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On Nonlinear Random Approximation of 3-Variable Cauchy Functional Equations

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Abstract. In this paper, we study to approximate the homomorphisms and derivations for 3-variable Cauchy functional equations in RC^*-algebras and Lie RC^*-algebras by the fixed point method..

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1. Introduction and Preliminaries

Most of well known results on stability of functional equations are given for those equations in several variables. In 1983, Hyers [14] surveyed the stability of isometries and, later, jointly with Rassias, the stability of homomorphisms in [15]. Another survey on Hyers–Ulam stability of functional equations in several

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variables is given by Forti [13]. Especially, in 2003, Agarwal et al. discussed Hyers–Ulam stability for functional equations in single variable including the forms of linear functional equations, nonlinear functional equations and iterative equations.

Recently, some authors have published some papers on stability of functional equations in several spaces by the direct method and the fixed point method, for example, fuzzy Menger normed algebras ([21]), fuzzy metric spaces ([25]), fuzzy normed spaces ([26]), non-Archimedian random Lie C^*-algebras ([16]), non-Archimedian Banach spaces ([24]), non-Archimedian random normed spaces ([31]), random multi-normed space ([2]), random lattice normed spaces ([17], [10]), random normed algebras ([23]), random normed spaces ([3], [4], [18], [22], [27], [28]). For more details on stability of functional equations on these spaces, see Cho et al. [8] and [9].

Let Ω^+ be the set of distribution mappings, i.e., the set of all mappings $G : \mathbb{R} \cup \{-\infty, \infty\} \to [0, 1]$ such that G is left-continuous and increasing on $\mathbb{R}$, $G(0)$ is zero and $G(+\infty)$ is one. $O^+ \subseteq \Omega^+$ includes all mappings $G \in \Omega^+$ for which $\ell^- G(+\infty)$ is one and $\ell^- g(x)$ is the left limit of the mapping g at the point x, i.e., $\ell^- g(x) = \lim_{t \to x^-} g(t)$.

In Ω^+, we define " $\leq$ " as follows:

$$H \leq F \iff H(s) \leq F(s)$$

for all s in $\mathbb{R}$ (partially ordered). Note that the function ω_u defined by

$$\omega_u(s) = \begin{cases} 0, & \text{if } s \leq u, \\ 1, & \text{if } s > u, \end{cases}$$

is a element of Ω^+ and ω_0 is the maximal element in this space (for some more details, see [7, 29, 30]).

Definition 1.1. ([29]) A *continuous triangular norm* (shortly, a *ct-norm*) is a continuous mapping κ from $[0, 1] \times [0, 1]$ to $[0, 1]$ such that

(a) $\kappa(r, t) = \kappa(r, t)$ and $\kappa(r, \kappa(t, s)) = \kappa(\kappa(r, t), s)$ for all $r, t, s \in [0, 1]$;
(b) $\kappa(r, 1) = r$ for all $r \in [0, 1]$;
(c) $\kappa(r, t) \leq \kappa(s, p)$ whenever $r \leq s$ and $t \leq p$ for all $r, t, s, p \in [0, 1]$.

Some examples of the t-norms are as follows:

(1) $\kappa_P(r, t) = rt$;
(2) $\kappa_M(r, t) = \min\{r, t\}$;
(3) $\kappa_L(r, t) = \max\{r + t - 1, 0\}$ (: the Lukasiewicz t-norm).

Also, we define

$$\Pi_{j=1}^n r_j = \kappa^{n-1}(r_1, \cdots, r_n).$$

Definition 1.2. ([30]) Suppose that κ is a *ct*-norm, V is a linear space and ξ is a mapping from V to O^+. In this case, the ordered tuple (V, ξ, T) is called a *random normed space* (in short, *RN-space*) if the following conditions are satisfied:

(RN1) $\xi_v(t) = \omega_0(t)$ for all $t > 0$ if and only if $v = 0$;
(RN2) $\xi_{\alpha v}(t) = \xi_v(\frac{t}{|\alpha|})$ for all $v \in V$ and $\alpha \in \mathbb{R}$ with $\alpha \neq 0$;
(RN3) $\xi_{u+v}(t+s) \geq \kappa(\xi_u(t), \xi_v(s))$ for all $u, v \in V$ and $t, s \geq 0$.

