Fixed Point of T_F-contractive Single-valued Mappings

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ABSTRACT. In this paper, we study the existence of fixed points for mappings defined on complete metric space (X, d) satisfying a general contractive inequality depended on another function. This conditions is analogous of Banach conditions and general contraction condition of integral type.

1. Introduction

The first important result on fixed points for contractive-type mapping was the well-known Banach's Contraction Principle appeared in explicit form in Banach's thesis in 1922, where it was used to establish the existence of a solution for an integral equation [1]. In the general setting of complete metric space this theorem runs as follows (see [4,Theorem 2.1] or [11,Theorem 1.2.2]).

Theorem 1.1. [***Banach's Contraction Principle***] *Let* (X, d) *be a complete metric space and* $f : X \longrightarrow X$ *be a contraction* (*there exists* $k \in (0, 1)$ *such*

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that for each $x, y \in X$; $d(fx, fy) \leq kd(x, y)$). Then f has a unique fixed point in X, and for each $x_0 \in X$ the sequence of iterates $\{f^n x_0\}$ converges to this fixed point.

After this classical result Kannan in [3] analyzed a substantially new type of contractive condition. Since then there have been many theorems dealing with mappings satisfying various types of contractive inequalities. Such conditions involve linear and nonlinear expressions (rational, irrational, and of general type). The interested reader who wants to know more about this matter is recommended to go deep into the survey articles by Rhoades [8,9,10] and Meszaros [6], and into the references therein. Another result on fixed points for contractive-type mapping is generally attributed to Edelstein (1962) who actually obtained slightly more general versions. In the general setting of compact metric spaces this result runs as follows (see [4, Theorem 2.2]).

Theorem 1.2. *Let (X, d) be a compact metric space and $f : X \longrightarrow X$ be a contractive mapping (for every $x, y \in X$ such that $x \neq y$; $d(fx, fy) < d(x, y)$). Then f has a unique fixed point in X, and for any $x_0 \in X$ the sequence of iterates $\{f^n x_0\}$ converges to this fixed point.*

Also in 2002 A. Branciari [2] analyzed the existence of fixed point for mapping f defined on a complete metric space (X, d) satisfying a contractive condition of integral type (see the following theorem).

Theorem 1.3. *Let (X, d) be a complete metric space, $\alpha \in (0, 1)$ and $f : X \longrightarrow X$ be a mapping such that for each $x, y \in X$, $\int_0^{d(fx, fy)} \phi(t)dt \leq \alpha \int_0^{d(x, y)} \phi(t)dt$, where $\phi : [0, +\infty) \longrightarrow [0, +\infty]$ is a Lebesgue-integrable mapping which is summable (i.e., with finite integral) on each compact subset of $[0, +\infty)$, nonnegative, and such that for each $\epsilon > 0$, $\int_0^\epsilon \phi(t)dt > 0$; then f has a unique fixed point $a \in X$ such that for each $x \in X$, $\lim_{n \to \infty} f^n x = a$.*

Fixed point theory has application in Fuzzy system too. For more information about Fuzzy system we can see [5] and [7]. The aim of this paper is to study the existence of fixed point for a mapping f defined on a complete metric space (X, d) such that it is a $T_F - contraction$. In particular, we extend the main theorem due to A. Branciari [2] and introduce a new class of contractive mappings. First we introduce the $T_F - contraction$ function and then extended A. Branciari Theorem and Banach-contraction principle, by the same method for proof of A. Branciari theorem. At the end of paper some examples and applications concerning this kind of contractions are given. In the 2002 A. Branciari gave an example [2; Example 3.6] such that it can be studied with the help of Theorem 1.2. (because $X = \{1/n : n \in \mathbb{N}\} \bigcup \{0\}$, with metric induced by $\mathbb{R}$, $d(x, y) = |x - y|$, is a compact metric space and f is a contractive mapping). In the end of this paper we give an example such that we can not study it with Theorem 1.1, Theorem 1.2 and Branciari theorem,

but we can use the main theorem (Theorem 2.5) in this paper. In the sequel, $\mathbb{N}$ will represent the set of natural numbers, $\mathbb{R}$ the set of real numbers and $\mathbb{R}_+$ the set of nonnegative real numbers.

2. Definitions and Main Result

The following theorem (Theorem 2.5) is the main result of this paper. First, we give some new definitions.

