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# A Numerical Method for Solving Stochastic Volterra-Fredholm Integral Equation

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Abstract. In this paper, we propose a numerical method based on the generalized hat functions (GHFs) and improved hat functions (IHFs) to find numerical solutions for stochastic Volterra-Fredholm integral equation. To do so, all known and unknown functions are expanded in terms of basic functions and replaced in the original equation. The operational matrices of both basic functions are calculated and embeded in the equation to achieve a linear system of equations which give the expansion coefficients of the solution. We prove that the rate of the convergence is $O(h^2)$ and $O(h^4)$ for these two different bases under some conditions. Two examples are solved and the results are compared with those of block pulse functions method (BPFs) to show the accuracy and reliability of the methods.

**Keywords:** Generalized hat functions, Improved hat functions, Stochastic operational matrix, Stochastic Volterra-Fredholm integral equation, Brownian motion.

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## 1. Introduction

Stochastic equations are one of the most important and applied topics in today's world. They arise in modelling of different problems in science such as finance [1, 2, 3], chemistry [4, 5, 6], mechanics [7], physics [8, 9, 10], mathematics and statistics [11, 12, 13], biology [14, 15, 16], etc.. This sometimes results in a stochastic Volterra-Fredholm integral equation and in many cases they have no explicit form of the solution [17]. Consequently, numerical methods come to solve the problem and find an appropriate approximation.
Consider the following stochastic Volterra-Fredholm integral equation,

$$X(t) = f(t) + \int_{\alpha}^{\beta} K_1(s,t)X(s)ds + \int_0^t K_2(s,t)X(s)\,ds + \int_0^t K_3(s,t)X(s)\,dB(s),$$

where $s, t \in [o, T), X, f, K_1, K_2$ and $K_3$ are the stochastic processes defined on the same probability space $(\Omega, F, P)$ and $X$ is unknown.
Also $\int_0^t K_3(s,t)X(s)\,dB(s)$, is the Itô integral and $B(t)$ is a Brownian motion [20].
Different basic functions have been used to find an approximation for stochastic integral equations such as block pulse functions [17, 18, 19], hat functions [21], modified hat functions [22, 23], triangular functions [24, 25], hybrid functions [26, 27], wavelet methods [28, 29], etc..
In this paper, we use both generalized hat functions (GHFs) and improved hat functions (IHFs) to find approximations of the solution of the original equation. In these methods, the operational matrices and approximations of all functions are found according to basic functions. They are replaced in the original equation and a linear system of equations is concluded. The rate of convergence is shown to be $O(h^2)$ and $O(h^4)$ respectively for these methods, which is acceptable.
This paper is organized as follows. In Section 2, we describe GHFs, their properties and operational matrices. In Section 3, IHFs, their properties and operational matrices are reviewed. In Section 4, the method of the solution is studied. In Section 5, the error analysis is discussed. Some numerical examples are solved and compared with those of BPFs in Section 6. And finally in Section 7, some tentative conclusions will be drawn.

## 2. Generalized hat functions (GHFs)

In this section, we get to know GHFs and their properties, function expansions and operational matrices [21, 30].


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Hat functions, also known as triangular or tent functions and whose graphs take the shape of triangles or hats, work to solve differential equations by Galerkin method. They are useful in signal processing and communication system engineering, and have applications in pulse code modulation for transmitting digital signals. These functions are continuous and defined on $[0, 1]$. Generalized hat functions (GHFs) are created by extending the domain of definition to $[0, T]$. To do so, we divide $[0, T]$ into $n$ equal subintervals $[ih, (i+1)h], i = 0, 1, \dots, n-1$, where $h = \frac{T}{n}$ and $n$ is an arbitrary positive integer and defines a set of GHFs as

$$h_0(t) = \begin{cases} \frac{h-t}{h}, & 0 \leq t \leq h \\ 0, & otherwise. \end{cases}$$

For $i = 1, 2, \dots, (n-1)$,

$$h_i(t) = \begin{cases} \frac{t-(i-1)h}{h}, & (i-1)h \leq t \leq ih \\ \frac{(i+1)h-t}{h}, & ih \leq t \leq (i+1)h, \\ 0, & otherwise, \end{cases}$$

and

$$h_n(t) = \begin{cases} \frac{t-(T-h)}{h}, & T-h \leq t \leq T \\ 0, & otherwise. \end{cases}$$

From the definition of GHFs, the following properties come as a result.

(1) They are linearly independent.

(2) $h_i(jh) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}.$

(3) $h_i(t)h_j(t) = 0, \ |i-j| \geq 2.$

(4) $\sum_{i=0}^{n} h_i(t) = 1.$

Suppose

$$\mathbf{H}(t) = [h_0(t), h_1(t), \dots, h_n(t)]^T, \tag{2.1}$$

then, we have

(5) $\mathbf{H}(t)\mathbf{H}^T(t) \simeq \begin{bmatrix} h_0(t) & 0 & \dots & 0 \\ 0 & h_1(t) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & h_n(t) \end{bmatrix}.$

(6) $\mathbf{H}(t)\mathbf{H}(t)^T F \simeq diag(F)\mathbf{H}(t)$, where $F$ is an $(n+1)$-column vector.

(7) Let $\mathbf{K}$ be an $(n+1) \times (n+1)$ matrix, then $\mathbf{H}(t)^T\mathbf{K}\mathbf{H}(t) \simeq \mathbf{H}(t)^T\tilde{\mathbf{K}}$, where $\tilde{\mathbf{K}}$ is a column vector with $(n+1)$ entries equal to the diagonal entries of matrix $\mathbf{K}$.



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An arbitrary real function $f \in L^2([0,T))$ can be expanded by these basic functions as

$$f(t) \simeq \sum_{i=0}^{n} f_i h_i(t) = \mathbf{F}^T \mathbf{H}(t) = \mathbf{H}^T(t)\mathbf{F}, \tag{2.2}$$

where $\mathbf{F} = [f_0, f_1, \ldots, f_n]^T$ and $\mathbf{H}(t)$ is defined in relation (2.1) and the coefficients in (2.2) are given by $f_i = f(ih), i = 0, 1, \ldots, n$.

Similarly, an arbitrary real function of two variables $g(s,t)$ defined on $L^2([0,T)\times[0,T))$ can also be expanded by these functions as

$$g(s,t) \simeq \mathbf{H}^T(s)\,\mathbf{G}\,\mathbf{H}(t) = \mathbf{H}^T(t)\,\mathbf{G}^T\,\mathbf{H}(s), \tag{2.3}$$

where $\mathbf{G}= [\mathbf{G}_{ij}]$ is an $(n+1)\times(n+1)$ GHFs coefficients matrix with entries $\mathbf{G}_{ij} = g(ih, jh)$, that $i, j = 0, 1, 2, \ldots, n$ and $h = \frac{T}{n}$.

