Iranian Journal of Mathematical Sciences and Informatics
Vol. 16, No. 2 (2021), pp 145-162
DOI: 10.29252/ijmsi.16.2.129

Ordered Γ-Semigroups and Fuzzy Γ-Ideals

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Abstract. We prove that fuzzy generalized bi-Γ-ideal and fuzzy interior Γ-ideal in a right weakly regular ordered Γ-semigroup are fuzzy Γ-ideal. We also show that every fuzzy generalized bi-Γ-ideal in a duo right weakly regular ordered Γ-semigroup is a fuzzy interior Γ-ideal. Then, by using fuzzy Γ-ideals, fuzzy bi-Γ-ideals, fuzzy generalized bi-Γ-ideals and fuzzy interior Γ-ideals, left simple, right simple and simple ordered Γ-semigroups have been characterized. Finally, we characterize right weakly regular ordered Γ-semigroups by its fuzzy Γ-ideals, fuzzy bi-Γ-ideals, fuzzy generalized bi-Γ-ideals and fuzzy interior Γ-ideals.

Keywords: Ordered Γ-semigroups, Right weakly regular ordered Γ-Semigroups, Fuzzy sets, Fuzzy Γ-ideals.

2010 Mathematics subject classification: 06F99, 08A72, 20N99.

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Received 20 March 2018; Accepted 22 September 2019

1. Introduction and Preliminaries

Zadeh [20], in 1965, introduced the concept of a fuzzy set while Rosenfeld [14] first introduced and studied the notion of a fuzzy subgroup. In 1979, Kuroki [8] introduced fuzzy sets in semigroup theory while in [9, 10, 11], Kuroki studied some properties of fuzzy ideals, fuzzy bi-ideals, fuzzy generalized bi-ideals and fuzzy semiprime bi-ideals of a semigroup. Fuzzy sets in ordered semigroups were first studied by Kehayopulu and Tsingelis in [4]. In 1986, Sen and Saha [18] introduced the notion of a Γ-semigroup as follows: Let S and Γ be two non-empty sets. S is called a Γ-semigroup if there exists a mapping from $S \times \Gamma \times S$ to S which maps $(a, \alpha, b) \to a\alpha b$ satisfying $(a\gamma b)\mu c = a\gamma(b\mu c)$ for all $a, b, c \in S$ and $\gamma, \mu \in \Gamma$. Later on in 1993, the notion of an ordered Γ-semigroup was introduced by Sen and Seth [19] as follow: Let S and Γ be non-empty sets. The triplet $(S, \Gamma, \leq)$ is called an ordered Γ-semigroup if S is a Γ-semigroup and $(S, \leq)$ is a partially ordered set such that $a \leq b \Rightarrow a\gamma c \leq b\gamma c$ and $c\gamma a \leq c\gamma b$ for all $a, b, c \in S$ and $\gamma \in \Gamma$. The notion of a Γ-semigroups has been extended to fuzzy settings by Sardar and Majumder [15, 16, 17]. They have studied fuzzy ideals, fuzzy prime ideals, fuzzy semiprime ideals and fuzzy ideal extensions in Γ-semigroups.

In 2014, Kanlaya and Iampan [2] had proved that in a regular (left regular, right regular) ordered Γ-semigroup S, every fuzzy generalized bi-ideal of S is a fuzzy bi-ideal. They also proved that in an intra-regular (left regular, right regular) ordered Γ-semigroup, every fuzzy interior ideal of S is a fuzzy ideal. In this paper we prove that if S is right weakly regular ordered Γ-semigroup, then every fuzzy interior Γ-ideal of S is a fuzzy Γ-ideal and every fuzzy generalized bi-Γ-ideal of S is a fuzzy bi-Γ-ideal. We also prove that if S is duo right weakly regular ordered Γ-semigroup, then every fuzzy generalized bi-Γ-ideal of S is a fuzzy interior Γ-ideal. We then characterize right weakly regular ordered Γ-semigroup by fuzzy (right Γ-ideal, left Γ-ideal) Γ-ideals, fuzzy quasi Γ-ideals, fuzzy generalized bi-Γ-ideals, fuzzy bi-Γ-ideals and fuzzy interior Γ-ideals. Lastly after proving that in a right weakly regular ordered Γ-semigroup S, for each fuzzy right Γ-ideal f and fuzzy Γ-ideal g of S, $f \circ g$ is a fuzzy quasi Γ-ideal, we investigate some properties of fuzzy interior ideals and fuzzy generalized bi-ideals of left simple, right simple and simple ordered Γ-semigroups.