Let $(V, \|\cdot\|)$ be a linear normed space. Then

$$\xi_v(s) = \frac{s}{s + \|v\|}$$

for all $s > 0$ defines a random norm and the ordered tuple (V, ξ, κ_M) is a *RN*-space.

Definition 1.3. ([21]) Assume the a algebraic structure on RN-space (V, ξ, T) is such that

(RN4) $\xi_{uv}(ts) \geq \kappa'(\xi_u(t), \xi_v(s))$ for all $u, v \in V$ and $t, s > 0$, where κ' is a *ct*-norm.

Then $(V, \xi, \kappa, \kappa')$ is a *random normed algebra*.

Suppose that $(V, \|\cdot\|)$ is a linear normed algebra. Then $(V, \xi, \kappa_M, \kappa_P)$ is a random normed algebra, where

$$\xi_v(s) = \frac{s}{s + \|v\|}$$

for all $s > 0$ if and only if

$$\|uv\| \leq \|u\|\|v\| + s\|v\| + t\|u\|$$

for all $u, v \in V$ and $t, s > 0$). For some more details, see [1, 6, 11, 16, 19, 20, 22].

Definition 1.4. A *random Banach $*$-algebra* $\mathcal{B}$ is a random Banach algebra $(\mathcal{B}, \xi, \kappa, \kappa')$ over the field of complex numbers with together an involution on $\mathcal{B}$ which is a mapping $g \to g^*$ from $\mathcal{B}$ into $\mathcal{B}$ satisfies the following conditions: for all $g, h \in \mathcal{B}$,

(RBA1) $g^{**} = g$;
(RBA2) $(ag + bh)^* = \overline{a}g^* + \overline{b}h^*$;
(RBA3) $(gh)^* = h^* g^*$.

If, in addition, $\mu_{g^*g}(ts) = T'(\mu_g(t), \mu_g(s))$ for all $g \in \mathcal{B}$ and $t, s > 0$, then $\mathcal{B}$ is a *random C^*–algebra* (in short, *RC^*–algebra*).

Assume that $\mathcal{B}$ is a random Banach $* -$ algebra. A *derivation* on $\mathcal{B}$ is a mapping ν from $\mathcal{B}$ to $\mathcal{B}$ such that:

$$\nu(\lambda g + h) = \lambda \nu(g) + \delta(h), \tag{1.1}$$

$$\nu(gh) = \nu(g)h + g\nu(h) \tag{1.2}$$

for each $g, h \in \mathcal{B}$. A mapping $\delta : \mathcal{B} \to \mathcal{B}$ is called a $*$-*derivation* on $\mathcal{B}$ when

$$\delta(g^*) = (g) = \delta(g)^*$$

for all $g \in \mathcal{B}$.

Note that

$$g(u + v) = g(u) + g(v), \tag{1.3}$$

$$g(u + v) + g(u - v) = 2g(u) + 2g(v) \tag{1.4}$$

Assume that Γ is a nonempty set. A mapping $\Delta : \Gamma \times \Gamma \to [0, \infty]$ is called a *generalized metric* (GM) on Γ if

(1) $\Delta(a, b) = 0$ if and only if $a = b$ for all $a, b \in \Gamma$;
(2) $\Delta(a, b) = \Delta(b, a)$ for all $a, b \in \Gamma$;
(3) $\Delta(a, b) \leq \Delta(a, c) + \Delta(c, b)$ for all $a, b, c \in \Gamma$.

The pair (Γ, Δ) is called a *generalized metric space* (shortly, *GM-space*).

Theorem 1.1. ([5, 12]). *Assume that (Γ, Δ) is a complete GM-space and a self-mapping J on Γ is a strictly contraction with Lipschitz constant $0 < L < 1$. Then, for all $a \in \Gamma$, either*

$$\Delta(J^n a, J^{n+1} a) = \infty$$

for each $n \geq 0$ or there exists $n_0 \in \mathcal{N}$ such that

(1) *$\Delta(J^n a, J^{n+1} a) < \infty$ for all $n \geq n_0$;*
(2) *the sequence $\{J^n a\}$ converges to a point $b^* \in \Gamma$;*
(3) *$J(b^*) = b^*$;*
(4) *$Jb^* = b^*$ and b^* is a unique fixed point in $\mathbb{E} = \{b \in \Gamma : \Delta(J^{n_0} a; b) < \infty\}$;*
(5) *$(1 - L)\Delta(b, b^*) \leq \Delta(b, Jb)$ for all $b \in \mathbb{E}$.*

Note that, in this paper, all random spaces have the *ct*-norm minimum.

