Iranian Journal of Mathematical Sciences and Informatics
Vol. 17, No. 2 (2022), pp 75-86
DOI: 10.52547/ijmsi.17.2.75

Linear Formulas in Continuous Logic

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Abstract. We prove that continuous sentences preserved by the ultramean construction (a generalization of the ultraproduct construction) are exactly those sentences which are approximated by linear sentences. Continuous sentences preserved by linear elementary equivalence are exactly those sentences which are approximated in the Riesz space generated by linear sentences. Also, characterizations for linear Δ_n-sentences and positive linear theories will be given.

Keywords: Continuous logic, Ultramean, Linear formula, Σ_n-formula, Positive formula.

2000 Mathematics subject classification: 03C40.

1. Introduction

First order model theory is a branch of mathematical logic which studies algebraic structures by logical tools. Continuous logic extends these tools and provides a logical framework for study of continuous structures such as metric groups, Banach spaces etc (see [5]). Part of the expressive power of first order logic is related to the ability to use arbitrary finitary connectives. In fact, the system $\{\wedge, \neg\}$ is complete and generates other connectives such as $\vee$ and $\rightarrow$.

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Received 19 October 2017; Accepted 15 October 2021

So, for example, the formula $x \neq 0 \to \exists y(xy = 1)$ states that the intended ring is a field. A similar (conditional) formula states that the intended field is algebraically closed. Continuous logic uses a similar complete system of connectives and generates a relatively strong expressive power for continuous structures. For example, using $\times$ as a connective, the parallelogram law

$$2\|x\|^2 + 2\|y\|^2 = \|x + y\|^2 + \|x - y\|^2$$

states that the intended Banach space is indeed a Hilbert space. Similarly, using absolute value as a connective, one can state that the given probability algebra is atomless.

Linear continuous logic is the sublogic of continuous logic obtained by restricting the connectives to addition and scalar multiplication (see below), hence reducing the expressive power considerably. This linearization leads to the linearization of most basic tools and techniques of continuous logic such as the ultraproduct construction, compactness theorem, saturation etc. (see [4]). Among consequences of the classical compactness theorem are preservation theorems which relate categorical properties of classes of structures to logic. In [3], some linear preservation theorems where deduced from the linear variant of compactness theorem. The goal of the present paper is first to characterize linear formulas among other continuous formulas as those which are preserved by the ultramean construction. It is well-known in first order logic that every Δ_n sentence is equivalent to a Boolean combination of Σ_n sentences. We prove that every linear Δ_n sentence is equivalent to a linear combination of linear Σ_n sentences. We also characterize positive theories as those preserved by surjective expanding homomorphisms. We start with reviewing the main notions and definitions of full and linear continuous logics.

2. Continuous logic

Continuous logic (see [5]) is usually presented as a variant of Łukasiewicz logic where the operations such as $\wedge, \vee$ and $1 - x$ on the unit interval are used as connectives. However, to obtain linear continuous logic as a sublgic of continuous logic, we use algebraic operations on the reals as connectives. Let L be a first order language consisting of function, relation and constant symbols. L is a *Lipschitz language* if it is assigned a Lipschitz constant $\lambda_F \geqslant 0$ to each function symbol F and respectively a Lipschitz constant $\lambda_R \geqslant 0$ as well as a bound $\mathbf{b}_R \geqslant 0$ to each relation symbol R . It is always assumed that L contains a distinguished binary relation symbol d which plays the role of $=$ in first order logic. Furthermore, $\mathbf{b}_d = 1$ and $\lambda_d = 1$. An L-structure is a metric space (M, d) on which the symbols of L are appropriately interpreted, i.e. for $F \in L$, the function $F^M : M^n \to M$ is λ_F-Lipschitz and for $R \in L$, the relation $R^M : M^n \to \mathbb{R}$ is λ_R-Lipschitz with $\|R^M\|_\infty \leqslant \mathbf{b}_R$. In particular, we must have that $diam(M) \leqslant 1$. Let L be a Lipschitz language. L-formulas are

[DOI: 10.52547/ijmsi.17.2.75]

defined as follows:

$$r, \quad d(t_1,t_2), \quad R(t_1,...,t_n), \quad \phi+\psi, \quad ,\phi\wedge\psi, \quad \phi\vee\psi$$

where $r\in\mathbb{R}$, $R\in L$ and $t_1,...t_n$ are L-terms. A formula without free variable is called a sentence. Expressions of the form $\phi\leqslant\psi$ are called *conditions*. A theory is a set of closed (without free variable) conditions. A formula in which the connectives $\wedge,\vee$ do not appear is called a *linear formula*. If $\phi(\bar{x})$ is a formula and M is a structure, the real value $\phi^M(\bar{a})$ is defined by induction on the complexity of ϕ. On can easily check that every map $\phi^M: M^n\to\mathbb{R}$ is bounded and Lipschitz. A wider framework is obtained if one replaces Lipschitz constants with moduli of uniform continuities (hence deducing that every ϕ^M is uniformly continuous). Here, we restrict ourselves to Lipschitz languages since we mainly deal with linear formulas whose properties are related to Lipschitzness.