Let S be an ordered Γ-semigroup. For a subset A of S define $(A] = \{x \in S \mid x \leq a \text{ for some } a \in A\}$. A non-empty subset T of S is said to be a Γ-subsemigroup of S if for all $x, y \in T$ and $\gamma \in \Gamma$, $x\gamma y \in T$. A non-empty subset A of an ordered Γ-semigroup S is called left (right) Γ-ideal of S if $S\Gamma A \subseteq A$ $(A\Gamma S \subseteq A)$ and for any $a \in A, b \in S$ such that $b \leq a$, then $b \in A$. A non-empty subset J of an ordered Γ-semigroup S is called a Γ-ideal of S if J is both a left Γ-ideal and a right Γ-ideal of S. An ordered Γ-semigroup S is called left (right) duo if every left (right) Γ-ideal of S is a Γ-ideal of S, and S is called duo if it is both left and right duo. An ordered Γ-semigroup S is called left (right) simple if it does not contain any proper left (right) Γ-ideal, and S is called simple if it does not contain any proper Γ-ideal. A Γ-subsemigroup B of an ordered Γ-semigroup S is called a bi-Γ-ideal of S if $B\Gamma S\Gamma B \subseteq B$ and $a \in B, (S \ni) b \leq a \Rightarrow b \in B$. A Γ-subsemigroup I of an ordered Γ-semigroup S is called an interior Γ-ideal of S if $S\Gamma I\Gamma S \subseteq I$ and $a \in I, (S \ni) b \leq a \Rightarrow b \in I$.

Let S be an ordered Γ-semigroup and let A be any non-empty subset of S. Then by $L(A), R(A), J(A), I(A)$ and $B(A)$, we denote the left Γ-ideal, the right Γ-ideal, the Γ-ideal, the interior Γ-ideal and the bi-Γ-ideal of S generated by A respectively. It is easy to verify that $L(A) = (A \cup S\Gamma A]$, $R(A) = (A \cup A\Gamma S]$, $J(A) = (A \cup S\Gamma A \cup A\Gamma S \cup S\Gamma A\Gamma S]$, $I(A) = (A \cup A\Gamma A \cup S\Gamma A\Gamma S]$ and $B(A) = (A \cup A\Gamma A \cup A\Gamma S\Gamma A]$.

If $A = \{a\}$, we write $(a]$ instead of $(\{a\}]$, $L(a)$ instead of $L(\{a\})$, $R(a)$ instead of $R(\{a\})$, $J(a)$ instead of $J(\{a\})$, $I(a)$ instead of $I(\{a\})$ and $B(a)$ instead of $B(\{a\})$ respectively in the sequel.

Lemma 1.1. [1] *Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup and A, B be non-empty subsets of S. Then*

(1) $A \subseteq (A]$;
(2) *If* $A \subseteq B$, *then* $(A] \subseteq (B]$;
(3) $((A]] = (A]$;
(4) $(A]\Gamma(B] \subseteq (A\Gamma B]$;
(5) *If L is left ideal and R a right ideal of S, then the set $(L\Gamma R]$ is an ideal of S;*
(6) *If A, B are ideals of S, then $(A\Gamma B]$, $(B\Gamma A]$, $A \cup B$, $A \cap B$ are ideals of S;*
(7) $(S\Gamma a]((a\Gamma S], (S\Gamma a\Gamma S])$ *is a left (right, two-sided) ideal of S for each $a \in S$;*
(8) $((A]\Gamma(B]] = (A\Gamma B]$.

An ordered Γ-semigroup S is called regular (left regular, right regular) if for each $x \in S$, there exist $y \in S$ and $\alpha, \beta \in \Gamma$ such that $x \leq x\alpha y\beta x (x \leq y\alpha x\beta x, x \leq x\alpha x\beta y)$; and S is called intra-regular if for each $x \in S$, there exist $y, z \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that $x \leq y\alpha x\beta x\gamma z$; S is called right weakly regular if $x \in (x\Gamma S\Gamma x\Gamma S]$ for each $x \in S$ and S is called semisimple if for each $x \in S$, there exist $y, z, a \in S$ and $\alpha, \beta, \gamma, \delta \in \Gamma$ such that $x \leq y\alpha x\beta a\gamma x\delta z$.

Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup. A mapping f from S to real closed interval $[0, 1]$ is called the fuzzy subset of S (or fuzzy set of S). We denote by f_A the characteristic function of a subset A of S, which is defined as the mapping of S into $[0, 1]$ by

$$f_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

So for any element a of S, characteristic function $f_{\{a\}}$ of the subset $\{a\}$ of S denoted by f_a in the sequel, is

$$f_a(x) = \begin{cases} 1 & \text{if } x = a \\ 0 & \text{if } x \neq a. \end{cases}$$