In this paper, we study to approximate the homomorphisms and derivations for 3-variable Cauchy functional equations in RC^*-algebras and Lie RC^*-algebras by the fixed point method.

2. Approximations of Homomorphisms in RC^*-Algebras

Let A is a C^*-algebra with the norm $\| \cdot \|_A$ and $\mathcal{A}$, $\mathcal{B}$ are a RC^*-algebras. Assume that a mapping $h : \mathcal{A} \to \mathcal{B}$ is defined by

$$D_\gamma h(u, v, w) := \gamma h(u + v + w) - h(\gamma u) - h(\gamma v) - h(\gamma w)$$

for all $\gamma \in \mathbf{S}^1 := \{\mu \in \mathbf{C} : |\mu| = 1\}$ and $u, v, w \in \mathcal{A}$.

If, for any **C**-linear mapping $F:\mathcal{A}\to\mathcal{B}$, we have

$$F(uv)=F(u)F(v),\quad F(u^*)=F(u)^*$$

for all $u,v\in\mathcal{A}$, then F is called a *homomorphism* in RC^*-algebras.

Now, we approximate the homomorphisms in RC^*-algebras for

$$D_\gamma h(u,v,w):=\gamma h(u+v+w)-h(\gamma u)-h(\gamma v)-h(\gamma w)=0.$$

Theorem 2.1. *Assume that $h:\mathcal{A}\to\mathcal{B}$ be a mapping for which there exists a mapping $\phi:\mathcal{A}^3\to O^+$ such that*

$$\xi_{D_\gamma h(u,v,w)}(t) \geq \phi(u,v,w,t), \tag{2.1}$$

$$\xi_{h(uv)-h(u)h(v)}(t) \geq \phi(u,v,0,t), \tag{2.2}$$

$$\xi_{h(u^*)-h(u)^*}(t) \geq \phi(u,u,u,t) \tag{2.3}$$

for all $\gamma\in\mathbf{S}^1$, $u,v,w\in\mathcal{A}$ and $t>0$. If there exists $K<1$ such that

$$\phi(u,v,w,3Kt)\geq\phi(\frac{u}{3},\frac{v}{3},\frac{w}{3},t)$$

for all $u,v,w\in\mathcal{A}$, then there exists a unique RC^*-algebra homomorphism $F:\mathcal{A}\to\mathcal{B}$ such that

$$\xi_{h(u)-F(u)}((3-3K)t)\geq\phi(u,u,u,t) \tag{2.4}$$

for all $u\in\mathcal{A}$ and $t>0$.

Proof. Assume that

$$\sum:=\{\sigma:\mathcal{A}\to\mathcal{B}:\text{ a mapping}\}$$

and define the generalized metric (GM) on $\sum$ as follows:

$$\Delta(\varrho,\sigma)=\inf\{\theta\in\mathbf{R}_+:\xi_{\varrho(u)-\sigma(u)}(\theta t)\geq\phi(u,u,u,t)\}$$

for all u in $\mathcal{A}$ and $t>0$. Miheţ and Radu [18] proved that $(\sum,\Delta)$ is a complete GM-space.

Now, define a linear function $\Lambda:\sum\to\sum$ such that

$$\Lambda\sigma(u):=\frac{1}{3}\sigma(3u)$$

for all $u\in\mathcal{A}$. Then we have

$$\Delta(\Lambda\sigma,\Lambda\varrho)\leq K\Delta(\sigma,\varrho)$$

for all $\varrho,\sigma\in\sum$. Putting $\gamma=1$ and $v=w=u$ in (2.1), we have

$$\xi_{h(3u)-3h(u)}(t)\geq\phi(u,u,u,t) \tag{2.5}$$

for all $u\in\mathcal{A}$ and $t>0$. So, we have

$$\xi h(u)-\frac{1}{3}h(3u)(t)\geq\phi(u,u,u,3t)$$

for all $u \in \mathcal{A}$. So, we have $\Delta(h, \Lambda h) \leq \frac{1}{3}$. Thus, by Theorem 1.1, there exists a function $F : \mathcal{A} \to \mathcal{B}$ such that the following (1)–(3) hold:

(1) $\Lambda F = F$, i.e., we have

$$F(3u) = 3F(u) \tag{2.6}$$

for all $u \in \mathcal{A}$. Also, F is a unique fixed point of Λ in the set

$$\Upsilon = \{\sigma \in \sum : \Delta(\varrho, \sigma) < \infty\}.$$

So, F is a unique function satisfying (2.6) such that there exists $\theta > 0$ satisfying

$$\xi_{F(u)-h(u)}(\theta t) \geq \phi(u, u, u, t)$$

for each $u \in \mathcal{A}$ and $t > 0$.

(2) $\Delta(\Lambda^n h, F) \to 0$ when $n \to \infty$. Then we have

$$\lim_{n\to\infty} \frac{h(3^n u)}{3^n} = F(u) \tag{2.7}$$

for each $u \in \mathcal{A}$.

(3) $\Delta(h, F) \leq \frac{1}{1-K}\Delta(h, \Lambda h)$. Then we have

$$\Delta(h, F) \leq \frac{1}{3 - 3K}$$

and so (2.4) holds.

From (2.1) and (2.7), it follows that

$$\begin{aligned}
&\xi_{F(u+v+w)-F(u)-F(v)-F(w)}(t) \\
&= \lim_{n\to\infty} \xi_{h(3^n(u+v+w))-h(3^n u)-h(3^n v)-h(3^n w)}(3^n t) \\
&\geq \lim_{n\to\infty} \phi(3^n u, 3^n v, 3^n w) = \omega_0(t)
\end{aligned}$$

for each $u, v, w \in \mathcal{A}$ and $t > 0$ and so

$$F(u + v + w) = F(u) + F(v) + F(w) \tag{2.8}$$

for each $u, v, w \in \mathcal{A}$ and $t > 0$. Put $w = 0$ in (2.8), we have

$$F(u + v) = F(u) + F(v) + F(0) = F(u) + F(v)$$

for each $u, v \in \mathcal{A}$. Put $v = w = u$ in (2.1), we have

$$\gamma h(3u) = h(3\gamma u)$$

for all $\gamma \in \mathbf{S}^1$ and $u \in \mathcal{A}$. Also, we have

$$\gamma F(3u) = F(3\gamma u)$$

for all $\gamma \in \mathbf{S}^1$ and $u \in \mathcal{A}$. Then we can conclude that $F : \mathcal{A} \to \mathcal{B}$ is a $\mathbf{C}$-linear mapping.

On the other hand, by (2.2), we have

$$\begin{aligned}\xi_{F(uv)-F(u)F(v)}(t) &= \lim_{n\to\infty} \xi_{h(9^n uv)-h(3^n u)h(3^n v)}(9^n t)\\ &\geq \lim_{n\to\infty} \phi(3^n u, 3^n v, 0, 9^n t)\\ &\geq \lim_{n\to\infty} \phi(3^n u, 3^n v, 0, 3^n t)\\ &= \omega_0(t)\end{aligned}$$

for all $u, v \in \mathcal{A}$ and $t > 0$. Then we have

$$F(uv) = F(u)F(v)$$

for all $u, v \in \mathcal{A}$. Also, (2.3) implies that

$$\begin{aligned}\xi_{F(u^*)-F(u)^*}(t) &= \lim_{n\to\infty} \xi_{h(3^n u^*)-h(3^n u)^*}(3^n t)\\ &\geq \lim_{n\to\infty} \phi(3^n u, 3^n x, 3^n u, t)\\ &= \omega_0(t)\end{aligned}$$

for all $u \in \mathcal{A}$ and $t > 0$, which implies that $F : \mathcal{A} \to \mathcal{B}$ satisfies

$$F(u^*) = F(u)^*$$

for all $u \in \mathcal{A}$ and is a RC^*-algebra homomorphism. This completes the proof. □