The logic based on the set of all formulas (stated above) is called continuous logic. Of course, thanks to the Stone-Weierstrass theorem, this logic is usually presented in an equivalent way where $[0,1]$ is taken as value space and $\{0,1,\frac{x}{2},\dot{-}\}$ is the system of connectives (see [5]). In contrast, by restricting to the class of linear formulas one obtains a weaker logic which we call linear continuous logic. In this logic, linear variants of several classical model theoretic results hold. In particular, the linear compactness theorem holds which will be discussed below. The purpose of the present paper is first to characterize linear formulas among other continuous formulas. Then, to characterize some special sorts of linear formulas (mainly Δ_n formulas and positive formulas) among other linear formulas. We first give a brief review of linear continuous logic. More details can be found in [4].

3. The logic of linear formulas

Two L-structures M,N, are *linearly elementarily equivalent*, $M\equiv_\ell N$, if for every linear sentence σ one has that $\sigma^M=\sigma^N$. The linear variant of elementary embedding is defined similarly. Note that these notions are weaker than the corresponding full versions defined in [5] where all continuous formulas are considered. It is not hard to check that linear variants of elementary joint embedding property and elementary amalgamation property hold.

The linear variant of the ultraproduct construction is the *ultramean construction*. Let $(M_i,d_i)_{i\in I}$ be a family of L-structures and $\wp: P(I)\to[0,1]$ an ultracharge (a maximal finitely additive probability measure on I). First define a pseudo-metric on $\prod_{i\in I}M_i$ by setting (see [9] for the definition of integral)

$$d(a,b)=\int d_i(a_i,b_i)d\wp.$$

Obviously, $d(a,b)=0$ defines an equivalence relation. The equivalence class of (a_i) is denoted by $[a_i]$. Let M be the set of equivalence classes. Then d induces

[DOI: 10.52547/ijmsi.17.2.75]

a metric on M by which is again denoted by d. So, $d([a_i],[b_i]) = \int d_i(a_i,b_i)d\wp$. Define an L-structure on (M,d) as follows:

$$c^M = [c^{M_i}]$$

$$F^M([a_i],...) = [F^{M_i}(a_i,...)]$$

$$R^M([a_i],...) = \int R^{M_i}(a_i,...)d\wp.$$

where $c, F, R \in L$. The structure M is called the ultramean of structures M_i and is denoted by $\prod_\wp M_i$. Note that an ultrafilter $\mathcal{F}$ corresponds to the $0-1$ valued ultracharge $\wp$ where $\wp(A) = 1$ if $A \in \mathcal{F}$ and $= 0$ otherwise. In this case, $\prod_\wp M_i$ is exactly the ultraproduct $\prod_\mathcal{F} M_i$ and by Łoś theorem, for every (linear or nonlinear) L-sentence σ one has that $\sigma^M = \lim_{i,\mathcal{F}} \sigma^{M_i}$. In the general case, we have the following variant of Łoś theorem (see [2]).

Theorem 3.1. *For every linear formula $\phi(x_1,\dots,x_n)$ and $[a_i^1],\dots,[a_i^n] \in M$*

$$\phi^M([a_i^1],\dots,[a_i^n]) = \int \phi^{M_i}(a_i^1,\dots,a_i^n)d\wp.$$

If $M_i = N$ for all i, the ultramean is denoted by $N^\wp$ and is called *power ultramean.* One checks that the map $a \mapsto [a]$, for $a \in N$, is an elementary embedding from N to $N^\wp$ (i.e. preserves linear formulas). Note that if $|N| \geqslant 2$ and $\wp$ is not an ultrafilter, $N^\wp$ is a proper extension of N. Also, if $I = \{1,2\}$ and $\wp(1) = \lambda, \wp(2) = 1-\lambda$ where $\lambda \in [0,1]$, the ultramean of $(M_i)_{i\in I}$ is denoted by $\lambda M_1 + (1-\lambda)M_2$. In this case, for each linear sentence σ we have that