Let f and g be two fuzzy subsets of S. Then $f \cap g, f \cup g$ and $f \circ g$ defined as

$$\begin{aligned} (f \cap g)(x) &= \min\{f(x), g(x)\} = f(x) \wedge g(x) \\ (f \cup g)(x) &= \max\{f(x), g(x)\} = f(x) \vee g(x) \end{aligned}$$

and

$$(f \circ g)(x) = \begin{cases} \bigvee\limits_{(y,z)\in A_x} \{f(y) \wedge g(z)\} & \text{if } A_x \neq \phi \\ 0 & \text{if } A_x = \phi, \end{cases}$$

where A_x is a relation on S defined as $A_x = \{(y, z) \in S \times S \mid x \leq y\alpha z$ for some $\alpha \in \Gamma\}$. We define an order relation $\preceq$ on the set of all fuzzy subsets of S by

$$f \preceq g \Leftrightarrow f(x) \leq g(x) \text{ for all } x \in S.$$

If f, g are fuzzy subsets of S such that $f \preceq g$, then, for every fuzzy subset h of S, $f \circ h \preceq g \circ h$ and $h \circ f \preceq h \circ g$. We denote by 1 the fuzzy subset of S defined by $1 : S \to [0, 1] | x \mapsto 1(x) = 1$. It may be easily checked that if S is an ordered Γ-semigroup, then the set of all fuzzy subsets of S with respect to multiplication "$\circ$" and the order "$\preceq$" is an ordered Γ-semigroup and the fuzzy subset 1 is the greatest element of the set of all fuzzy subsets of S.

Definition 1.2. Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup. A fuzzy subset f of S is called a fuzzy Γ-subsemigroup of S if for all $x, y \in S$ and $\alpha \in \Gamma$ such that

$$f(x\alpha y) \geq \min\{f(x), f(y)\}.$$

Definition 1.3. Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup. A fuzzy subset f of S is called a fuzzy left (right) Γ-ideal of S if

(1) $f(x\alpha y) \geq f(y)(f(x\alpha y) \geq f(x))$ for all $x, y \in S$ and $\alpha \in \Gamma$ and
(2) for any $x, y \in S$, $x \leq y$ implies $f(x) \geq f(y)$.

A fuzzy subset f of S is called a fuzzy Γ-ideal of S if it is both a fuzzy left and a fuzzy right Γ-ideal of S.

Definition 1.4. Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup. A fuzzy Γ-subsemigroup f of S is called a fuzzy bi-Γ-ideal of S if

(1) $f(x\alpha z\beta y) \geq \min\{f(x), f(y)\}$ for all $x, y, z \in S$ and $\alpha, \beta \in \Gamma$ and
(2) for any $x, y \in S$, $x \leq y$ implies $f(x) \geq f(y)$.

Definition 1.5. Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup. A fuzzy subset f of S is called a fuzzy generalized bi-Γ-ideal of S if

(1) $f(x\alpha z\beta y) \geq \min\{f(x), f(y)\}$ for all $x, y, z \in S$ and $\alpha, \beta \in \Gamma$ and
(2) for any $x, y \in S$, $x \leq y$ implies $f(x) \geq f(y)$.

Definition 1.6. Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup. A fuzzy Γ-subsemigroup f of S is called a fuzzy interior Γ-ideal of S if

(1) $f(x\alpha z\beta y) \geq f(z)$ for all $x, y, z \in S$ and $\alpha, \beta \in \Gamma$ and
(2) for any $x, y \in S$, $x \leq y$ implies $f(x) \geq f(y)$.

Definition 1.7. An ordered Γ-semigroup S is called fuzzy left (right) duo if every fuzzy left (right) Γ-ideal of S is a fuzzy Γ-ideal of S, and S is called fuzzy duo if it is both fuzzy left and fuzzy right duo.

Definition 1.8. A fuzzy subset f of an ordered Γ-semigroup S is called idempotent if $f \circ f = f$.

2. Coincidence of Fuzzy Γ-Ideals

Remark 2.1. In an ordered Γ-semigroup S every fuzzy Γ-ideal is a fuzzy interior Γ-ideal but the converse need not be true in general.

Example 2.2. Let $S = \{0, a, b, c\}$ and $\Gamma = \{\alpha, \beta\}$ be the non-empty sets. Define binary operations as:

α	0	a	b	c
0	0	0	0	0
a	0	b	0	a
b	0	b	0	c
c	0	0	0	b

β	0	a	b	c
0	0	0	0	0
a	a	a	a	a
b	0	0	0	0
c	a	a	a	c

Define order relation on S as, $\leq = \{(0,0), (a,a), (b,b), (c,c), (0,a), (0,b), (0,c)\}$. Clearly S is an ordered Γ-semigroup. The fuzzy set $\mu : S \to [0,1]$ defined by $\mu(0) = 0.9, \mu(a) = 0.7, \mu(b) = 0.6, \mu(c) = 0.1$ is a fuzzy interior Γ-ideal of S but not a fuzzy Γ-ideal since $\mu(a\alpha a) = \mu(b) = 0.6 \ngeq \mu(a)$.