Corollary 2.2. *Assume that $\rho < 1$ and $\tau > 0$. Suppose that $h : A \to \mathcal{B}$ be a function satisfies the following:*

$$\xi_{D_\gamma h(u,v,w)}(t) \geq \frac{t}{t + \tau(\|u\|^\rho + \|v\|^\rho + \|w\|^\rho)}, \tag{2.9}$$

$$\xi_{h(uv)-h(u)h(v)}(t) \geq \frac{t}{t + \tau(\|u\|^\rho + \|v\|^\rho)}, \tag{2.10}$$

$$\xi_{h(u^*)-h(u)^*}(t) \quad \geq \quad \frac{t}{t + 3\tau\|u\|^\rho} \tag{2.11}$$

for all γ in $\mathbf{S}^1$, u, v, w in A and $t > 0$. Then there exists a unique RC^*-algebra homomorphism $F : A \to \mathcal{B}$ such that

$$\xi_{h(u)-F(u)}(t) \geq \frac{t}{t + \frac{3\tau}{3-3^\rho}\|u\|^\rho} \tag{2.12}$$

for all $u \in A$ and $t > 0$.

Proof. Apply Theorem 2.1 and put

$$\phi(u, v, w) := \frac{t}{t + \tau(\|u\|^\rho + \|v\|^\rho + \|w\|^\rho)}$$

for all $u, v, w \in A$ and $t > 0$. Then $K = 3^{\rho-1}$. □

Theorem 2.3. *Assume that $h : \mathcal{A} \to \mathcal{B}$ be a function for which there exists a mapping $\phi : \mathcal{A}^3 \to O^+$ satisfying the conditions* (2.1), (2.2) *and* (2.3). *If there exists $K < 1$ such that*

$$\phi(u, v, w, Kt) \geq \phi(3u, 3v, 3w, 3t)$$

for all $u, v, w \in \mathcal{A}$ and $t > 0$, then there exists a unique RC*-algebra homomorphism $F : \mathcal{A} \to \mathcal{B}$ such that

$$\xi_{h(u)-F(u)}(Kt) \geq \phi(u, u, u, (3 - 3K)t) \tag{2.13}$$

for all $u \in \mathcal{A}$ and $t > 0$.

Proof. Define a linear function $\Lambda : \sum \to \sum$ such that

$$\Lambda\sigma(u) := 3\sigma\left(\frac{u}{3}\right)$$

for all $u \in \mathcal{A}$. Note that (2.5) implies that

$$\xi_{h(u)-3h(\frac{u}{3})}(Kt) \geq \phi\left(\frac{u}{3}, \frac{u}{3}, \frac{u}{3}\right) \geq \phi(u, u, u, 3t)$$

for all $u \in \mathcal{A}$ and $t > 0$. Hence $\Delta(h, \Lambda h) \leq \frac{K}{3}$, which, applying Theorem 2.1, implies that there exists a function $F : \mathcal{A} \to \mathcal{B}$ such that

(1) $\Lambda F = F$, $F(3u) = 3F(u)$ for all $u \in \mathcal{A}$ and there exists $\theta > 0$ such that

$$\xi_{F(u)-h(u)}(\theta t) \geq \phi(u, u, u, t)$$

for all $u \in \mathcal{A}$ and $t > 0$.

(2) $\Delta(\Lambda^n h, F) \to 0$ as $n \to \infty$ which we conclude that the equality

$$\lim_{n\to\infty} 3^n h\left(\frac{u}{3^n}\right) = F(u)$$

for all $u \in \mathcal{A}$.

(3) $\Delta(h, F) \geq \frac{1}{1-K}\Delta(h, \Lambda h)$ which we conclude that the inequality

$$\Delta(h, F) \leq \frac{K}{3 - 3K}$$

and then the inequality (2.3) holds and the reminder is similar of the proof of Theorem 2.1. This completes the proof. □

Corollary 2.4. *Assume that $\rho > 2$ and $\tau > 0$. Suppose that $h : A \to \mathcal{B}$ be a function satisfying the conditions* (2.9), (2.10) *and* (2.11). *Then there exists a unique RC*-algebra homomorphism $F : A \to \mathcal{B}$ such that*

$$\xi_{h(u)-F(u)}(t) \geq \frac{t}{t + \frac{\tau}{3^\rho - 3}\|u\|^\rho}$$

for all $u \in A$ and $t > 0$.