$$\sigma^{\prod_\wp M_i} = \lambda\sigma^{M_1} + (1-\lambda)\sigma^{M_2}.$$

A *condition* is an expression of the form $\phi \leqslant \psi$ where ϕ and ψ are formulas. It is a linear condition if ϕ, ψ are linear formulas. It is a closed condition if ϕ, ψ are sentence. The expression $\phi = \psi$ is an abbreviation for $\{\phi \leqslant \psi, \psi \leqslant \phi\}$. M is model of a closed condition $\phi \leqslant \psi$ if $\phi^M \leqslant \psi^M$. A set of closed linear conditions is called a *linear theory.* The *linear closure* of a theory T is the set of all conditions $\sum_i r_i\phi_i \leqslant \sum_i r_i\psi_i$ where $0 \leqslant r_1,...,r_n$ and $\phi_1 \leqslant \psi_1,\ ...,\ \phi_n \leqslant \psi_n$ belong to T. A linear theory T is *linearly closed* if it coincides with its linear closure. T is *linearly satisfiable* if every condition in its linear closure has a model. It can be proved by a linear variant of Henkin's method (see [3]) that

Theorem 3.2. *(Linear compactness) Every linearly satisfiable linear theory is satisfiable.*

Let Γ be a set of L-formulas. A formula $\phi(\bar{x})$ is approximated by formulas in Γ if for each $\epsilon > 0$, there is a formula $\theta(\bar{x})$ in Γ such that

$$M \vDash |\phi(\bar{a}) - \theta(\bar{a})| \leqslant \epsilon$$

for each model M and $\bar{a} \in M$.

[DOI: 10.52547/ijmsi.17.2.75]

Let call two linear L-sentences σ, η equivalent, $\sigma \equiv \eta$, if $\sigma^M = \eta^M$ for every M. We identify equivalent sentences. Up to this equivalence, the set of linear L-sentences, denoted by $\mathbb{D}$, forms a partially ordered real vector space where $\sigma \leqslant \eta$ if $\sigma^M \leqslant \eta^M$ for every M. It is also normed by

$$\|\sigma\| = \sup_M \sigma^M.$$

A linear theory T is *linearly complete* if for each sentence σ, there is a unique r such that $\phi = r \in T$. In this case, r is denoted by $T(\sigma)$. Then, the function $\sigma \mapsto T(\sigma)$ is linear and positive. Note also that $\sup_{\|\sigma\| \leqslant 1} |T(\sigma)| = 1$. So, the linearly complete linear theory T can be regarded as a positive linear functional on $\mathbb{D}$ with $\|T\|_\infty = 1$. Conversely, it is easy to show by linear compactness theorem that every positive linear functional $T : \mathbb{D} \to \mathbb{R}$ having norm 1 is of this form. So, we may identify linearly complete theories with the norm 1 positive linear functionals on $\mathbb{D}$. Note that, regarding theories as functionals, $M \vDash T$ means that $\sigma^M = T(\sigma)$ for every linear sentence σ. Let $\mathcal{K}$ denote the set of all linearly complete linear theories. So, indeed $\mathcal{K} \subseteq \mathbb{D}^*$. Put the weak* topology of $\mathbb{D}^*$ on $\mathcal{K}$. So, every $T \in \mathcal{K}$ is continuous as a functional.

Proposition 3.3. *$\mathcal{K}$ is a compact convex Hausdorff space.*

Proof. It is clear that for $T_1, T_2 \in \mathcal{K}$ and $0 \leqslant \lambda \leqslant 1$, $\lambda T_1 + (1-\lambda)T_2 \in \mathcal{K}$. So, $\mathcal{K}$ is convex. For compactness, note that $\mathcal{K}$ is a closed subset of the unit ball of $\mathbb{D}^*$ hence compact by Alaoglu's theorem (see [7] Th. 5.18). $\square$

A function $f : \mathcal{K} \to \mathbb{R}$ is called affine if for every $T_1, T_2 \in \mathcal{K}$ and $0 \leqslant \lambda \leqslant 1$

$$f(\lambda T_1 + (1-\lambda)T_2) = \lambda f(T_1) + (1-\lambda) f(T_2).$$

The set of all affine continuous functions on $\mathcal{K}$ is denoted by $A(\mathcal{K})$. This is a Banach space.