At first we introduce the notation

$\Psi := \big\{ F : \mathbb{R}_+ \longrightarrow \mathbb{R}_+ : F$ is nondecreasing continuous from the right and $F^{-1}(0) = \{0\} \big\}$.

Definition 2.1. Let (X, d) be a metric space, let $f, T : X \longrightarrow X$ be two functions and let $F \in \Psi$. A mapping f is said to be a $T_F - contraction$ if there exists $\alpha \in [0, 1)$ such that for all $x, y \in X$

$$F\big(d(Tfx, Tfy)\big) \leq \alpha F\big(d(Tx, Ty)\big).$$

Remark 2.2. By taking $Tx \equiv x$ and $F(x) \equiv x$, $T_F - contraction$ and contraction are equivalent. Also by taking $Fx \equiv x$ we can define $T - contraction$ and by taking $Tx \equiv x$ we can define $I_F - contraction$ (I is identify function).

Example 2.3. *Let $X = [1, +\infty)$ endowed with the Euclidean metric. We consider two mappings $T, f : X \longrightarrow X$ by $Tx = \frac{1}{x} + 1$ and $fx = 2x$. Obviously f is not a contraction but f is a $T_F - contraction$ where $F(x) \equiv x$.*

Definition 2.4. Let (X, d) be a metric space. A mapping $T : X \longrightarrow X$ is said to be graph closed if for every sequence $\{x_n\}$ such that $\lim\limits_{n \to \infty} Tx_n = a$ then for some $b \in X$ $Tb = a$. For example the identity function on X is graph closed.

Theorem 2.5. *Let (X, d) be a complete metric space, $\alpha \in [0, 1)$, $T, f : X \longrightarrow X$ be two mappings such that T is one-to-one and graph closed and f is a $T_F - contraction$ where $F \in \Psi$; then f has a unique fixed point $a \in X$. Also for every $x \in X$, the sequence of iterates $\{Tf^n x\}$ converges to Ta.*

Proof. Let $\alpha \in [0, 1)$ such that for all $x, y \in X$

$$(1) \qquad F\big(d(Tfx, Tfy)\big) \leq \alpha F\big(d(Tx, Ty)\big).$$

So if for $a, b > 0$, $F(a) \leq \alpha F(b)$ then $a < b$. Also

$$(2) \qquad F(\varepsilon) > 0,$$

for all $\varepsilon > 0$.

Let $x \in X$. We break the argument into 4 steps.

Step 1. $\lim_{n \to \infty} d(Tf^{n+1}x, Tf^n x) = 0$.
Proof. Since (1) holds, for all $n \in \mathbb{N}$:

$$F\big(d(Tf^{n+1}x, Tf^n x)\big) \leq \alpha^n F\big(d(Tfx, Tx)\big) \qquad (x \in X).$$

As a consequence, since $\alpha \in [0, 1)$, we further have

$$(3) \qquad F\big(d(Tf^{n+1}x, Tf^n x)\big) \to 0^+ \qquad as \qquad n \to \infty.$$

Since (2) and (3) hold

(4)
$$\lim_{n \to \infty} d(Tf^{n+1}x, Tf^n x) = 0.$$

Step 2. $\{Tf^n x\}$ is a bounded sequence.

Proof. If $\{Tf^n x\}_{n=1}^{\infty}$ is an unbounded sequence then, we choose the subsequence $\{n(k)\}_{k=1}^{\infty}$ such that $n(1) = 1$ and for each $k \in \mathbb{N}$, $n(k+1)$ is "minimal" in the sense that

$$d(Tf^{n(k+1)}x, Tf^{n(k)}x) > 1$$

and

$$d(Tf^m x, Tf^{n(k)}x) \le 1 \ \forall \ m = n(k) + 1, n(k) + 2, ..., n(k+1) - 1.$$

So, using the triangular inequality

$$
\begin{aligned}
1 \quad &< \quad d(Tf^{n(k+1)}x, Tf^{n(k)}x) \\
&\le \quad d(Tf^{n(k+1)}x, Tf^{n(k+1)-1}x) + d(Tf^{n(k+1)-1}x, Tf^{n(k)}x) \\
&\le \quad d(Tf^{n(k+1)}x, Tf^{n(k+1)-1}x) + 1
\end{aligned}
$$

(5)