Proposition 2.3. *Let $(S, \Gamma, \leq)$ be a right weakly regular ordered Γ-semigroup. Then every fuzzy interior Γ-ideal of S is a fuzzy Γ-ideal of S.*

Proof. Let S be a right weakly regular ordered Γ-semigroup. Let $a, b \in S$, since S is right weakly regular, there exist $x, y, u, v \in S$ and $\alpha, \beta, \gamma, \eta, \zeta, \delta \in \Gamma$ such that $a \leq a\alpha x\beta a\gamma y$ and $b \leq b\eta u\zeta b\delta v$. Now for any $\lambda \in \Gamma$, $a\lambda b \leq a\alpha x\beta a\gamma y\lambda b = (a\alpha x)\beta a\gamma(y\lambda b)$. Then for any fuzzy interior Γ-ideal f of S, we have

$$f(a\lambda b) \geq f((a\alpha x)\beta a\gamma(y\lambda b)) \geq f(a).$$

Also, as $a\lambda b \leq a\lambda b\eta u\zeta b\delta v = a\lambda b\eta(u\zeta b\delta v)$, we have

$$f(a\eta b) \geq f(a\lambda b\eta(u\zeta b\delta v)) \geq f(b).$$

Hence f is a fuzzy Γ-ideal of S. □

Remark 2.4. In an ordered Γ-semigroup, every fuzzy bi-Γ-ideal is a fuzzy generalized bi-Γ-ideal but the converse need not be true in general.

EXAMPLE 2.5. Let $S = \{0, a, b, c\}$ and $\Gamma = \{\alpha\}$ be two non-empty sets. Define a binary operation as:

α	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	0	a	a
c	0	0	a	a

Also define order relation on S as, $\leq := \{(0,0), (a,a), (b,b), (c,c)(0,a), (0,b)\}$. Clearly S is an ordered Γ-semigroup. The fuzzy set $\mu : S \to [0,1]$ defined by $\mu(0) = 0.6, \mu(a) = 0, \mu(b) = 0.3, \mu(c) = 0$ is a fuzzy generalized bi-Γ-ideal of S but not a fuzzy bi-Γ-ideal since $\mu(b\alpha b) = \mu(a) = 0 \ngeq \min\{\mu(b), \mu(b)\}$.

Proposition 2.6. *Let $(S, \Gamma, \leq)$ be a right weakly regular ordered Γ-semigroup. Then every fuzzy generalized bi-Γ-ideal of S is a fuzzy bi-Γ-ideal of S.*

Proof. Let S be a right weakly regular ordered Γ-semigroup. Let $a, b \in S$, since S is right weakly regular, there exist $x, y \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that $a \leq a\alpha x\beta a\gamma y$. Now for any $\eta \in \Gamma$, $a\eta b \leq a\alpha x\beta a\gamma y\eta b = a\alpha(x\beta a\gamma y)\eta b$. Then for any fuzzy generalized bi-Γ-ideal f of S, we have

$$f(a\eta b) \geq f(a\alpha(x\beta a\gamma y)\eta b) \geq \min\{f(a), f(b)\}.$$

Hence f is a fuzzy bi-Γ-ideal of S. □

Remark 2.7. In an ordered Γ-semigroup S, every fuzzy right Γ-ideal (resp. fuzzy left Γ-ideal, fuzzy Γ-ideal) is a fuzzy generalized bi-Γ-ideal(fuzzy bi-Γ-ideal) but the converse need not be true in general.

EXAMPLE 2.8. Let $S = \{0, a, b, c\}$ and $\Gamma = \{\alpha, \beta, \gamma\}$ be the non-empty sets, define binary operations as:

α	0	a	b	c	β	0	a	b	c	γ	0	a	b	c
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
a	a	a	a	a	a	0	a	0	0	a	0	b	0	a
b	0	0	0	b	b	0	0	b	0	b	0	b	0	c
c	0	0	0	c	c	0	0	0	c	c	0	0	0	b

Define order relation on S as, $\leq := \{(0,0), (a,a), (b,b), (c,c), (a,0), (b,0), (c,0)\}$. Clearly S is an ordered Γ-semigroup. The fuzzy set $\mu : S \to [0,1]$ defined by $\mu(0) = 0.6, \mu(a) = 0.7, \mu(b) = 0.8, \mu(c) = 0.9$ is a fuzzy generalized bi-Γ-ideal (and fuzzy bi-Γ-ideal), but not a fuzzy right Γ-ideal (left Γ-ideal) of S, since $\mu(b\beta c) = \mu(0) = 0.6 \ngeq 0.8 = \mu(b)(\mu(a\alpha b) = \mu(a) = 0.7 \ngeq 0.8 = \mu(b))$.

Proposition 2.9. *Let $(S, \Gamma, \leq)$ be a left duo (right duo, duo) right weakly regular ordered Γ-semigroup. Then every fuzzy generalized bi-Γ-ideal of S is a fuzzy right Γ-ideal (fuzzy left Γ-ideal, fuzzy Γ-ideal) of S.*

Proof. Let f be any fuzzy generalized bi-Γ-ideal of S and $x, y \in S$, $\gamma \in \Gamma$. Since S is left duo right weakly regular, $(S\Gamma x]$ is a Γ-ideal of S. Therefore

$$\begin{aligned} x\gamma y &\in (x\Gamma S\Gamma x\Gamma S]\Gamma S \\ &\subseteq (x\Gamma(S\Gamma x]\Gamma S]\Gamma S \\ &\subseteq (x\Gamma(S\Gamma x]]\Gamma(S] \\ &\subseteq (x\Gamma(S\Gamma x]\Gamma S] \\ &\subseteq ((x]\Gamma(S\Gamma x]] \\ &= (x\Gamma S\Gamma x]. \end{aligned}$$