Proof. Apply Theorem 2.3 and put

$$\phi(u,v,w) := \frac{t}{t+\tau(\|u\|^\rho + \|v\|^\rho + \|w\|^\rho)}$$

for all $u, v, w \in A$ and $t > 0$. Then $K = 3^{1-\rho}$. □

3. Approximations of Derivations on RC^*-algebras

For any $\mathbf{C}$-linear function $\delta : \mathcal{A} \to \mathcal{A}$, if we have

$$\delta(uv) = \delta(u)v + u\delta(v)$$

for all $u, v \in \mathcal{A}$, then δ is called a *derivation* on A

Now, we show the approximation of derivations on RC^*-algebras for $D_\gamma h(u,v,w) = 0$.

Theorem 3.1. *Assume that $h : \mathcal{A} \to \mathcal{A}$ be a function for which there exists a function $\phi : \mathcal{A}^3 \to O^+$ such that*

$$\xi_{D_\gamma h(u,v,w)}(t) \geq \phi(u,v,w,t), \tag{3.1}$$

$$\xi_{h(uv)-h(u)v-uh(v)}(t) \geq \phi(u,v,0,t) \tag{3.2}$$

for all $\gamma \in \mathbf{S}^1$, $u, v, w \in \mathcal{A}$ and $t > 0$. If there exists $K < 1$ such that

$$\phi(u,v,w,3Kt) \geq \phi(\frac{u}{3}, \frac{v}{3}, \frac{w}{3})$$

for all $u, v, w \in \mathcal{A}$. Then there exists a unique derivation $\nu : \mathcal{A} \to \mathcal{A}$ such that

$$\xi_{h(u)-\nu(u)}(t) \geq \phi(u,u,u,(3-3K)t) \tag{3.3}$$

for all $u \in \mathcal{A}$ and $t > 0$.

Proof. Using the similar method presented in Theorem 1.1, we can conclude that there exists a unique involutive $\mathbf{C}$-linear mapping $\nu : \mathcal{A} \to \mathcal{A}$ satisfying (3.3) in which

$$\nu(u) = \lim_{n\to\infty} \frac{f(3^n u)}{3^n}$$

for all $u \in \mathcal{A}$.

On the other hand, we have

$$\begin{aligned}
&\xi_{\nu(uv)-\nu(u)v-u\nu(v)}(t)\\
&\quad = \lim_{n\to\infty} \xi_{h(9^n uv)-h(3^n u)\cdot 3^n v - 3^n u h(3^n v)}(9^n t)\\
&\quad \geq \lim_{n\to\infty} \phi(3^n u, 3^n v, 0, 9^n t) \geq \lim_{n\to\infty} \phi(3^n u, 3^n v, 0, 3^n t) = \omega_0(t)
\end{aligned}$$

for all $u, v \in \mathcal{A}$ and $t > 0$. Then it follows that

$$\delta(uv) = \delta(u)v + u\delta(v)$$

for all $u \in \mathcal{A}$. This completes the proof. □

Corollary 3.2. *Assume that $\rho < 1$ and $\tau > 0$. Suppose that $h : \mathcal{A} \to \mathcal{A}$ be a function such that*

$$\xi_{D_\gamma h(u,v,w)}(t) \geq \frac{t}{t + \tau(\|u\|^\rho + \|v\|^\rho + \|w\|^\rho)}, \tag{3.4}$$

$$\xi_{h(uv)-h(u)v-uh(v)}(t) \geq \frac{t}{t + \tau(\|u\|^\rho + \|v\|^\rho)} \tag{3.5}$$

for all $\gamma \in \mathbf{S}^1$, $u, v, w \in \mathcal{A}$ and $t > 0$. Then there exists a unique derivation $\nu : \mathcal{A} \to \mathcal{A}$ such that

$$\xi_{h(u)-\nu(u)}(t) \geq \frac{t}{t + \frac{3\tau}{3-3^\rho}\|u\|^\rho} \tag{3.6}$$

for all $u \in \mathcal{A}$ and $t > 0$.