Hence, from (4) and (5),

(6)
$$d(Tf^{n(k+1)}x, Tf^{n(k)}x) \to 1^+ \quad as \quad k \to +\infty.$$

Using (1) and using the triangular inequality,

$$
\begin{aligned}
1 \quad &< \quad d(Tf^{n(k+1)}x, Tf^{n(k)}x) \\
&\le \quad d(Tf^{n(k+1)-1}x, Tf^{n(k)-1}x) \\
&\le \quad d(Tf^{n(k+1)-1}x, Tf^{n(k)}x) + d(Tf^{n(k)}x, Tf^{n(k)-1}x) \\
&\le \quad 1 + d(Tf^{n(k)}x, Tf^{n(k)-1}x).
\end{aligned}
$$

(7)

So from (4) and (7),

(8)
$$d(Tf^{n(k+1)-1}x, Tf^{n(k)-1}x) \to 1^+ \quad as \quad k \to +\infty.$$

Since $F\big(d(Tf^{n(k+1)}x, Tf^{n(k)}x)\big) \le \alpha F\big(d(Tf^{n(k+1)-1}x, Tf^{n(k)-1}x)\big)$, F is continuous from the right and (6) and (8) hold, $F(1) \le \alpha F(1)$. So $F(1) = 0$ and this is a contradiction.

Step 3. $\{Tf^n x\}_{n=1}^{\infty}$ is a Cauchy sequence.

Proof. Using (1), for every $m, n \in \mathbb{N} \ (m > n)$,

(9)
$$F\big(d(Tf^m x, Tf^n x)\big) \le \alpha^n F\big(d(Tf^{m-n}x, Tx)\big).$$

From step 2, inequality (9) and $\alpha \in [0, 1)$,

(10)
$$\lim_{m,n \to \infty} F\big(d(Tf^m x, Tf^n x)\big) = 0$$

Since F is nondecreasing and (2) holds $\lim_{m,n \to \infty} d(Tf^m x, Tf^n x) = 0$, and this shows that $\{Tf^n x\}_{n=1}^{\infty}$ is a Cauchy sequence.

Step 4. f has a unique fixed point.

Proof. Since $\{Tf^n x\}_{n=1}^{\infty}$ is a Cauchy sequence and T is graph closed there exists $a \in X$ such that $\lim_{n \to \infty} Tf^n x = Ta$. Also

(11)
$$F\big(d(Tf^{n+1}x, Tfa)\big) \le \alpha F\big(d(Tf^n x, Ta)\big).$$

Since $d(Tf^nx, Ta) \longrightarrow 0^+$ we conclude that $F\big(d(Tf^nx, Ta)\big) \longrightarrow 0$ as $n \longrightarrow +\infty$. So, from (11) $F\big(d(Tf^{n+1}x, Tfa)\big) \longrightarrow 0$. Hence from (2), $\lim_{n\to\infty} Tf^{n+1}x = Tfa$. So $Ta = Tfa$. Since T is one-to-one $a = fa$. Therefore f has a fixed point. Since T is one-to-one and f is a $T_F - contraction$, f has a unique fixed point. $\square$

3. Examples and Applications

In this section, we give some applications and some examples concerning these type of contractive mappings, which clarify the connection between our result and the classical ones.

Remark 3.1. Theorem 2.5 is a generalization of Banach's contraction principle (Theorem 1.1); letting $F(x) \equiv x$ and $Tx \equiv x$, we have

$$
\begin{aligned}
F\big(d(Tfx, Tfy)\big) &= d(fx, fy) \\
&\leq \alpha d(x, y) \\
&= \alpha F\big(d(Tx, Ty)\big).
\end{aligned}
$$

Remark 3.2. Theorem 2.5 is a generalization of the Branciari theorem (Theorem 1.3); letting $Tx = x$ for each $x \in X$ and $F(s) = \int_0^s \phi(t)dt$ (obviously $F \in \Psi$), so

$$
\begin{aligned}
F\big(d(Tfx, Tfy)\big) = \int_0^{d(Tfx, Tfy)} \phi(t)dt &= \int_0^{d(fx, fy)} \phi(t)dt \\
&\leq \alpha \int_0^{d(x,y)} \phi(t)dt \\
&= \alpha \int_0^{d(Tx, Ty)} \phi(t)dt = \alpha F\big(d(Tx, Ty)\big).
\end{aligned}
$$

Using Theorem 2.5, by taking $F(x) \equiv x$, we conclude the following theorem which extends Banach contraction principle.