We present $\mathbf{P}$ and $\mathbf{P}_s$ as the operational matrix and stochastic operational matrix of integration for GHFs respectively, where [21]

$$\mathbf{P} = \frac{h}{2}\begin{bmatrix} 0 & 1 & 1 & \ldots & 1 & 1 \\ 0 & 1 & 2 & \ldots & 2 & 2 \\ 0 & 0 & 1 & \ldots & 2 & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 2 \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{bmatrix}, \tag{2.4}$$

and

$$\mathbf{P}_s = \begin{bmatrix} 0 & \beta_0(h) & \beta_0(h) & \ldots & \beta_0(h) & \beta_0(h) \\ 0 & B(h)+\beta_1(h) & \gamma_1(h) & \ldots & \gamma_1(h) & \gamma_1(h) \\ 0 & 0 & B(2h)+\beta_2(h) & \ldots & \gamma_2(h) & \gamma_2(h) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & B((n-1)h)+\beta_{n-1}(h) & \gamma_{n-1} \\ 0 & 0 & 0 & \ldots & 0 & B(T)+\beta_n(h) \end{bmatrix}, \tag{2.5}$$

with

$$\beta_0(h) = \frac{1}{h}\int_0^h B(\tau)\,d\tau,$$

$$\beta_i(h) = \frac{-1}{h}\int_{(i-1)h}^{ih} B(\tau)\,d\tau,\; i = 1, 2, \ldots, n,$$

$$\gamma_i(h) = \frac{-1}{h}\left(\int_{(i-1)h}^{ih} B(\tau)\,d\tau - \int_{ih}^{(i+1)h} B(\tau)\,d\tau\right),\; i = 1, 2, \ldots, n-1,$$

**Theorem 2.1.** *Let* $\mathbf{H}(t)$ *be the vector defined in relation* (2.1)*, then*

$$\int_0^T \mathbf{H}(\tau)\mathbf{H}^T(\tau)\,d\tau = \boldsymbol{P}_*,$$



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*where* $\boldsymbol{P}_*$ *is the following* $(n+1)\times(n+1)$ *matrix,*

$$\boldsymbol{P}_* = \frac{h}{3}\begin{bmatrix} 1 & \frac{1}{2} & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \frac{1}{2} & 2 & \frac{1}{2} & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 2 & \frac{1}{2} & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & \frac{1}{2} & 2 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & \frac{1}{2} & 1 \end{bmatrix}, \tag{2.6}$$

*Proof.* The proof comes after integrating the elements of $\mathbf{H}(t)\mathbf{H}^T(t)$ from 0 to $T$. □

## 3. Improved hat functions (IHFs)

IHFs and their properties, function expansions and operational matrices are studied in this section [23, 30].

Let $n \geq 3$ be a multiple of 3 and $h = \frac{T}{n}$. Also assume that the interval $[0, T)$ is divided into $\frac{n}{3}$ equal subintervals $[ih, (i+3)h], i = 0, 3, \dots, (n-3)$. Moreover, let $X_n$ be the set of all continuous functions that are the third degree polynomials when restricted to the above subintervals. Each element of $X_n$ being completely determined by its values at $(n+1)$ nodes $ih, i = 0, 1, \dots, n$ eventuates $(n+1)$ is the dimension of $X_n$ and $f \in C([0, T))$ can be approximated by a linear combination of the following set of functions,

$$m_0(t) = \begin{cases} \frac{-1}{6h^3}(t-h)(t-2h)(t-3h), & 0 \leq t \leq 3h \\ 0, & otherwise. \end{cases}$$

If $i = 3k-2$ and $1 \leq k \leq \frac{n}{3}$,

$$m_i(t) = \begin{cases} \frac{1}{2h^3}(t-(i-1)h)(t-(i+1)h)(t-(i+2)h), & (i-1)h \leq t \leq (i+2)h \\ 0, & otherwise. \end{cases}$$

If $i = 3k-4$ and $2 \leq k \leq \frac{n}{3}+1$,

$$m_i(t) = \begin{cases} \frac{-1}{2h^3}(t-(i-2)h)(t-(i-1)h)(t-(i+1)h), & (i-2)h \leq t \leq (i+1)h \\ 0, & otherwise, \end{cases}$$

and if $i = 3k$ and $1 \leq k \leq \frac{n}{3}-1$,

$$m_i(t) = \begin{cases} \frac{1}{6h^3}(t-(i-3)h)(t-(i-2)h)(t-(i-1)h), & (i-3)h \leq t \leq ih \\ \frac{-1}{6h^3}(t-(i+1)h)(t-(i+2)h)(t-(i+3)h), & ih \leq t \leq (i+3)h \\ 0, & otherwise, \end{cases}$$

and

$$m_n(t) = \begin{cases} \frac{1}{6h^3}(t-(T-h))(t-(T-2h))(t-(T-3h)), & T-3h \leq t \leq T \\ 0, & otherwise. \end{cases}$$



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The following properties come as a result of above definition.
1) They are linearly independent.

2) $m_i(jh) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$.

3) $\sum_{i=0}^{n} m_i(t) = 1$.

Suppose

$$\mathbf{M}(t) = [m_0(t), m_1(t), \ldots, m_n(t)]^T, \tag{3.1}$$

then, we have

4) $\mathbf{M}(t)\mathbf{M}^T(t) \simeq \begin{bmatrix} m_0(t) & 0 & \ldots & 0 \\ 0 & m_1(t) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & m_n(t) \end{bmatrix}$.

5) $\mathbf{M}(t)\mathbf{M}(t)^T F \simeq diag(F)\mathbf{M}(t)$, where $F$ is an $(n+1)$-column vector.

6) Let $\mathbf{K}$ be an $(n+1) \times (n+1)$ matrix, then $\mathbf{M}(t)^T\mathbf{K}\mathbf{M}(t) \simeq \mathbf{M}(t)^T\tilde{\mathbf{K}}$, where $\tilde{\mathbf{K}}$ is a column vector with $(n+1)$ entries equal to the diagonal entries of matrix $\mathbf{K}$.