Therefore there exist $z \in S$ and $\alpha, \beta \in \Gamma$ such that $x\gamma y \leq x\alpha z\beta x$. As f is fuzzy generalized bi-Γ-ideal of S, $f(x\gamma y) \geq f(x\alpha z\beta x) \geq \min\{f(x), f(x)\} = f(x)$. Hence f is a fuzzy right Γ-ideal of S. □

Proposition 2.10. *Let $(S, \Gamma, \leq)$ be a duo right weakly regular ordered Γ-semigroup. Then every fuzzy generalized bi-Γ-ideal of S is a fuzzy interior Γ-ideal of S.*

Proof. Let f be any fuzzy generalized bi-Γ-ideal of S and $x, a, y \in S, \alpha \in \Gamma$. Since S is duo right weakly regular, $(x\Gamma S]$ is a Γ-ideal of S. Therefore we have

$$\begin{aligned} x\alpha y &\in (x\Gamma S\Gamma x\Gamma S]\Gamma y \\ &\subseteq ((x\Gamma S]\Gamma S\Gamma S]\Gamma y \\ &\subseteq ((x\Gamma S]\Gamma S]\Gamma y \\ &\subseteq ((x\Gamma S]]\Gamma y \\ &\subseteq (x\Gamma S]\Gamma(y] \\ &\subseteq (x\Gamma S\Gamma y]. \end{aligned}$$

Thus there exist $z \in S$ and $\beta, \gamma \in \Gamma$, such that $x\alpha y \leq x\beta z\gamma y$. As f is fuzzy generalized bi-Γ-ideal of S, $f(x\alpha y) \geq f(x\beta z\gamma y) \geq \min\{f(x), f(y)\}$. Therefore f is fuzzy Γ-subsemigroup. Again, as S is duo right weakly regular, $(a\Gamma S], (S\Gamma a]$ are a Γ-ideal of S, we have

$$\begin{aligned} x\alpha a\beta y &\in x\alpha(a\Gamma S\Gamma a\Gamma S]\beta y = x\alpha((a\Gamma S]\Gamma(a\Gamma S]]\beta y \\ &\subseteq S\Gamma((a\Gamma S]\Gamma(a\Gamma S]]\Gamma S \\ &= (S]\Gamma((a\Gamma S]\Gamma(a\Gamma S]]\Gamma(S] \\ &\subseteq (S\Gamma(a\Gamma S]\Gamma(a\Gamma S]\Gamma S] \\ &\subseteq ((a\Gamma S]\Gamma(a\Gamma S]] \\ &= (a\Gamma S\Gamma a\Gamma S] \\ &\subseteq (a\Gamma(S\Gamma a]\Gamma S] \\ &\subseteq ((a]\Gamma(S\Gamma a]] \\ &= (a\Gamma S\Gamma a]. \end{aligned}$$

This implies that there exist $b \in S$ and $\gamma, \eta \in \Gamma,$ such that $x\alpha a\beta y \leq a\gamma z\eta a$. As f is a fuzzy generalized bi-Γ-ideal of S, $f(x\alpha a\beta y) \geq f(a\gamma z\eta a) \geq \min\{f(a), f(a)\}$ $= f(a)$. Hence f is fuzzy interior Γ-ideal of S. □

Lemma 2.11. [2] *Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup. Then A is a left Γ-ideal (resp. right Γ-ideal, Γ-ideal, generalized bi-Γ-ideal, bi-Γ-ideal, interior Γ-ideal) if and only if fuzzy subset f_A is fuzzy left Γ-ideal (resp. fuzzy right Γ-ideal, fuzzy Γ-ideal, fuzzy generalized bi-Γ-ideal, fuzzy bi-Γ-ideal, fuzzy interior Γ-ideal) of S.*

Proposition 2.12. *Let $(S, \Gamma, \leq)$ be a right weakly regular ordered Γ-semigroup. Then S is left duo (right duo, duo) if and only if S is fuzzy left duo (fuzzy right duo, fuzzy duo).*

Proof. Let S be any left duo right weakly regular ordered Γ-semigroup and f be any fuzzy left Γ-ideal of S. Let $a, b \in S$ and $\gamma \in \Gamma$. Since S is left duo right weakly regular, $(S\Gamma a]$ is a Γ-ideal of S. Therefore

$$\begin{aligned} a\gamma b &\in (a\Gamma S\Gamma a\Gamma S]\Gamma b \\ &\subseteq (a\Gamma(S\Gamma a]\Gamma S]\Gamma S \\ &\subseteq (x\Gamma(S\Gamma a]]\Gamma S \\ &\subseteq (S\Gamma(S\Gamma a]]\Gamma S \\ &\subseteq ((S\Gamma a]]\Gamma S \\ &= (S\Gamma a]\Gamma S \\ &\subseteq (S\Gamma a]. \end{aligned}$$

Thus there exists $x \in S$ and $\alpha \in \Gamma$ such that $a\gamma b \leq x\alpha a$. As f is a fuzzy left Γ-ideal of S, $f(a\gamma b) \geq f(x\alpha a) \geq f(a)$. Therefore f is a fuzzy right Γ-ideal of S. Hence f is fuzzy left duo.