Theorem 3.3. *Let (X, d) be a complete metric space and $T : X \longrightarrow X$ be one-to-one and graph closed. Then for every $T - contraction$ function $f : X \longrightarrow X$, f has a unique fixed point.*

Example 3.4. *Let $X = [1, +\infty)$ endowed with the Euclidean metric. Since X is a closed subset of $\mathbb{R}$, it is a complete metric space. We define $T, f : X \longrightarrow X$ by $Tx = \ln x + 1$ and $fx = k\sqrt{x}$, where $k \geq 1$ be a fixed element of $\mathbb{R}$. Obviously f is not a contraction but f is a $T_F - contraction$, where $F(t) = t$. Therefore, by using Theorem 2.5, f has a unique fixed point.*

The following is our main example which shows that the conditions in Theorem 2.5 is weaker than of Theorems 1.1, 1.2, 1.3 and 3.3.

Example 3.5. *Let $X := \{\frac{1}{n} \mid n \in \mathbb{N}\} \bigcup \{0\}$ endowed with the Euclidean metric. We consider a mapping $f : X \longrightarrow X$ defined by*

$$
fx = \begin{cases} \frac{1}{n+3} & ; x = \frac{1}{n},\ n \text{ is odd} \\ 0 & ; x = 0 \\ \frac{1}{n-1} & ; x = \frac{1}{n},\ n \text{ is even} \end{cases}
$$

By taking $n = 2$ and $m = 4$, $|f(1/m) - f(1/n)| > |1/m - 1/n|$, so f is not a contraction and is not a contractive mapping in the sense of Edelstein either. Hence, we can not conclude that, f has a fixed point by Theorem 1.1 and Theorem 1.2 and Branciari theorem. Because if

$$\int_0^{|fx - fy|} \phi(t)dt \leq \alpha \int_0^{|x - y|} \phi(t)dt$$

for all $x, y \in X$ and for some ϕ and $\alpha \in [0, 1)$, then we must have $|f(x) - f(y)| \leq |x - y|$, but this is false.

Now we defined $\phi : [0, +\infty) \longrightarrow [0, +\infty)$ by

$$\phi(t) = \begin{cases} t^{\frac{1}{t} - 2}[1 - \log t] & ; 0 < t < e \\ 0 & ; t = 0 \, and \, t \geq e \end{cases}$$

and $F(\tau) = \int_0^\tau \phi(t)dt$ and define $T : X \longrightarrow X$ by

$$Tx = \begin{cases} \frac{1}{n-1} & ; x = \frac{1}{n}, \, n \text{ is even} \\ 0 & ; x = 0 \\ \frac{1}{n+1} & ; x = \frac{1}{n}, \, n \text{ is odd} \end{cases}$$

Obviously $F(\tau) = \int_0^\tau \phi(t)dt = \tau^{\frac{1}{\tau}}$ for $0 < \tau < e$ and $F(\tau) = e^{\frac{1}{e}}$ for $\tau \geq e$, T is one-to-one and graph closed and

$$Tfx = \begin{cases} \frac{1}{n+2} & ; x = \frac{1}{n}, \, n \text{ is odd} \\ 0 & ; x = 0 \\ \frac{1}{n} & ; x = \frac{1}{n}, \, n \text{ is even} \end{cases}$$

Since $\sup \frac{|Tfx - Tfy|}{|Tx - Ty|}| = 1$, f is not a $T-$contraction, and so we can not use Theorem 3.3 in this case. Now we show that the conditions of Theorem 2.5 hold. We show that

$$(12) \qquad\qquad F\big(|Tfx - Tfy|\big) \leq \frac{1}{2}F\big(|Tx - Ty|\big)$$

for all $x, y \in X$. There are 5 cases.

Case 1. *Let $x = \frac{1}{m}, y = \frac{1}{n}$ and m and n are even. Then*

$$F\big(|Tfx - Tfy|\big) \leq \frac{1}{2}F\big(|Tx - Ty|\big)$$

$$\Leftrightarrow \qquad |\frac{1}{m} - \frac{1}{n}|^{|\frac{1}{|\frac{1}{m} - \frac{1}{n}|}|} \leq \frac{1}{2}|\frac{1}{m-1} - \frac{1}{n-1}|^{|\frac{1}{|\frac{1}{m-1} - \frac{1}{n-1}|}|}$$

$$\Leftrightarrow \qquad |\frac{m-n}{mn}|^{|\frac{mn}{m-n}|} \cdot |\frac{(m-1)(n-1)}{m-n}|^{|\frac{(m-1)(n-1)}{m-n}|} \leq \frac{1}{2}$$

$$\Leftrightarrow \qquad |\frac{(m-1)(n-1)}{mn}|^{|\frac{(m-1)(n-1)}{m-n}|} \cdot |\frac{(m-n)}{mn}|^{|\frac{(m+n-1)}{m-n}|} \leq \frac{1}{2}$$