An arbitrary real function $f$ on $[0, T)$ can be expanded by these basic functions as

$$f(t) \simeq \sum_{i=0}^{n} f_i m_i(t) = \mathbf{F}^T\mathbf{M}(t) = \mathbf{M}^T(t)\mathbf{F}, \tag{3.2}$$

where $\mathbf{F} = [f_0, f_1, \ldots, f_n]^T$ and $\mathbf{M}(t)$ is defined in relation (3.1) and the coefficients in (3.2) are given by $f_i = f(ih), i = 0, 1, \ldots, n$.

Similarly, an arbitrary real function of two variables $g$ on $[0, T) \times [0, T)$ can also be expanded by these basis functions as

$$g(s, t) \simeq \mathbf{M}^T(s)\,\mathbf{G}\,\mathbf{M}(t) = \mathbf{M}^T(t)\,\mathbf{G}^T\,\mathbf{M}(s), \tag{3.3}$$

where $G = [G_{ij}]$ is an $(n+1) \times (n+1)$ matrix and $G_{ij} = g(ih, jh)$ for $i, j = 0, 1, 2, \ldots, n$.



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We introduce $\mathbf{P}'$ and $\mathbf{P}'_s$, the operational matrix of integration of the vector $\mathbf{M}(t)$ defined in relation (3.3) and stochastic operational matrix of Itô integration of the vector $\mathbf{M}(t)$ as

$$\mathbf{P}' = \frac{h}{24}\begin{bmatrix} 0 & a_1 & a_2 & a_2 & a_2 & \dots & a_2 \\ \mathbf{0} & p'_1 & p'_2 & p'_3 & p'_3 & \dots & p'_3 \\ \mathbf{0} & \mathbf{0} & p'_1 & p'_2 & p'_3 & \dots & p'_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & p'_1 & p'_2 & \dots & p'_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & p'_1 & \dots & p'_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & p'_1 \end{bmatrix},$$

where

$a_1 = \begin{bmatrix} 9 & 8 & 9 \end{bmatrix}$, $a_2 = \begin{bmatrix} 9 & 9 & 9 \end{bmatrix}$, $p'_1 = \begin{bmatrix} 19 & 32 & 27 \\ -5 & 8 & 27 \\ 1 & 0 & 9 \end{bmatrix}$,

$p'_2 = \begin{bmatrix} 27 & 27 & 27 \\ 27 & 27 & 27 \\ 18 & 17 & 18 \end{bmatrix}$, $p'_3 = \begin{bmatrix} 27 & 27 & 27 \\ 27 & 27 & 27 \\ 18 & 18 & 18 \end{bmatrix}$,

and $\mathbf{0}$, based on its location in this matrix is a $3 \times 3$ zero matrix or a 3-vector. It is noteworthy that the operational matrix is not presented appropriately in [23].

As well

$$\mathbf{P}'_s = \begin{bmatrix} 0 & a_{s_1} & a_{s_2} & a_{s_2} & a_{s_2} & \dots & a_{s_2} \\ \mathbf{0} & p'_{s_1} & p'_{s_2} & p'_{s_3} & p'_{s_3} & \dots & p'_{s_3} \\ \mathbf{0} & \mathbf{0} & p'_{s_1} & p'_{s_2} & p'_{s_3} & \dots & p'_{s_3} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & p'_{s_1} & p'_{s_2} & \dots & p'_{s_3} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & p'_{s_1} & \dots & p'_{s_3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & p'_{s_1} \end{bmatrix},$$

where $a_{s_1} = \begin{bmatrix} \theta_{0,1} & \theta_{0,2} & \theta_{0,3} \end{bmatrix}$, $a_{s_2} = \begin{bmatrix} \theta_{0,3} & \theta_{0,3} & \theta_{0,3} \end{bmatrix}$, and

$p'_{s_1} = \begin{bmatrix} B(ih) + \theta_{i,i} & \theta_{i,i+1} & \theta_{i,i+2} \\ \delta_{i,i-1} & B(ih) + \delta_{i,i} & \delta_{i,i+1} \\ \xi_{i,i-2} & \xi_{i,i-1} & B(ih) + \xi_{i,i} \end{bmatrix}$,

$p'_{s_2} = \begin{bmatrix} \theta_{i,i+2} & \theta_{i,i+2} & \theta_{i,i+2} \\ \delta_{i,i+1} & \delta_{i,i+1} & \delta_{i,i+1} \\ \xi_{i,i+1} & \xi_{i,i+2} & \xi_{i,i+3} \end{bmatrix}$, $p'_{s_3} = \begin{bmatrix} \theta_{i,i+2} & \theta_{i,i+2} & \theta_{i,i+2} \\ \delta_{i,i+1} & \delta_{i,i+1} & \delta_{i,i+1} \\ \xi_{i,i+3} & \xi_{i,i+3} & \xi_{i,i+3} \end{bmatrix}$,

with

$\theta_{0,j}(h) = \dfrac{1}{6h^3}\displaystyle\int_0^{jh} (3\tau^2 - 12\tau h + 11h^2)B(\tau)\, d\tau$, $j = 1, 2, 3.$

if $i = 3k - 2$ and $1 \le k \le \frac{n}{3}$



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$\theta_{i,j}(h) = -\frac{1}{2h^3}\int_{(i-1)h}^{jh}(3\tau^2 - (6i+4)h\tau + (3i^2+4i-1)h^2)B(\tau)\,d\tau,$
$j = i, i+1, i+2,$

if $i = 3k-4$ and $2 \leq k \leq \frac{n}{3}+1$
$\delta_{i,j}(h) = \frac{1}{2h^3}\int_{(i-2)h}^{jh}(3\tau^2 - (6i-4)h\tau + (3i^2-4i-1)h^2)B(\tau)\,d\tau,$
$j = i-1, i, i+1,$
if $i = 3k$ and $1 \leq k \leq \frac{n}{3}-1$
$\xi_{i,j}(h) = \frac{-1}{6h^3}\int_{(i-3)h}^{jh}(3\tau^2 - (6i-12)h\tau + (3i^2-12i+11)h^2)B(\tau)\,d\tau,$
$j = i-2, i-1, i,$
and

$$\begin{aligned}\xi_{i,j}(h) = \frac{-1}{6h^3}\Big(&\int_{(i-3)h}^{ih}(3\tau^2 - (6i-12)h\tau + (3i^2-12i+11)h^2)B(\tau)\,d\tau\\ &-\int_{ih}^{jh}(3\tau^2 - (6i+12)h\tau + (3i^2+12i+11)h^2)B(\tau)\,d\tau\Big),\end{aligned}$$

$j = i+1, i+2, i+3,$
and $\mathbf{0}$, based on its location in this matrix is a $3\times 3$ zero matrix or a 3-vector. It is noteworthy that the stochastic operational matrix is not presented well in [23].