Theorem 3.4. *([10] Corollary 1.1.12) Let $\mathcal{K}$ be a compact convex subset of a locally convex space E. Any subspace of $A(\mathcal{K})$ which contains the constants and separates the points of $\mathcal{K}$ is dense in $A(\mathcal{K})$.*

Let ϕ be a (not necessarily linear) sentence in the Lipschitz language L. We say ϕ is preserved by ultrameans if for every ultracharge space $(I, \wp)$ and models M_i, $i \in I$, one has that $\phi^M = \int \phi^{M_i} d\wp$. Linear sentences are preserved by ultrameans. It was proved in [8] that linear continuous logic is maximal with the properties linear compactness and the linear variant of elementary chain theorem. A consequence of maximality is that every formula preserved by ultrameans is approximated by linear formulas. Here, we give a simpler proof for this result using Theorem 3.4. Both proofs are based on the unproved assumption that a linear variant of Shelah-Keisler theorem holds, i.e. if $M \equiv_\ell N$, then there are ultracharges $(I_1, \wp_1)$, $(I_2, \wp_2)$ such that $M^{\wp_1} \simeq N^{\wp_2}$.

[DOI: 10.52547/ijmsi.17.2.75]

Theorem 3.5. *Assume the linear variant of Shelah-Keisler theorem holds. Then, if ϕ is preserved by ultrameans, ϕ is approximated by linear sentences.*

Proof. For each linear sentence σ define a function f_σ on $\mathcal{K}$ by setting

$$f_\sigma(T) = T(\sigma).$$

Clearly, f_σ is affine and continuous. Let

$$X = \{f_\sigma : \ \ \sigma \text{ a linear } L\text{-sentence}\}.$$

X is a linear subspace of $A(\mathcal{K})$ which contains constant functions. Moreover, if $T_1 \neq T_2$, there is a linear sentence σ such that $T_1(\sigma) \neq T_2(\sigma)$. So, $f_\sigma(T_1) \neq f_\sigma(T_2)$. This shows that X separates points. By Theorem 3.4, X is dense in $A(\mathcal{K})$. Define similarly $f_\phi(T) = \phi^M$ where $M \vDash T$. Note that if $M \equiv_\ell N$, by the assumption, for some ultracharges $\wp_1, \wp_2$ one has that $M^{\wp_1} \simeq N^{\wp_2}$. Hence,

$$\phi^M = \phi^{M^{\wp_1}} = \phi^{N^{\wp_2}} = \phi^N.$$

So, f_ϕ is well-defined. Let us show that f_ϕ is affine. Let $\lambda \in [0,1]$ and $T_1, T_2 \in \mathcal{K}$. Let $M_1 \vDash T_1$ and $M_2 \vDash T_2$. Then, $M = \lambda M_1 + (1-\lambda)M_2$ is a model of the theory $\lambda T_1 + (1-\lambda)T_2$. Moreover, since ϕ is preserved by ultrameans, we have that

$$f_\phi(\lambda T_1 + (1-\lambda)T_2) = \phi^M = \lambda\phi^{M_1} + (1-\lambda)\phi^{M_2} = \lambda f_\phi(T_1) + (1-\lambda) f_\phi(T_2).$$

So, f_ϕ is affine. Note also that f_ϕ is continuous, i.e. for each r the sets

$$\{T \in \mathcal{K} : \ f_\phi(T) \leqslant r\}, \qquad \{T \in \mathcal{K} : \ f_\phi(T) \geqslant r\}$$

are closed. For example, assume $T_k \to T$ in the weak* topology and $f_\phi(T_k) \leqslant r$ for each k. We show that $f_\phi(T) \leqslant r$. Take a nonprincipal ultrafilter $\mathcal{F}$ on $\mathbb{N}$. Let $M_k \vDash T_k$ and $M = \prod_{\mathcal{F}} M_k$. Then, we have that $M \vDash T$. As a consequence,

$$f_\phi(T) = \phi^M = \lim_{k,\mathcal{F}} \phi^{M_k} = \lim_{k,\mathcal{F}} f_\phi(T_k) \leqslant r.$$

We conclude that $f_\phi \in A(\mathcal{K})$. So, since X is dense, for each $\epsilon > 0$ there is a linear sentence σ such that for every $T \in \mathcal{K}$, $|f_\phi(T) - f_\sigma(T)| \leqslant \epsilon$. In other words, for every M, $|\phi^M - \sigma^M| \leqslant \epsilon$. $\square$

An L-sentence σ is preserved by linear elementary equivalence if for every M, N, whenever $M \equiv_\ell N$, one has that $\sigma^M = \sigma^N$. Note that if σ, η are preserved by $\equiv_\ell$ then so does $\sigma \wedge \eta$ and $\sigma \vee \eta$. In fact, every sentence in the Riesz space generated by the set of linear sentences is preserved by $\equiv_\ell$. We denote this Riesz space by Λ.