Conversely, assume that S be a fuzzy left duo. Let A be any left Γ-ideal of S. Then, by Lemma 2.11, characteristic function f_A of A is fuzzy left Γ-ideal of S. By hypothesis, f_A is fuzzy right Γ-ideal of S. Again, by Lemma 2.11, A is right Γ-ideal of S. Hence S is left duo. □

Lemma 2.13. [3] *Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup. If S is left simple (right simple, simple), then $S = (S\Gamma a](S = (a\Gamma S],\ S = (S\Gamma a\Gamma S])$ for each $a \in S$.*

Proposition 2.14. *Let $(S, \Gamma, \leq)$ be a left (right) simple ordered Γ-semigroup. Then every fuzzy interior Γ-ideal of S is fuzzy left Γ-ideal (resp. fuzzy right Γ-ideal) of S.*

Proof. Since S is a left simple ordered Γ-semigroup, $S = (S\Gamma a]$ for each $a \in S$. Let f be any fuzzy interior Γ-ideal of S and $x, y \in S$. Since S is left simple, there exists $z \in S$ and $\alpha \in \Gamma$ such that $x \leq z\alpha y$. Then $x\gamma y \leq z\alpha y\gamma y$. So $f(x\gamma y) \geq f(z\alpha y\gamma y) \geq f(y)$. Hence f is a fuzzy left Γ-ideal of S. □

Proposition 2.15. *Let $(S, \Gamma, \leq)$ be a left and right simple ordered Γ-semigroup. Then every fuzzy generalized bi-Γ-ideal of S is fuzzy interior Γ-ideal of S.*

Proof. Let f be any fuzzy generalized bi-Γ-ideal of S and $x, y \in S$. Since S is left simple, there exists $z \in S$ and $\alpha \in \Gamma$ such that $y \leq z\alpha x$. Therefore $x\eta y \leq x\eta z\alpha x$, implies that $f(x\eta y) \geq f(x\eta z\alpha x) \geq \min\{f(x), f(x)\} = f(x) \geq \min\{f(x), f(y)\}$. Therefore f is fuzzy Γ-subsemigroup of S.

Next, take any $x, a, y \in S$. Since S is left and right simple ordered Γ-semigroup, there exist $r, s \in S$ and $\alpha, \beta \in \Gamma$ such that $x \leq a\alpha r$ and $y \leq s\beta a$. Now for any $\eta, \xi \in \Gamma$, $x\eta a\xi y \leq a\alpha r\eta a\xi s\beta a = a\alpha(r\eta a\xi s)\beta a$, implies that $f(x\eta a\xi y) \geq f(a\alpha(r\eta a\xi s)\beta a) \geq \min\{f(a), f(a)\} = f(a)$. Hence f is a fuzzy interior Γ-ideal of S. □

The proof of the following proposition is straightforward.

Proposition 2.16. *Let $(S, \Gamma, \leq)$ be a left (right) simple ordered Γ-semigroup. Then every fuzzy generalized bi-Γ-ideal of S is fuzzy bi-Γ-ideal of S.*

3. Characterizations of Right Weakly Regular Ordered Γ-Semigroups

Theorem 3.1. [12] *Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup. Then the following are equivalent:*

(i) *S is right weakly regular;*
(ii) *Each right Γ-ideal of S is idempotent;*
(iii) *$A \cap B = (B\Gamma A]$ for each right Γ-ideal B and Γ-ideal A of S.*

Corollary 3.2. [6] *Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup, f a fuzzy right Γ-ideal and g a fuzzy left Γ-ideal of S. Then $f \circ g \preceq f \wedge g$.*

The results of following Corollary is well known and is easy to prove.

Corollary 3.3. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup and A, B are subsets of S. Then the following are true:*

(1) *$A \subseteq B$ if and only if $f_A \preceq f_B$;*
(2) *$f_A \cap f_B = f_{A\cap B}$;*
(3) *$f_A \circ f_B = f_{(A\Gamma B]}$.*

Theorem 3.4. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup. Then S is right weakly regular if and only if for each fuzzy right Γ-ideal f and fuzzy Γ-ideal g of S, $f \cap g = f \circ g$.*

Proof. Let S be a right weakly regular ordered Γ-semigroup. Let f be a fuzzy right Γ-ideal and g be a fuzzy Γ-ideal of S. Take any $a \in S$. Since S is right weakly regular, there exist $x, y \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that $a \leq a\alpha x\beta a\gamma y$

i.e., $(a\alpha x, a\gamma y) \in A_a$. Then we have

$$\begin{aligned}(f \circ g)(a) &= \bigvee_{(p,q)\in A_a} \{f(p) \wedge f(q)\} \\ &\geq f(a\alpha x) \wedge g(a\gamma y) \\ &\geq f(a) \wedge g(a) = (f \cap g)(a).\end{aligned}$$

This implies $f \cap g \subseteq f \circ g$. Thus, by Corollary 3.2, $f \cap g = f \circ g$.