**Theorem 3.1.** *Let* $\mathbf{M}(t)$ *be the vector defined in relation* (3.2), *then*

$$\int_0^T \mathbf{M}(\tau)\mathbf{M}^T(\tau)\,d\tau = \boldsymbol{P}'_*,$$

*where* $\boldsymbol{P}'_*$ *is the following* $(n+1)\times(n+1)$ *matrix*

$$\boldsymbol{P}'_* = \frac{h}{35}\begin{bmatrix}
8 & \frac{99}{16} & \frac{-9}{4} & \frac{19}{16} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\frac{99}{16} & \frac{81}{2} & \frac{-81}{16} & \frac{-9}{4} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\frac{-9}{4} & \frac{-81}{16} & \frac{81}{2} & \frac{99}{16} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\frac{19}{16} & \frac{-9}{4} & \frac{99}{16} & 16 & \frac{99}{16} & \frac{-9}{4} & \frac{19}{16} & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & \frac{99}{16} & \frac{81}{2} & \frac{-81}{16} & \frac{-9}{4} & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & \frac{-9}{4} & \frac{-81}{16} & \frac{81}{2} & \frac{99}{16} & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & \frac{19}{16} & \frac{-9}{4} & \frac{99}{16} & 16 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\vdots & & & \ddots & \ddots & \ddots & \ddots & \ddots & & & & & & & \vdots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \frac{19}{16} & \frac{-9}{4} & \frac{99}{16} & 16 & \frac{99}{16} & \frac{-9}{4} & \frac{19}{16}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \frac{99}{16} & \frac{81}{2} & \frac{-81}{16} & \frac{-9}{4}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \frac{-9}{4} & \frac{-81}{16} & \frac{81}{2} & \frac{99}{16}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \frac{19}{16} & \frac{-9}{4} & \frac{99}{16} & 8
\end{bmatrix},$$

*Proof.* The proof is easy and it comes after integrating the elements of $\mathbf{M}(t)\mathbf{M}^T(t)$ from 0 to $T$. □

## 4. Method of solution

As stated in the introduction, our aim is to solve the next stochastic Volterra-Fredholm integral equation

$$X(t) = f(t) + \int_{\alpha}^{\beta} K_1(s,t)X(s)ds + \int_0^t K_2(s,t)X(s)\,ds + \int_0^t K_3(s,t)X(s)\,dB(s), \tag{4.1}$$

where $t \in [0,T)$. Without loss of generality we can set $[0,T]$ instead of $[\alpha,\beta]$. By approximating $X, f, K_1, K_2, K_3$ through GHFs expansions as mentioned in relations (2.2) and (2.3), we have

$$X(t) \simeq \mathbf{X}^T\mathbf{H}(t) = \mathbf{H}(t)^T\mathbf{X},$$

$$f(t) \simeq \mathbf{F}^T\mathbf{H}(t) = \mathbf{H}(t)^T\mathbf{F},$$

$$K_1(s,t) \simeq \mathbf{H}(t)^T\mathbf{K}_1^T\mathbf{H}(s) = \mathbf{H}(s)^T\mathbf{K_1}\mathbf{H}(t),$$

$$K_2(s,t) \simeq \mathbf{H}(t)^T\mathbf{K_2}^T\mathbf{H}(s) = \mathbf{H}(s)^T\mathbf{K_2}\mathbf{H}(t),$$

$$K_3(s,t) \simeq \mathbf{H}(t)^T\mathbf{K_3}^T\mathbf{H}(s) = \mathbf{H}(s)^T\mathbf{K_3}\mathbf{H}(t).$$

Substituting above approximations in equation (4.1), we obtain

$$\begin{aligned}\mathbf{H}(t)^T\mathbf{X} \simeq \mathbf{H}(t)^T\mathbf{F} + \int_0^T \mathbf{H}(t)^T\mathbf{K}_1^T\mathbf{H}(s)\mathbf{H}(s)^T\mathbf{X}\,ds \\ + \int_0^t \mathbf{H}(t)^T\mathbf{K}_2^T\mathbf{H}(s)\mathbf{H}(s)^T\mathbf{X}\,ds + \int_0^t \mathbf{H}(t)^T\mathbf{K}_3^T\mathbf{H}(s)\mathbf{H}(s)^T\mathbf{X}\,dB(s),\end{aligned}$$

by applying the 6-th property of GHFs and Theorem 2.1, we have

$$\begin{aligned}\mathbf{H}(t)^T\mathbf{X} \simeq \mathbf{H}(t)^T\mathbf{F} + \mathbf{H}^T(t)\mathbf{K}_1^T\mathbf{P}_*\mathbf{X} + \mathbf{H}^T(t)\mathbf{K}_2^T diag(\mathbf{X})\left(\int_0^t \mathbf{H}(s)\,ds\right) \\ + \mathbf{H}^T(t)\mathbf{K}_3^T diag(\mathbf{X})\left(\int_0^t \mathbf{H}(s)\,dB(s)\right),\end{aligned} \tag{4.2}$$

where $\mathbf{P}_*$ is defined in relation (2.6). Using operational matrices defined in relations (2.4) and (2.5) , we get

$$\begin{aligned}\mathbf{H}(t)^T\mathbf{X} \simeq \mathbf{H}(t)^T\mathbf{F} + \mathbf{H}^T(t)\mathbf{K}_1^T\mathbf{P}_*\mathbf{X} + \mathbf{H}^T(t)\mathbf{K}_2^T diag(\mathbf{X})\mathbf{P}\mathbf{H}(t) \\ + \mathbf{H}^T(t)\mathbf{K}_3^T diag(\mathbf{X})\mathbf{P}_s\mathbf{H}(t).\end{aligned}$$

The following relation is obtained by using property 7 of GHFs in the previous relation,

$$\mathbf{H}^T(t)\mathbf{X} \simeq \mathbf{H}^T(t)\mathbf{F} + \mathbf{H}^T\mathbf{K}_1^T\mathbf{P}_*\mathbf{X} + \mathbf{H}^T(t)\tilde{\mathbf{A}} + \mathbf{H}^T(t)\tilde{\mathbf{B}},$$



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where $\mathbf{A} = \mathbf{K}_2^T diag(\mathbf{X})\mathbf{P}$ and $\mathbf{B} = \mathbf{K}_3^T diag(\mathbf{X})\mathbf{P_s}$. Eliminating $\mathbf{H}^T(t)$ and replacing $\simeq$ by $=$, we obtain

$$\mathbf{X} = \mathbf{F} + \mathbf{K}_1^T\mathbf{P}_*\mathbf{X} + \tilde{\mathbf{A}} + \tilde{\mathbf{B}}, \tag{4.3}$$

which is a linear system of equations that its solution is easily found by mathematical softwares.
The solution method of IHFs is just like GHFs with the difference that the basic functions and their operational matrices are changed to $\mathbf{M}, \mathbf{P}', \mathbf{P}'_s$ and $\mathbf{P}'_*$.