Proof. Apply Theorem 3.1, define

$$\phi(u, v, w, t) := \frac{t}{t + \tau(\|u\|^\rho + \|v\|^\rho)}$$

for all $u, v, w \in \mathcal{A}$ and $t > 0$ and $K = 3^{\rho-1}$. Then we have the conclusion. □

Theorem 3.3. *Assume that $h : \mathcal{A} \to \mathcal{A}$ be a function for which there exists a function $\phi : \mathcal{A}^3 \to O^+$ such that (3.1) and (3.2) hold. If there exists $K < 1$ such that*

$$\phi(u, v, w, Kt) \geq \phi(3u, 3v, 3w, 3t)$$

for all $u, v, w \in \mathcal{A}$ and $t > 0$, then there exists a unique derivation $\nu : \mathcal{A} \to \mathcal{A}$ such that

$$\xi_{h(u)-\nu(u)}(t) \geq \phi\left(u, u, u, \frac{(3-3K)t}{K}\right) \tag{3.7}$$

for all $u \in \mathcal{A}$ and $t > 0$.

Proof. The proof is similar to the proof of Theorem 3.1. □

Corollary 3.4. *Assume that $\rho < 1$ and $\tau > 0$. Suppose that $h : \mathcal{A} \to \mathcal{A}$ be a function satisfying the conditions (3.4) and (3.5). Then there exists a unique derivation $\nu : \mathcal{A} \to \mathcal{A}$ such that*

$$\xi_{h(u)-\nu(u)}(t) \geq \frac{t}{t + \frac{\tau}{3^\rho-3}\|u\|^\rho} \tag{3.8}$$

for all $u \in \mathcal{A}$ and $t > 0$.

Proof. The proof follows by using Theorem 3.3, defining

$$\phi(u, v, w, t) := \frac{t}{t + \tau(\|u\|^\rho + \|v\|^\rho + \|w\|^\rho)}$$

for all $u, v, w \in \mathcal{A}$ and $t > 0$ and $K = 3^{1-\rho}$. □

4. Stability of Homomorphisms in Lie RC^*-algebras

We remember the *random Lie RC^*-algebra* which is a Lie product

$$[u,v] := \frac{uv - vu}{2}$$

on RC^*-algebra.

Definition 4.1. Assume that $\mathcal{A}$ and $\mathcal{B}$ are Lie RC^*-algebras. If $F : A \to B$ satisfies

$$F([u,v]) = [F(u), F(v)]$$

for all $u, v \in \mathcal{A}$, then F is called a *Lie C^*-algebra homomorphism.*

Theorem 4.2. *Assume that $h : \mathcal{A} \to \mathcal{B}$ be a function for which there exists a function $\phi : \mathcal{A}^3 \to O^+$ with (2.1) such that*

$$\xi_{h([u,v])-[h(u),h(v)]}(t) \quad \geq \quad \phi(u,v,0,t) \tag{4.1}$$

for all $u, v \in \mathcal{A}$ and $t > 0$. If there exists $K < 1$ such that

$$\phi(u,v,w,3Kt) \geq \phi\left(\frac{u}{3}, \frac{v}{3}, \frac{w}{3}\right)$$

for all $u, v, w \in \mathcal{A}$ and $t > 0$, then there exists a unique Lie RC^*-algebra homomorphism $F : \mathcal{A} \to \mathcal{B}$ satisfying (2.4).

Proof. Applying the similar method presented in Theorem 1.1, there exists a unique **C**-linear mapping $h : \mathcal{A} \to \mathcal{A}$ satisfies (4.1). The mapping $F : \mathcal{A} \to \mathcal{B}$ is given by

$$F(u) = \lim_{n\to\infty} \frac{h(3^n u)}{3^n}$$

for all $u \in \mathcal{A}$.

On the other hand, we have

$$\begin{aligned} \xi_{F([u,v])-[F(u),F(v)]}(t) &= \lim_{n\to\infty} \xi_{h(9^n[u,v])-[h(3^n u),h(3^n v)]}(9^n t) \\ &\geq \lim_{n\to\infty} \phi(3^n u, 3^n v, 0, 9^n t) \\ &\geq \lim_{n\to\infty} \phi(3^n u, 3^n u, 0, 3^n t) \\ &= \omega_0(t) \end{aligned}$$

for all $u, v \in \mathcal{A}$ and $t > 0$. So, we have

$$F([u,v]) = [F(u), F(v)]$$

for all $u, v \in \mathcal{A}$. Hence $F : \mathcal{A} \to \mathcal{B}$ is a Lie RC^*-algebra homomorphism satisfying (4.1). This completes the proof. □

Corollary 4.3. *Assume that* $\rho < 1$ *and* $\tau > 0$. *Suppose that* $h : \mathcal{A} \to \mathcal{A}$ *is a function satisfying* (4.1) *and*

$$\xi_{h([u,v])-[h(u),h(v)]}(t) \geq \frac{t}{t + \tau(\|u\|^{\rho} + \|v\|^{\rho})} \tag{4.2}$$

for all $u, v \in \mathcal{A}$ *and* $t > 0$. *Then there exists a unique Lie* RC^**-algebra homomorphism* $F : \mathcal{A} \to \mathcal{B}$ *satisfying* (2.4).