Obviously the last inequality holds, because

$$|\frac{m-n}{mn}|^{|\frac{m+n-1}{m-n}|} \leq \frac{1}{2}$$

and

$$|\frac{(m-1)(n-1)}{mn}| \leq 1 \ and \ |\frac{(m-1)(n-1)}{m-n}| \geq 1$$

and so

$$|\frac{(m-1)(n-1)}{mn}|^{|\frac{(m-1)(n-1)}{m-n}|} \leq 1.$$

Therefore for this case (12) holds.

Case 2. *Let* $x = \frac{1}{m}, y = \frac{1}{n}$ *where* m *and* n *are odd.*

Case 3. *Let* $x = \frac{1}{m}, y = \frac{1}{n}$ *where* m *is even and* n *is odd.*
By the same argument in case 1 we conclude that (12) holds for case 2 and case 3.

Case 4. *Let* $x = 0, y = \frac{1}{n}$ *where* n *is even. Then*

$$F\big(|Tfx - Tfy|\big) \leq \frac{1}{2}F\big(|Tx - Ty|\big)$$
$$\Leftrightarrow (\frac{1}{n})^n \leq \frac{1}{2}(\frac{1}{n-1})^{n-1}$$
$$\Leftrightarrow (\frac{1}{n})^n (n-1)^{n-1} \leq \frac{1}{2}$$
$$\Leftrightarrow (\frac{n-1}{n})^{n-1}.\frac{1}{n} \leq \frac{1}{2}.$$

The last inequality holds, because

$$(\frac{n-1}{n})^{n-1} \leq 1 \qquad and \qquad \frac{1}{n} \leq \frac{1}{2}.$$

Therefore (12) is true for this case.

Case 5. *Let* $x = 0 and \ \ y = \frac{1}{n}$ *where* n *is odd. By the same argument in case 4 we conclude that (12) holds for this case.*
Hence, (12) holds for all $x, y \in X$. *So* f *is* $T_F - contraction$ *and the conditions of Theorem 2.5 hold. Therefore* f *has a unique fixed point.*

References

[1] S. Banach, Sur Les Operations Dans Les Ensembles Abstraits et Leur Application Aux Equations Integrales, *Fund. Math.*, **3** (1922), 133-181 (French).
[2] A. Branciari, A fixed point theorem for mapping satisfying a general contractive condition of integral type, *Int. J. M. and M. Sciences*, **29**(9) (2002), 531-536.
[3] R. Kannan, Some results on fixed points, *Bull.Calcutta Math. Soc.*, **60** (1968),71-76.
[4] Kazimierz Goebel and W. A. Kirk, *Topics in Metric Fixed Point Theory*, Combridge University Press, New York, 1990.
[5] S. Kutukcu and S. Sharma, A fixed point theorem in fuzzy metric spaces, *Int. Journal of Math. Analysis*, **1**(18) (2007), 861-872.
[6] J. Meszaros, A Comparison of Various Definitions of Contractive Type Mappings, *Bull. Calcutta Math. Soc.*, **84**(2) (1992), 167-194.
[7] A.Nagoorgani , P. Palaniammal, A fuzzy production model with probabilistic re-salable returns, *Iranian Journal of Mathematical Sciences and Informatics*, **3**(1) (2008), 77-86.

[8] B. E. Rhoades, A Comparison of Various Definitions of Contractive Mappings, *Trans. Amer. Math. Soc.*, **226** (1977), 257-290.

[9] B. E. Rhoades, *Contractive definitions revisited: Topological Methods in Nonlinear Functional Analysis* (Toronto, Ont.,1982), Contemp. Math., Vol. 21, American Mathematical Society, Rhode Island, 1983, pp. 189-205.

[10] B. E. Rhoades, *Contractive Definitions: Nonlinear Analysis*, World Science Publishing, Singapore, 1987, pp. 513-526.

[11] D. R. Smart, *Fixed Point Theorems*, Cambridge University Press, London, 1974.