## 5. Error analysis

In this section, we prove that the rate of convergence for GHFs and IHFs methods are $O(h^2)$ and $O(h^4)$ respectively, in solving stochastic Volterra-Fredholm integral equation.

**Theorem 5.1.** [21] *Let $f \in C^2([0,T))$ and $e_n(t) = f(t) - f_n(t)$, $t \in [0,T)$, where $f_n(t) = \sum_{i=0}^n f(ih)h_i(t)$ is the GHFs expansion of $f$, then we have*

$$\|e_n\| \leq \frac{h^2}{2}\|f^{(2)}\|,$$

*where $\|.\|$ denotes the sup-norm.*

**Theorem 5.2.** [21] *Let $g(s,t) \in C^2([0,T) \times [0,T))$ and $e_n(s,t) = g(s,t) - g_n(s,t)$ for $(s,t) \in [0,T)\times[0,T)$, where $g_n(s,t) = \sum_{i=0}^n \sum_{j=0}^n g(ih,jh)h_i(s)h_j(t)$, is the GHFs expansion of $g(s,t)$, then we have*

$$\|e_n\| \leq \frac{h^2}{2}\left(\|f_s^{(2)}\| + 2\|f_{s,t}^{(1+1)}\| + \|f_t^{(2)}\|\right),$$

*and so $\|e_n\| = O(h^2)$.*

**Theorem 5.3.** *Let $X$ be the exact solution of equation (4.1) and $\hat{Z}_n$ be the solution by GHFs method then*

$$\|X - \hat{Z}_n\| \leq \frac{h^2}{2}\left(\|L^{-1}\|(1 + (\beta-\alpha)N_1 + TN_2 + N_3\|B\|)\|X^{(2)}\| + \|Z^{(2)}\|\right),$$

*where $X(t) \simeq \hat{X}_n(t)$ and $Z(t) \simeq \hat{Z}_n(t)$, so $\|X - \hat{Z}_n\| = O(h^2)$, where $N_i = \sup_{s,t\in[0,T)} |K_i(s,t)|$ for $i = 1,2,3$ and $L$ is the matrix that satisfies in relation (4.3) as $L\mathbf{X} = F$.*

*Proof.* We know

$$\|X - \hat{Z}_n\| \leq \|X - Z\| + \|Z - \hat{Z}_n\|, \tag{5.1}$$

by the fact that $X(t) = L^{-1}f(t)$ and $Z(t) = L^{-1}\hat{f}(t)$, we obtain

$$\|X - Z\| \leq \|L^{-1}\|\|f - \hat{f}\|. \tag{5.2}$$

Applying inequality (5.2) and Theorem 5.1 in relation (5.1), we get

$$\|X - \hat{Z}_n\| \leq \|L^{-1}\|\|f - \hat{f}\| + \frac{h^2}{2}\|Z^{(2)}\|. \tag{5.3}$$

From the original equation (4.1) and for $t \in [0, T)$, we have

$$f(t) = X(t) - \int_\alpha^\beta K_1(s,t)X(s)ds - \int_0^t K_2(s,t)X(s)\,ds - \int_0^t K_3(s,t)X(s)\,dB(s),$$

and

$$\hat{f}(t) = \hat{X}_n(t) - \int_\alpha^\beta K_1(s,t)\hat{X}_n(t)ds - \int_0^t K_2(s,t)\hat{X}_n(t)\,ds - \int_0^t K_3(s,t)\hat{X}_n(t)\,dB(s).$$

Thus, we obtain

$$\begin{aligned} f(t) - \hat{f}(t) &= X(t) - \hat{X}_n(t) - \int_\alpha^\beta K_1(s,t)(X(s) - \hat{X}_n(s))ds \\ &\quad - \int_0^t K_2(s,t)(X(s) - \hat{X}_n(s))\,ds - \int_0^t K_3(s,t)(X(s) - \hat{X}_n(s))\,dB(s), \end{aligned}$$

therefore

$$\begin{aligned} \sup_{t\in[0,T)} |f(t) - \hat{f}(t)| &\leq \sup_{t\in[0,T)} |X(t) - \hat{X}_n(t)| + \sup_{t\in[0,T)} \left|\int_\alpha^\beta K_1(s,t)(X(s) - \hat{X}_n(s))ds\right| \\ &+ \sup_{t\in[0,T)} \left|\int_0^t K_2(s,t)(X(s) - \hat{X}_n(s))\,ds\right| + \sup_{t\in[0,T)} \left|\int_0^t K_3(s,t)(X(s) - \hat{X}_n(s))\,dB(s)\right|. \end{aligned}$$

Hence, the following inequality is obtained

$$\begin{aligned} \sup_{t\in[0,T)} |f(t) - \hat{f}(t)| &\leq \sup_{t\in[0,T)} |X(t) - \hat{X}_n(t)| + \sup_{t\in[0,T)} \int_\alpha^\beta |K_1(s,t)||X(s) - \hat{X}_n(s)|ds \\ &+ \sup_{t\in[0,T)} \int_0^t |K_2(s,t)||X(s) - \hat{X}_n(s)|\,ds + \sup_{t\in[0,T)} \left|\int_0^t |K_3(s,t)||X(s) - \hat{X}_n(s)|\,dB(s)\right|, \end{aligned}$$

thus

$$\begin{aligned} \sup_{t\in[0,T)} |f(t) - \hat{f}(t)| &\leq \sup_{t\in[0,T)} |X(t) - \hat{X}_n(t)| + (\beta - \alpha)N_1 \sup_{t\in[0,T)} |X(t) - \hat{X}_n(t)| \\ &\quad + TN_2 \sup_{t\in[0,T)} |X(t) - \hat{X}_n(t)| + N_3 \sup_{t\in[0,T)} |B(t)| \sup_{t\in[0,T)} |X(t) - \hat{X}_n(t)| \\ &\leq \frac{h^2}{2}\|X^{(2)}\|(1 + (\beta - \alpha)N_1 + TN_2 + N_3\|B\|). \end{aligned} \tag{5.4}$$

Using relation (5.4) in inequality (5.3), we have

$$\|X - \hat{Z}_n\| \leq \frac{h^2}{2}\|L^{-1}\|\|X^{(2)}\|(1 + (\beta - \alpha)N_1 + TN_2 + N_3\|B\|) + \frac{h^2}{2}\|Z^{(2)}\|,$$

therefore

$$\|X - \hat{Z}_n\| \leq \frac{h^2}{2}(\|L^{-1}\|(1 + (\beta - \alpha)N_1 + TN_2 + N_3\|B(t)\|)\|X^{(2)}\| + \|Z^{(2)}\|),$$



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and this completes the proof. □

IHFs error analysis is reviewed in next theorems.

