectively.

*Proof.* (a)  $\Rightarrow$  (b). Let  $N = N_1 \times N_2$  be a strongly 2-irreducible submodule of  $M$  such that  $N_2 = M_2$ . From our hypothesis,  $N$  is proper, so  $N_1 \neq M_1$ . Set  $\hat{M} = M/(0 \times M_2)$ . One can see that  $\hat{N} = N/(0 \times M_2)$  is a strongly 2-irreducible submodule of  $\hat{M}$ . Also, observe that  $\hat{M} \cong M_1$  and  $\hat{N} \cong N_1$ . Thus  $N_1$  is a strongly 2-irreducible submodule of  $M_1$ . By a similar argument as in the previous case,  $N_2$  is a strongly 2-irreducible submodule of  $M_2$ , where,  $N_1 = M_1$ . Now suppose that  $N_1 \neq M_1$  and  $N_2 \neq M_2$ . We show that  $N_1$  is a irreducible submodule of  $M_1$ . Suppose that  $H_1 \cap K_1 \subseteq N_1$  for some submodules  $H_1$  and  $K_1$  of  $M_1$ . Then

$$(H_1 \times M_2) \cap (M_1 \times 0) \cap (K_1 \times M_2) \subseteq (H_1 \cap K_1) \times 0 \subseteq N_1 \times N_2.$$

Thus by assumption, either  $(H_1 \times M_2) \cap (M_1 \times 0) \subseteq N_1 \times N_2$  or  $(H_1 \times M_2) \cap (K_1 \times M_2) \subseteq N_1 \times N_2$  or  $(M_1 \times 0) \cap (K_1 \times M_2) \subseteq N_1 \times N_2$ . Therefore,  $H_1 \subseteq N_1$  or  $K_1 \subseteq N_1$  since  $N_2 \neq M_2$ . Thus  $N_1$  is a strongly irreducible submodule of  $M_1$ . Similarly, we can show that  $N_2$  is strongly irreducible submodule of  $M_2$ .

(b)  $\Rightarrow$  (a). Suppose that  $N = N_1 \times M_2$ , where  $N_1$  is a strongly 2-irreducible submodule of  $M_1$ . Then it is clear that  $N$  is a strongly 2-irreducible submodule of  $M$ . Now, assume that  $N = N_1 \times N_2$ , where  $N_1$  and  $N_2$  are strongly irreducible submodules of  $M_1$  and  $M_2$ , respectively. Hence  $(N_1 \times M_2) \cap (M_1 \times N_2) = N_1 \times N_2 = N$  is a strongly 2-irreducible submodule of  $M$ , by Lemma 2.15.  $\square$

**Theorem 2.28.** *Let  $R = R_1 \times R_2 \times \cdots \times R_n$  ( $2 \leq n < \infty$ ) be a decomposable ring and  $M = M_1 \times M_2 \cdots \times M_n$  be an  $R$ -module, where for every  $1 \leq i \leq n$ ,  $M_i$  is an  $R_i$ -module, respectively. Then for a proper submodule  $N$  of  $M$  the following conditions are equivalent:*

- (a)  $N$  is a strongly 2-irreducible submodule of  $M$ ;
- (b) Either  $N = \times_{i=1}^n N_i$  such that for some  $k \in \{1, 2, \dots, n\}$ ,  $N_k$  is a strongly 2-irreducible submodule of  $M_k$ , and  $N_i = M_i$  for every  $i \in \{1, 2, \dots, n\} \setminus \{k\}$  or  $N = \times_{i=1}^n N_i$  such that for some  $k, m \in \{1, 2, \dots, n\}$ ,  $N_k$  is a strongly irreducible submodule of  $M_k$ ,  $N_m$  is a strongly irreducible submodule of  $M_m$ , and  $N_i = M_i$  for every  $i \in \{1, 2, \dots, n\} \setminus \{k, m\}$ .

*Proof.* We use induction on  $n$ . For  $n = 2$  the result holds by Theorem 2.27. Now let  $3 \leq n < \infty$  and suppose that the result is valid when  $K = M_1 \times \dots \times M_{n-1}$ . We show that the result holds when  $M = K \times M_n$ . By Theorem 2.27,  $N$  is a strongly 2-irreducible submodule of  $M$  if and only if either  $N = L \times M_n$  for some strongly 2-irreducible submodule  $L$  of  $K$  or  $N = K \times L_n$  for some strongly 2-irreducible submodule  $L_n$  of  $M_n$  or  $N = L \times L_n$  for some strongly irreducible submodule  $L$  of  $K$  and some strongly irreducible submodule  $L_n$  of  $M_n$ . Note that a proper submodule  $L$  of  $K$  is a strongly irreducible submodule of  $K$  if and only if  $L = \times_{i=1}^{n-1} N_i$  such that for some  $k \in \{1, 2, \dots, n-1\}$ ,  $N_k$  is a strongly irreducible submodule of  $M_k$ , and  $N_i = M_i$  for every  $i \in \{1, 2, \dots, n-1\} \setminus \{k\}$ . Consequently the claim is now verified.  $\square$

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