Proposition 3.6. *ϕ is preserved by linear elementary equivalence if and only if it is approximated by the Riesz space Λ generated by the set of linear sentences.*

[DOI: 10.52547/ijmsi.17.2.75]

Proof. As in the proof of Theorem 3.5, for each $\sigma \in \Lambda$, define $f_\sigma : \mathcal{K} \to \mathbb{R}$ by

$$f_\sigma(T) = \sigma^M$$

where $M \vDash T$ is arbitrary. Let

$$X = \{f_\sigma : \ \sigma \in \Lambda\}.$$

Then, X is a sublattice of $\mathbf{C}(\mathcal{K})$ which contains 1 and separates points. In particular, $-f_\sigma = f_{-\sigma}$, $f_\sigma + f_\eta = f_{\sigma+\eta}$ and $f_\sigma \wedge f_\eta = f_{\sigma\wedge\eta}$. Note that, by the assumption, the function $f_\phi(T) = \phi^M$ for $M \vDash T$ is well-defined. Since ϕ is preserved by ultraproducts, it is shown similar to the proof of Proposition 3.5 that f_ϕ is continuous. So, by the lattice version of Stone-Weierstrass theorem (see [1] Th. 9.12), f_ϕ is approximated by elements of Λ. □

In the proof of Theorem 3.5 one needs the linear variant of Shelah-Keisler theorem to show that ϕ is preserved by linear elementary equivalence. So one deduces (without this assumption) that if ϕ is preserved by ultrameans and linear elementary equivalence, then it is approximated by linear sentences.

4. Δ_n sentences

From now on, by formula (sentence, theory etc) we mean a linear one. A formula ϕ is $\Sigma_0 = \Pi_0$ if it contains no quantifiers. A formula ϕ is Σ_{n+1} (resp. Π_{n+1}) if $\phi = \sup_{x_1 \dots x_m} \psi$ (resp. $\phi = \inf_{x_1 \dots x_m} \psi$) where ψ is Π_n (resp. Σ_n). We may extend a bit the terminology and say that ϕ is Σ_n (resp Π_n) if it is equivalent to a Σ_n (resp Π_n) formula. The notion of Σ_n extension generalizes the notion of embedding. If M is a subset of N, then N is a Σ_n extension of M if for each Σ_n-formula $\phi(\bar{x})$ and $\bar{a} \in M$ one has that $\phi^M(\bar{a}) \leqslant \phi^N(\bar{a})$. So, Σ_0 extension is the same as embedding.

Lemma 4.1. *Let $M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots$ be a Σ_n-chain of L-structures. Let $M = \cup_{\alpha<\omega} M_\alpha$. Then*

(i) M is a Σ_n extension of each M_α.

(ii) For each Π_{n+1} sentence ϕ, if $r \leqslant \phi^{M_\alpha}$ for all α, then $r \leqslant \phi^M$.

Proof. (i) The claim holds for $n = 0$. Assume it holds for $n-1$. Let $\phi(\bar{x}) = \sup_{\bar{y}} \psi(\bar{x}, \bar{y})$ where ψ is Π_{n-1}. Let $\phi^{M_\alpha}(\bar{a}) = r$. Then, for each $\epsilon > 0$, there exists $\bar{b} \in M_\alpha$ such that $r - \epsilon \leqslant \psi^{M_\alpha}(\bar{a}, \bar{b})$. Consider the Σ_n chain

$$(M_\alpha, \bar{a}, \bar{b}) \subseteq (M_{\alpha+1}, \bar{a}, \bar{b}) \subseteq \cdots .$$

Since $r - \epsilon \leqslant \psi(\bar{a}, \bar{b})$ holds in every model of this chain, by the induction hypothesis, we have that $r - \epsilon \leqslant \psi^{(M,\bar{a},\bar{b})}(\bar{a}, \bar{b})$. Hence, $r - \epsilon \leqslant \sup_{\bar{y}} \psi^M(\bar{a}, \bar{y})$. Since ϵ is arbitrary, we have that $r \leqslant \sup_{\bar{y}} \psi^M(\bar{a}, \bar{y})$.