**Theorem 5.4.** [23] *Let $t_j = jh, j = 0, 1, \ldots, n, f \in C^4([0,T))$ and $f_n(t)$ be the IHFs expansion of $f(t)$, that is defined as $f_n(t) = \sum_{j=0}^{n} f(t_j)m_j(t)$. Also assume that $e_n(t) = f(t) - f_n(t)$, for $t \in [0,T)$, then we have*

$$\|e_n\| \leq \frac{3h^4}{128}\|f^{(4)}\|.$$

**Theorem 5.5.** [23] *Let $s_i = t_i = ih$, $i = 0, 1, \ldots, n, K \in C^4([0,T) \times [0,T))$ and $K_n(s,t) = \sum_{i=0}^{n}\sum_{j=0}^{n} K(s_i,t_j)m_i(s)m_j(t)$, be the IHFs expansion of $K(s,t)$. Also assume that $e_n(s,t) = K(s,t) - K_n(s,t)$ for $s, t \in [0,T)$, then we have*

$$\|e_n\| \leq \frac{3h^4}{128}\left(\|K_s^{(4)}\| + \|K_t^{(4)}\|\right) + \frac{9h^8}{16384}\|K_{s,t}^{(4+4)}\|.$$

**Theorem 5.6.** *Let $X(t)$ be the exact solution of equation (4.1) and $\hat{Y}_n(t)$ be the solution obtained by the proposed method. So $X(t) \simeq \hat{X}_n(t)$ and $Y(t) \simeq \hat{Y}_n(t)$:*

$$\|X - \hat{Y}_n\| \leq \frac{3h^4}{128}(\|L^{-1}\|(1 + (\beta - \alpha)M_1 + TM_2 + M_3\|B(t)\|)\|X^{(4)}\| + \|Y^{(4)}\|)$$

*for $t \in [0,T)$ and hence, $\|X - \hat{Y}_n\| = O(h^4)$, where $M_i = \sup\limits_{s,t\in[0,T)} |K_i(s,t)|$ for $i = 1, 2, 3$ and $L$ is the matrix that satisfies in relation (4.3) as $L\mathbf{X} = F$.*

*Proof.* From equation (4.1), we have

$$f(t) = X(t) - \int_\alpha^\beta K_1(s,t)X(s)ds - \int_0^t K_2(s,t)X(s)\,ds - \int_0^t K_3(s,t)X(s)\,dB(s),$$

and

$$\hat{f}(t) = \hat{X}_n - \int_\alpha^\beta K_1(s,t)\hat{X}_n(t)ds - \int_0^t K_2(s,t)\hat{X}_n(t)\,ds - \int_0^t K_3(s,t)\hat{X}_n(t)\,dB(s),$$

for all $t \in [0,T)$, where $\hat{X}_n(t)$ is the expansion of $X(t)$ by IHFs. So

$$\begin{aligned} f(t) - \hat{f(t)} = X(t) - \hat{X}_n(t) - \int_\alpha^\beta K_1(s,t)(X(s) - \hat{X}_n(s))ds \\ -\int_0^t K_2(s,t)(X(s)-\hat{X}_n(s))\,ds - \int_0^t K_3(s,t)(X(s)-\hat{X}_n(s))\,dB(s), \end{aligned}$$

therefore

$$\begin{aligned} \sup_{t\in[0,T)}|f(t)-\hat{f}(t)| \leq \sup_{t\in[0,T)}|X(t)-\hat{X}_n(t)| + \sup_{t\in[0,T)}\left|\int_\alpha^\beta K_1(s,t)(X(s)-\hat{X}_n(s))ds\right| \\ + \sup_{t\in[0,T)}\left|\int_0^t K_2(s,t)(X(s)-\hat{X}_n(s))\,ds\right| + \sup_{t\in[0,T)}\left|\int_0^t K_3(s,t)(X(s)-\hat{X}_n(s))\,dB(s)\right|, \end{aligned}$$



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thus

$$\sup_{t\in[0,T)}|f(t)-\hat{f}(t)|\leq\sup_{t\in[0,T)}|X(t)-\hat{X}_n(t)|+\sup_{t\in[0,T)}\int_\alpha^\beta|K_1(s,t)||X(s)-\hat{X}_n(s)|ds$$
$$+\sup_{t\in[0,T)}\int_0^t|K_2(s,t)||X(s)-\hat{X}_n(s)|\,ds+\sup_{t\in[0,T)}|\int_0^t|K_3(s,t)||X(s)-\hat{X}_n(s)|\,dB(s).$$

Applying the assumptions, we have

$$\sup_{t\in[0,T)}|f(t)-\hat{f}(t)|\leq\sup_{t\in[0,T)}|X(t)-\hat{X}_n(t)|+(\beta-\alpha)M_1\sup_{t\in[0,T)}|X(t)-\hat{X}_n(t)|$$
$$+TM_2\sup_{t\in[0,T)}|X(t)-\hat{X}_n(t)|+M_3\sup_{t\in[0,T)}|B(t)|\sup_{t\in[0,T)}|X(t)-\hat{X}_n(t)|$$
$$\leq\frac{3h^4}{128}\|X^{(4)}\|(1+(\beta-\alpha)M_1+TM_2+M_3\|B\|). \tag{5.5}$$

By the fact that $X(t)=L^{-1}f(t)$ and $Y(t)=L^{-1}\hat{f}(t)$, we obtain

$$\sup_{t\in[0,T)}|X(t)-Y(t)|\leq\|L^{-1}\|\sup_{t\in[0,T)}|f(t)-\hat{f}(t)|. \tag{5.6}$$

We also have

$$\sup_{t\in[0,T)}|X(t)-\hat{Y}_n(t)|\leq\sup_{t\in[0,T)}|X(t)-Y(t)|+\sup_{t\in[0,T)}|Y(t)-\hat{Y}_n(t)|. \tag{5.7}$$

Applying inequalities (5.5) and (5.6) in (5.7), we get

$$\sup_{t\in[0,T)}|X(t)-\hat{Y}_n(t)|\leq\|L^{-1}\|\frac{3h^4}{128}\|X^{(4)}\|(1+(\beta-\alpha)M_1+TM_2+M_3\|B\|)$$
$$+\frac{3h^4}{128}\|Y^{(4)}\|,$$

hence

$$\|X-\hat{Y}_n\|\leq\frac{3h^4}{128}(\|L^{-1}\|(1+(\beta-\alpha)M_1+TM_2+M_3\|B\|)\|X^{(4)}\|+\|Y^{(4)}\|),$$

and the proof is complete. □

## 6. Numerical examples

To show the accuracy of these two methods, we consider some examples. The computations associated with the examples are performed using Matlab 7 and [31].