Conversely assume that $f \cap g = f \circ g$ for each fuzzy right Γ-ideal f and each fuzzy Γ-ideal g of S. Let $a \in B \cap A$, where B and A are right Γ-ideal and Γ-ideal of S respectively. Therefore, by Lemma 2.11, the fuzzy subsets χ_B and $_A$ are fuzzy right Γ-ideal and Γ-ideal of S respectively. By hypothesis

$$\begin{aligned}&(\chi_B {\cap}_A)(a) \leq (\chi_B {\circ}_A)(a) \\ &\Rightarrow \chi_B(a) \wedge \_A(a) \leq (\chi_B {\circ}_A)(a).\end{aligned}$$

Since $a \in B$ and $a \in A$, we have $_B(a) = 1 =_A (a)$. Thus $_B(a) {\wedge}_A(a) = 1 \wedge 1 = 1$. Therefore $(_B {\circ}_A)(a) \geq 1$. So $(_B {\circ}_A)(a) = 1$. By Corollary 3.3, $_B {\circ}_A =_{(B\Gamma A]}$. Thus we have $_{(B\Gamma A]}(a) = 1$. So $a \in (B\Gamma A]$. Therefore $B \cap A \subseteq (B\Gamma A]$. Also as $(B\Gamma A] \subseteq B \cap A$, $B \cap A = (B\Gamma A]$. Hence by Theorem 3.1, S is right weakly regular. □

Theorem 3.5. [12] *Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup. Then S is right weakly regular if and only if $B \cap I \subseteq (B\Gamma I]$ for each bi-Γ-ideal B and each interior Γ-ideal I of S.*

Theorem 3.6. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup. Then S is right weakly regular if and only if for each fuzzy bi-Γ-ideal f and each fuzzy interior Γ-ideal g of S, $f \cap g \subseteq f \circ g$.*

Proof. Let S be a right weakly regular ordered Γ-semigroup, f a fuzzy bi-Γ-ideal and g a fuzzy interior Γ-ideal of S. Take any $a \in S$. Since S is right weakly regular, there exist $x, y \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that $a \leq a\alpha x\beta a\gamma y$. This implies that $a \leq (a\alpha x)\beta(a\alpha x\beta a\gamma y)\gamma y = (a\alpha x\beta a)\alpha(x\beta a\gamma y\gamma y)$. So $(a\alpha x\beta a, x\beta a\gamma y\gamma y) \in A_a$. Now

$$\begin{aligned}(f \circ g)(a) &= \bigvee_{(p,q)\in A_a} \{f(p) \wedge f(q)\} \\ &\geq f(a\alpha x\beta a) \wedge g(x\beta a\gamma y\gamma y) \\ &\geq f(a) \wedge g(a) = (f \cap g)(a).\end{aligned}$$

So $f \cap g \subseteq f \circ g$.

Conversely assume that $f \cap g \subseteq f \circ g$ for each fuzzy bi-Γ-ideal f and each fuzzy interior Γ-ideal g of S. As each fuzzy right Γ-ideal of S is a fuzzy bi-Γ-ideal and each fuzzy Γ-ideal of S is a fuzzy interior Γ-ideal. Therefore, by hypothesis, $f \cap g \subseteq f \circ g$ for each fuzzy right Γ-ideal f and each fuzzy Γ-ideal g of S. Hence by Theorem 3.4, S is right weakly regular. □

Theorem 3.7. [12] *Let* $(S, \Gamma, \leq)$ *be an ordered* Γ*-semigroup. Then* S *is right weakly regular if and only if* $B \cap I \cap R \subseteq (B\Gamma I\Gamma R]$ *for each bi-*Γ*-ideal* B*, each interior* Γ*-ideal* I *and each right* Γ*-ideal* R *of* S*.*

Theorem 3.8. *Let* $(S, \Gamma, \leq)$ *be an ordered* Γ*-semigroup. Then* S *is right weakly regular if and only if for each fuzzy bi-*Γ*-ideal* f*, each fuzzy interior* Γ*-ideal* g *and each fuzzy right* Γ*-ideal* h *of* S*,* $f \cap g \cap h \subseteq f \circ g \circ h$*.*

Proof. Let S be a right weakly regular ordered Γ-semigroup, f a fuzzy bi-Γ-ideal, g a fuzzy interior Γ-ideal and h a fuzzy right Γ-ideal of S. Take any $a \in S$. Since S is right weakly regular, there exist $x, y \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that $a \leq a\alpha x\beta a\gamma y$. Now $a \leq (a\alpha x)\beta(a\alpha x\beta a\gamma y)\gamma y \leq (a\alpha x\beta a)\alpha(x\beta a\alpha x)\beta(a\gamma y\gamma y\gamma y)$. So $(a\alpha x\beta a, x\beta a\alpha x\beta a\gamma y\gamma y\gamma y) \in A_a$. Now

$$\begin{aligned}(f \circ g \circ h)(a) &= \bigvee_{(p,q)\in A_a} \{f(p) \wedge (g \circ h)(q)\} \\ &\geq f(a\alpha x\beta a) \wedge (g \circ h)(x\beta a\alpha x\beta a\gamma y\gamma y\gamma y) \\ &\geq f(a) \wedge \{\bigvee_{(r,s)\in A_{x\beta a\alpha x\beta a\gamma y\gamma y\gamma y}} \{g(r) \wedge h(s)\}\}.\end{aligned}$$