(ii) Let $\phi = \inf_{\bar{x}} \psi(\bar{x})$ where ψ is Σ_n. Assume $r \leqslant \phi^{M_\alpha}$ for all α. Let $\bar{a} \in M$. Then $\bar{a} \in M_\beta$ for some β and $r \leqslant \psi^{M_\beta}(\bar{a})$. So, by (i), $r \leqslant \psi^M(\bar{a})$. We conclude that $r \leqslant \phi^M$. □

[DOI: 10.52547/ijmsi.17.2.75]

The following result is the linear variant of Theorem 3.1.11 of [6]:

Theorem 4.2. *The following are equivalent (for $n \geqslant 1$):*
(i) ϕ is approximated by both Σ_{n+1} sentences and Π_{n+1} sentences.
(ii) ϕ is approximated by linear combinations of Σ_n sentences.

Proof. (ii)⇒(i) is easy. (i)⇒(ii): We first prove the following claim for each M, N.

Claim: If $\theta^M = \theta^N$ for each Σ_n sentence θ, then $\phi^M = \phi^N$.
Proof of the claim: Assume M, N satisfy the hypothesis of the claim. We construct a Σ_n-chain

$$M = M_0 \subseteq N_0 \subseteq M_1 \subseteq N_1 \subseteq \cdots \subseteq M_k \subseteq N_k \subseteq \cdots$$

such that for all k

$$M_k \equiv M, \qquad N_k \equiv N. \qquad (1)$$

Suppose that

$$M_0 \subseteq N_0 \subseteq \cdots \subseteq M_m \subseteq N_m$$

has been constructed such that (1) holds for $k \leqslant m$. Let T be the set of all conditions $0 \leqslant \sigma$ holding in N_k where σ is a Σ_n sentence in $L \cup \{c_b : b \in N_m\}$. Clearly, T is linearly closed. For $0 \leqslant \sigma(\bar{b})$ in T, the condition $0 \leqslant \sup_{\bar{y}} \sigma(\bar{y})$ holds in N_m and hence in M_m by (1) and assumption of the claim. So, $T \cup Th(M)$ has a model, say M_{m+1}. One checks that

$$M_0 \subseteq N_0 \subseteq \cdots \subseteq M_m \subseteq N_m \subseteq M_{m+1}$$

is a Σ_n chain. Similarly, one obtain a Σ_n chain

$$M_0 \subseteq N_0 \subseteq \cdots \subseteq M_m \subseteq N_m \subseteq M_{m+1} \subseteq N_{m+1}$$

in which the conditions (1) hold for $k \leqslant m+1$. So, the required infinite chain is obtained. Now, let $r \leqslant \phi$ hold in M. Then, it holds in every M_k. Since ϕ is approximated by Π_{n+1} sentences, by Lemma 4.1 (ii), $r \leqslant \phi$ holds in $\cup M_k = \cup N_k$. Suppose $r \leqslant \phi$ does not hold in N. Then $-r+\epsilon \leqslant -\phi$ holds in N for some $\epsilon > 0$. Since $-\phi$ is approximated by Π_{n+1} formulas, again by Lemma 4.1, $-r+\epsilon \leqslant -\phi$ must hold in $\cup_k N_k$ which is a contradiction. Similarly, if $r \leqslant \phi$ holds in N, it must hold in M too. We conclude that $\phi^M = \phi^N$. ▲

Proof of the main theorem:
Let Γ_n be the set of linear combinations of Σ_n sentences (hence a vector space). Let $\mathcal{K}_n$ be the set of all maximal satisfiable sets T of conditions $\sigma = 0$ where $\sigma \in \Gamma_n$. As in Section 3, each $T \in \mathcal{K}_n$ is regarded as a positive norm one linear functional on Γ_n, i.e. $T(\sigma) = \sigma^M$ for $\sigma \in \Gamma_n$ and some (or any) $M \vDash T$. It is easily checked (like proposition 3.3) that $\mathcal{K}_n$ is compact convex and Hausdorff. For $\sigma \in \Gamma_n$ set

$$f_\sigma(T) = T(\sigma).$$

[DOI: 10.52547/ijmsi.17.2.75]

Let

$$X = \{f_\sigma : \ \sigma \in \Gamma_n\}.$$

Then, X is a linear subspace of $A(\mathcal{K}_n)$ (the set of affine continuous real valued functions on $\mathcal{K}_n$) which contains constant functions. Assume $T_1 \neq T_2$. Then, there is a Σ_n sentence σ such that $T_1(\sigma) \neq T_2(\sigma)$. So, $f_\sigma(T_1) \neq f_\sigma(T_2)$. This shows that X separates points. By Theorem 3.4, X is dense in $A(\mathcal{K}_n)$.