Example 6.1. Let [17]

$$X(t)=f(t)+\int_0^1\cos(s+t)X(s)\,ds+\int_0^t(s+t)X(s)ds$$
$$+\int_0^t exp(-3(s+t))X(s)dB(s)$$



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be a linear stochastic Volterra-Fredholm integral equation and $s, t \in [0, 1)$, $f(t) = t^2 + \sin(1+t) - 2\cos(1+t) - 2\sin(t) - \frac{7t^4}{12} + \frac{1}{40}B(t)$. $X(t)$ is an unknown stochastic process defined on the probability space$(\Omega, F, P)$ and $B(t)$ is a Brownian motion process. The numerical results for the above mentioned basic functions for $m = 15$, $m = 30$ and $k = 20$ are inserted in TABLEs 1 and 2, where $k$ is the number of iterations. According to the error analysis studied in Section 5 and the numerical results shown in TABLEs 1 and 2, it can be concluded that IHFs method is more accurate than BPFs and GHFs. Also, the number of basic functions has an important role in accuracy. Curves in FIGUREs 1 and 2 show the solutions computed by GHFs and IHFs for $m = 15$ and $m = 30$.

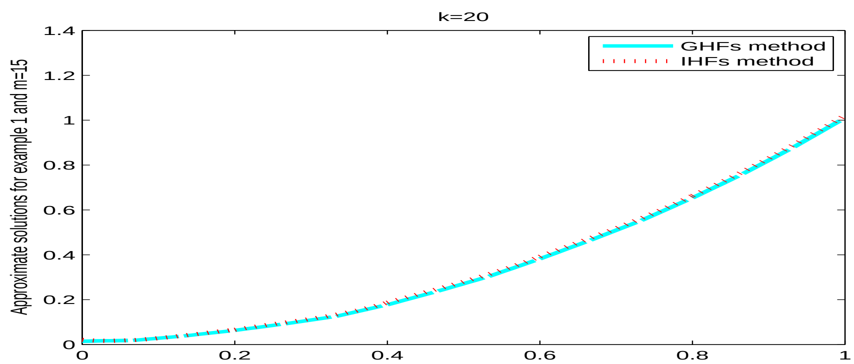


FIGURE 1. Numerical results for Example 7.1 by GHFs and IHFs methods with m=15.

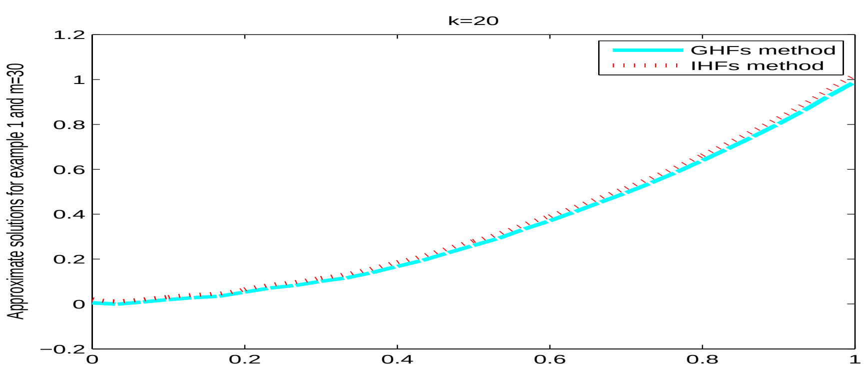


FIGURE 2. Numerical results for Example 7.1 by GHFs and IHFs methods with m=30.



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| $nodes\, t_i$ | $m = 15$ | | |
|---|---|---|---|
| | $BPFs\, in\, [17]$ | $GHFs$ | $IHFs$ |
| 0 | 0.0189981383 | 0.0153842767 | 0.0210118224 |
| 0.1 | 0.0443427347 | 0.0285856363 | 0.0286495434 |
| 0.2 | 0.1036134219 | 0.0633603756 | 0.0660030915 |
| 0.3 | 0.1036134219 | 0.1079519530 | 0.1148579113 |
| 0.4 | 0.2223270323 | 0.1775973036 | 0.1869844526 |
| 0.5 | 0.1666985014 | 0.2549383742 | 0.2801345118 |
| 0.6 | 0.4314041184 | 0.3819749524 | 0.3918766851 |
| 0.7 | 0.2854952711 | 0.4666445831 | 0.5182189623 |
| 0.8 | 0.7172045224 | 0.6534792502 | 0.6633343506 |
| 0.9 | 0.4314041184 | 0.7582855445 | 0.8301212786 |
| 1 | 0.6106764696 | 1.1040791318 | 1.0214181806 |

Table 1. Numerical results for Example 7.1 and $k = 20$ and $m = 15$.

| $nodes\, t_i$ | m=30 | | |
|---|---|---|---|
| | $BPFs\, in\, [17]$ | $GHFs$ | $IHFs$ |
| 0 | −0.0232084950 | 0.0052161010 | 0.0202046201 |
| 0.1 | −0.0016913095 | 0.0194995661 | 0.0311602020 |
| 0.2 | 0.0352166406 | 0.0531067848 | 0.0652089729 |
| 0.3 | 0.0866090271 | 0.1011729439 | 0.1160211920 |
| 0.4 | 0.1575206061 | 0.1675756213 | 0.1862674199 |
| 0.5 | 0.2523749931 | 0.2608108343 | 0.2803509939 |
| 0.6 | 0.3620734327 | 0.3708175101 | 0.3909921800 |
| 0.7 | 0.4852043944 | 0.4953250063 | 0.5163329361 |
| 0.8 | 0.6290333441 | 0.6393069770 | 0.6621374997 |
| 0.9 | 0.7913292795 | 0.8025816039 | 0.8283788832 |
| 1 | 0.9160050969 | 0.9834906651 | 1.0199870218 |

Table 2. Numerical results for Example 7.1 and $k = 20$ and $m = 30$.