Proof. The proof follows by using Theorem 4.2, defining

$$\phi(u, v, w, t) := \frac{t}{t + \tau(\|u\|^{\rho} + \|v\|^{\rho} + \|w\|^{\rho})}$$

for all $u, v, w \in \mathcal{A}$ and $t > 0$ and $K = 3^{\rho-1}$. □

5. Approximations of Lie Derivations on RC^*-algebras

First, we give a definition of a Lie derivation in a Lie C^*-algebra $\mathcal{A}$.

Definition 5.1. In a Lie C^*-algebra $\mathcal{A}$, a **C**-linear function $\nu : \mathcal{A} \to \mathcal{A}$ with

$$\nu([u, v]) = [\nu(u), v] + [u, \nu(v)]$$

for all $u, v, w \in \mathcal{A}$ is a *Lie derivation.*

Theorem 5.2. *Assume that* $h : \mathcal{A} \to \mathcal{A}$ *be a function for which there exists a function* $\phi : \mathcal{A}^3 \to O^+$ *with* (3.1) *such that*

$$\xi_{h([u,v])-[h(u),v]-[u,h(v)]}(t) \quad \geq \quad \phi(u, v, 0, t) \tag{5.1}$$

for all $u, v \in \mathcal{A}$ *and* $t > 0$. *If there exists* $K < 1$ *such that*

$$\phi(u, v, w, 3Kt) \geq \phi\left(\frac{u}{3}, \frac{v}{3}, \frac{w}{3}\right)$$

for all $u, v, w \in \mathcal{A}$ *and* $t > 0$, *then there exists a unique Lie derivation* $\nu : A \to A$ *satisfying* (3.3).

Proof. Applying the similar method presented in Theorem 3.1, it follows that there exists a unique involutive **C**-linear function $\nu : \mathcal{A} \to \mathcal{A}$ satisfying (3.3) in which

$$\nu(u) = \lim_{n\to\infty} \frac{h(3^n u)}{3^n}$$

for all $u \in \mathcal{A}$.

On the other hand, we have

$$\begin{aligned} & xi_{\nu([u,v])-[\nu(u),v]-[u,\nu(v)]}(t) \\ = & \lim_{n\to\infty} \xi_{h(9^n[u,v])-[h(3^n u),3^n v]-[3^n u,h(3^n v)]}(9^n t) \\ \geq & \lim_{n\to\infty} \phi(3^n u, 3^n v, 0, 9^n t) \\ \geq & \lim_{n\to\infty} \phi(3^n u, 3^n v, 0, 3^n t) \\ = & \omega_0(t) \end{aligned}$$

for all $u, v \in \mathcal{A}$ and $t > 0$. Thus we have

$$\nu([u,v]) = [\nu(u), v] + [u, \nu(v)]$$

for all $u, v \in \mathcal{A}$. This completes the proof. □

Corollary 5.3. *Assume that* $\rho < 1$ *and* $\tau > 0$. *Suppose that* $h : \mathcal{A} \to \mathcal{A}$ *is a function satisfying* (3.2) *and*

$$\xi_{h([u,v])-[h(u),v]-[u,h(v)]}(t) \geq \frac{t}{t + \tau(\|u\|^\rho + \|v\|^\rho)} \tag{5.2}$$

for all $u, v \in \mathcal{A}$ *and* $t > 0$. *Then there exists a unique Lie derivation* $\nu : \mathcal{A} \to \mathcal{A}$ *satisfying* (3.3).

Proof. The proof follows by using Theorem 5.2, defining

$$\phi(u, v, w, t) := \frac{t}{t + \tau(\|u\|^\rho \ + \|v\|^\rho + \|w\|^\rho)}$$

for all $u, v, w \in \mathcal{A}$ and $t > 0$ and $K = 3^{\rho-1}$. □

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