Define similarly $f_\phi(T) = \phi^M$ where $M \vDash T$. By the above claim, f_ϕ is well-defined. It is clearly affine. The proof of continuity of f_ϕ is as in the proof of theorem 3.5. Assume $T_k \to T$ in the weak* topology of $\mathcal{K}_n$ and $T_k(\phi) \leqslant r$ for each k. Take a nonprincipal ultrafilter $\mathcal{F}$ on $\mathbb{N}$. Let $M_k \vDash T_k$ and $M = \prod_{\mathcal{F}} M_k$. Then, we have that $M \vDash T$. As a consequence,

$$f_\phi(T) = \phi^M = \lim_{k,\mathcal{F}} \phi^{M_k} = \lim_{k,\mathcal{F}} T_k(\phi) \leqslant r.$$

Hence $f_\phi \in A(\mathcal{K}_n)$. We conclude that for each $\epsilon > 0$ there is a $\sigma \in \Gamma_n$ such that for every $T \in \mathcal{K}_n$, $|f_\phi(T) - f_\sigma(T)| \leqslant \epsilon$. In other words, for every M, $|\phi^M - \sigma^M| \leqslant \epsilon$. □

5. Positive axiomatization

Two main preservation theorems are characterization of universal conditions and universal-existential conditions. A condition is *universal* if it is of the form $0 \leqslant \inf_{\bar{x}} \phi(\bar{x})$ where ϕ is quantifier-free. A condition is *universal-existential* (or $\forall\exists$ for short) if it is of the form $0 \leqslant \inf_{\bar{x}} \sup_{\bar{y}} \phi(\bar{x}, \bar{y})$ where ϕ is quantifier-free. A theory T is preserved under substructure if any substructure of a model of T is a models of T. It is inductive if whenever $M_0 \subseteq M_1 \subseteq \cdots$ and every M_n is a model of T then $\cup M_n$ is a model of T. It was proved in [3] that a theory T is preserved under substructure if and only if it has a set of universal axioms. It is inductive if and only if it is axiomatized by $\forall\exists$-conditions. In this sections we study two other preservation theorems, namely characterization of theories preserved by expanding and contracting surjective homomorphisms.

Definition 5.1. An *expanding* (resp. *contracting*) homomorphism is a function $f : M \to N$ such that

- for each $c \in L$, $f(c^M) = c^N$
- for each $F \in L$ and $\bar{a} \in M$, $f(F^M(\bar{a})) = F^N(f(\bar{a}))$
- for each $R \in L$ (including d) and $\bar{a} \in M$, $R^M(\bar{a}) \leqslant R^N(f(\bar{a}))$ (resp. $R^N(f(\bar{a})) \leqslant R^M(\bar{a})$).

A formula is *positive* (resp. *negative*) if it is built up from atomic (resp. negative atomic) formulas (including the reals $r \in \mathbb{R}$ in both cases) using the connectives $+$, $s\cdot$ for $s \geqslant 0$ and the quantifiers inf, sup. So, positive formulas are built as follows:

$$r, \quad d(t_1, t_2), \quad R(t_1, ..., t_n), \quad \phi + \psi, \quad s\phi, \quad \sup_x \phi, \quad \inf_x \phi$$

[DOI: 10.52547/ijmsi.17.2.75]

where $r \in \mathbb{R}$ and $s \in \mathbb{R}^+$. Obviously, ϕ is negative if and only if $-\phi$ is equivalent to a positive formula. A *positive axiom* is a condition of the form $0 \leqslant \sigma$ where σ is positive. Similarly, if σ is negative, $0 \leqslant \sigma$ is called a *negative axiom*.

It is not hard to check that a surjective function $f : M \to N$ is an expanding (resp. contracting) homomorphism if and only if for every positive (resp. negative) formula $\phi(\bar{x})$ and $\bar{a} \in M$ one has that $\phi^M(\bar{a}) \leqslant \phi^N(f(\bar{a}))$.

Lemma 5.2. *Let Δ be a linearly closed set of closed conditions of the form $0 \leqslant \phi$. Assume $0 \leqslant \phi + r \in \Delta$ whenever $0 \leqslant \phi \in \Delta$ and $0 \leqslant r$. Then for each theory T the following are equivalent:*

(i) *T is axiomatized by Δ-conditions.*

(ii) *If $M \vDash T$ and every Δ-condition which holds in M holds in N, then $N \vDash T$.*

Proof. (i)⇒(ii) is obvious. (ii)⇒(i): Let

$$T_\Delta = \{0 \leqslant \phi \in \Delta : \quad T \vDash 0 \leqslant \phi\}.$$