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Example 6.2. Let [17]

$$X(t) = f(t) + \int_0^1 (s+t)X(s)\,ds + \int_0^t (s-t)X(s)ds + \int_0^t \frac{1}{125}\sin(s+t))X(s)dB(s),$$

be a linear stochastic Volterra-Fredholm integral equation and $s, t \in [0,1)$, $f(t) = 2 - \cos(1) - (1+t)\sin(1) + \frac{1}{250}\sin(B(t))$. $X(t)$ is an unknown stochastic process defined on the probability space$(\Omega, F, P)$, and $B(t)$ is a Brownian motion process. The numerical results are inserted in TABLEs 3 and 4 for $m = 15$, $m = 30$ and $k = 20$, where $k$ is the number of iterations. Also curves in FIGUREs 3 and 4 show the solutions computed by GHFs and IHFs method for $m = 15$ and $m = 30$.

| | $m=15$ | | |
|---|---|---|---|
| $nodes\, t_i$ | $BPFs\, in\, [17]$ | $GHFs$ | $IHFs$ |
| 0 | 0.8887010585 | 1.0043538668 | 1.0067965374 |
| 0.1 | 0.8739398501 | 0.9994807132 | 1.0028942472 |
| 0.2 | 0.8330559947 | 0.9860670105 | 0.9893671679 |
| 0.3 | 0.8330559947 | 0.9641075472 | 0.9658206983 |
| 0.4 | 0.7446350666 | 0.9293127652 | 0.9332182224 |
| 0.5 | 0.7770668749 | 0.8953335497 | 0.8915672703 |
| 0.6 | 0.6264915336 | 0.8357127003 | 0.8401305971 |
| 0.7 | 0.7086942516 | 0.7924107845 | 0.7791483715 |
| 0.8 | 0.4814490330 | 0.7059055898 | 0.7108899589 |
| 0.9 | 0.6264915336 | 0.6572112277 | 0.6359036879 |
| 1 | 0.5314586850 | 0.4984763159 | 0.5552917396 |

Table 3. Numerical results for Example 7.2 and $k = 20$ and $m = 15$.

| $nodes\, t_i$ | m=30 | | |
|---|---|---|---|
| | $BPFs\, in\, [17]$ | $GHFs$ | $IHFs$ |
| 0 | 0.9333907554 | 1.0064981428 | 1.0070405950 |
| 0.1 | 0.9170933336 | 1.0024669678 | 1.0031060950 |
| 0.2 | 0.8921163336 | 0.9889465339 | 0.9896906808 |
| 0.3 | 0.8581147160 | 0.9655639138 | 0.9663324513 |
| 0.4 | 0.8159938269 | 0.9328257670 | 0.9336072689 |
| 0.5 | 0.7657945036 | 0.8911282982 | 0.8919054816 |
| 0.6 | 0.7074524310 | 0.8397311733 | 0.8405694614 |
| 0.7 | 0.6404782622 | 0.7787328004 | 0.7797028588 |
| 0.8 | 0.5683453181 | 0.7102683321 | 0.7113601989 |
| 0.9 | 0.4903056104 | 0.6350338002 | 0.6363025241 |
| 1 | 0.4362719187 | 0.5570677486 | 0.5557721735 |

TABLE 4. Numerical results for Example 7.2 and $k = 20$ and $m = 30$.

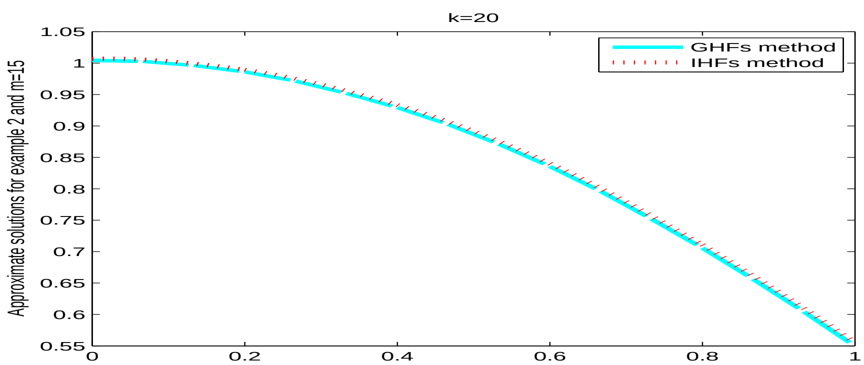


FIGURE 3. Numerical results for Example 7.2 by GHFs and IHFs methods with m=15.



[ DOI: 10.52547/ijmsi.18.1.145 ]

## 7. Conclusion

In this paper, computational methods based on GHFs and IHFs were proposed to solve stochastic Volterra-Fredholm integral equation. Substituting the approximations of all known and unknown functions in the original equation and applying operational matrices resulted in a linear system of algebraic equations which were simply solved by mathematical softwares. Convergence and error

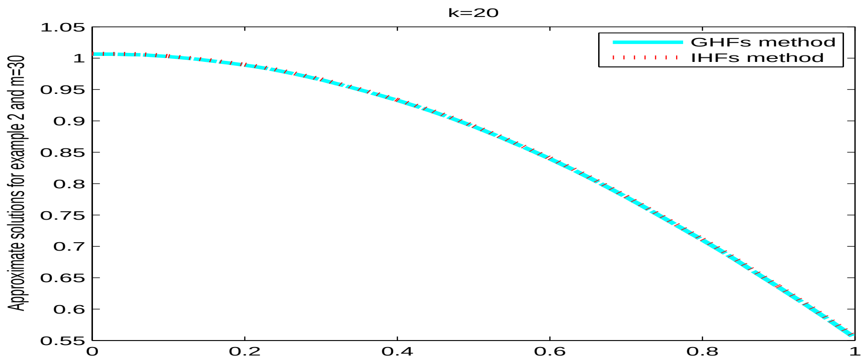


Figure 4. Numerical results for Example 7.2 by GHFs and IHFs methods with m=30.

analysis of these two methods were investigated. According to the error analysis studied in Section 5, IHFs and GHFs rate of convergence are $O(h^4)$ and $O(h^2)$ respectively. So, it can be concluded that IHFs is more accurate than GHFs and BPFs.

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