Every model of T is a model of T_Δ. Conversely assume $N \vDash T_\Delta$. Let

$$\Sigma = \{\phi + \epsilon \leqslant 0 : \quad 0 \leqslant \phi \in \Delta,\ 0 < \epsilon \text{ and } N \vDash \phi + \epsilon \leqslant 0\}.$$

Note that Σ is linearly closed. We show that $T \cup \Sigma$ is satisfiable. It is sufficient to show that $T \cup \{\phi + \epsilon \leqslant 0\}$ is satisfiable for each $\phi + \epsilon \leqslant 0 \in \Sigma$. Suppose not. Then for some $\phi + \epsilon \leqslant 0 \in \Sigma$ and $0 < \delta < \epsilon$, we have that $T \vDash \delta \leqslant \phi + \epsilon$. So, $N \vDash \delta \leqslant \phi + \epsilon$ which is a contradiction. Let M be a model of $T \cup \Sigma$. Then, every Δ-condition holding in M holds in N. So, by (ii), $N \vDash T$. □

Let

$$ediag^+(M) = \{0 \leqslant \phi(\bar{a}) :\ 0 \leqslant \phi^M(\bar{a}),\ \bar{a} \in M,\ \phi(\bar{x}) \text{ is positive}\}$$
$$ediag^-(M) = \{0 \leqslant \phi(\bar{a}) :\ 0 \leqslant \phi^M(\bar{a}),\ \bar{a} \in M,\ \phi(\bar{x}) \text{ is negative}\}$$
$$ediag(M) = \{0 \leqslant \phi(\bar{a}) :\ 0 \leqslant \phi^M(\bar{a}),\ \bar{a} \in M,\ \phi(\bar{x}) \text{ is arbitrary}\}$$

Following [6] (p.151), let Mpos N mean that every positive closed condition holding in M holds in N. In other words, $\sigma^M \leqslant \sigma^N$ for every positive sentence σ.

Theorem 5.3. *A theory T is preserved under surjective expanding homomorphisms if and only if it has a set of positive axioms.*

Proof. We prove the nontrivial direction which is a linearized variant of the proof of Theorem 3.2.4. in [6]. Assume T is preserved by surjective expanding homomorphisms. One first proves that if Mpos N then there is an elementary extension $N \preccurlyeq N'$ and a mapping $f : M \to N'$ such that $(M, a)_{a \in M}$pos $(N', f(a))_{a \in M}$. For this purpose one checks that $ediag^+(M) \cup ediag(N)$ is linearly satisfiable.

[DOI: 10.52547/ijmsi.17.2.75]

Similarly, if Mpos N, then there is an elementary extension $M \preccurlyeq M'$ and a mapping $g : N \to M'$ such that $(M', g(b))_{b\in N}$pos $(N, b)_{b\in N}$. For this purpose, one checks that $ediag^{-}(N) \cup ediag(M)$ is linearly satisfiable. Now assume $M_0 \vDash T$ and M_0pos N_0. Iterate the arguments to find chains

$$M_0 \preccurlyeq M_1 \preccurlyeq \ldots, \qquad N_0 \preccurlyeq N_1 \preccurlyeq \ldots$$

and maps

$$f_i : M_i \to N_{i+1}, \qquad g_i : N_i \to M_i$$

such that

$$(M_0, a)_{a\in M_0}\text{pos } (N_1, f_0a)_{a\in M_0}$$

$$(M_1, a, g_1b)_{a\in M_0,\ b\in N_1}\text{pos } (N_1, f_0a, b)_{a\in M_0, b\in N_0}$$

and so forth. In particular, $f_i : M_i \to N_{i+1}$ is an expanding homomorphism and $f_i \subseteq f_{i+1}$, $g_{i+1}^{-1} \subseteq f_{i+1}$. Set $\bar{M} = \cup_i M_i$ and $\bar{N} = \cup_i N_i$. Then $M_0 \preccurlyeq \bar{M}$, $N_0 \preccurlyeq \bar{N}$ and $\cup f_i : M \to N$ is a surjective expanding homomorphism. By the assumption of proposition, we must have that $N_0 \vDash T$. Let Δ be the set of all positive L-conditions. Thus, we have proved that the clause (ii) of Lemma 5.2 holds for Δ. We conclude T is axiomatized by a set of positive conditions. $\square$

A similar proof shows that

Proposition 5.4. *A theory T is preserved under surjective contracting homomorphisms if and only if it has a set of negative axioms.*

Acknowledgments

The authors wish to thank the referees for their valuable comments which helped to improve the paper.

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[DOI: 10.52547/ijmsi.17.2.75]