Since $(x\beta a\alpha x, a\gamma y\gamma y\gamma y) \in A_{x\beta a\alpha x\beta a\gamma y\gamma y\gamma y}$,

$$\begin{aligned}\bigvee_{(r,s)\in A_{x\beta a\alpha x\beta a\gamma y\gamma y\gamma y}} \{g(r) \wedge h(s)\} &\geq g(x\beta a\alpha x) \wedge h(a\gamma y\gamma y\gamma y) \\ &\geq g(a) \wedge h(a).\end{aligned}$$

So we get $(f \circ g \circ h)(a) \geq f(a) \wedge g(a) \wedge h(a)$. Hence $f \cap g \cap h \subseteq f \circ g \circ h$.

Conversely assume that $f \cap g \cap h \subseteq f \circ g \circ h$, for each fuzzy bi-Γ-ideal f, each fuzzy interior Γ-ideal g and each fuzzy right Γ-ideal h of S. Take any $a \in B \cap I \cap R$, where B is a bi-Γ-ideal, I is an interior Γ-ideal and R is a right Γ-ideal of S. By Lemma 2.11, χ_B, χ_I and χ_R are fuzzy bi-Γ-ideal, fuzzy interior Γ-ideal and fuzzy right Γ-ideal of S respectively. Then, by hypothesis

$$\begin{aligned}&(\chi_B \cap \chi_I \cap_A)(a) \leq (\chi_B \circ \chi_I \circ \chi_R)(a) \\ &\Rightarrow \chi_B(a) \wedge \chi_I(a) \wedge \chi_(a) \leq (\chi_B \circ \chi_I \circ \chi_R)(a).\end{aligned}$$

Since $a \in B, a \in I$ and $a \in R$, we have $\chi_B(a) = 1 = \chi_I(a) = \chi_R(a)$. Thus $\chi_B(a) \wedge \chi_I(a) \wedge \chi_(a) = 1 \wedge 1 \wedge 1 = 1$. Therefore $(\chi_B \circ \chi_I \circ \chi_R)(a) \geq 1$. So $(\chi_B \circ \chi_I \circ \chi_R)(a) = 1$. Now, by Corollary 3.3, $\chi_B \circ \chi_I \circ \chi_R = \chi_{(B\Gamma I\Gamma A]}$. So $\chi_{(B\Gamma I\Gamma R]}(a) = 1$ implies $a \in (B\Gamma I\Gamma R]$. Therefore $B \cap I \cap R \subseteq (B\Gamma I\Gamma R]$. Hence by Theorem 3.7, S is right weakly regular. □

Theorem 3.9. *Let* $(S, \Gamma, \leq)$ *be an ordered* Γ*-semigroup. Then* S *is right weakly regular if and only if each fuzzy right* Γ*-ideal of* S *is idempotent.*

Proof. Let S be a right weakly regular ordered Γ-semigroup and f a fuzzy right Γ-ideal of S. Take any $a \in S$. Since S is right weakly regular, there exist

$x, y \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that $a \leq a\alpha x\beta a\gamma y$ i.e., $(a\alpha x, a\gamma y) \in A_a$. Now

$$\begin{aligned}(f \circ f)(a) &= \bigvee_{(p,q)\in A_a} \{f(p) \wedge f(q)\} \\ &\geq f(a\alpha x) \wedge f(a\gamma y) \\ &\geq f(a) \wedge f(a) = f(a).\end{aligned}$$

This implies $f \subseteq f \circ f$. Also, as f is fuzzy right Γ-ideal of S, $f \circ f \subseteq f$. Therefore $f = f \circ f$.

Conversely assume that each fuzzy right Γ-ideal of S is idempotent. Take any $a \in S$. Let $A = (a \cup a\Gamma S]$, the right Γ-ideal generated by the element a of S. By Lemma 2.11, χ_A is fuzzy right Γ-ideal of S. By hypothesis, $(\chi_A \circ \chi_A)(a) = \chi_A(a) = 1$ as $\chi_{(A\Gamma A]}(a) = (\chi_A \circ \chi_A)(a)$, $\chi_{(A\Gamma A]}(a) = 1$. Therefore $a \in (A\Gamma A]$. Now, we have

$$\begin{aligned}a \in (A\Gamma A] &= ((a \cup a\Gamma S]\Gamma(a \cup a\Gamma S]] \\ &= ((a \cup a\Gamma S)\Gamma(a \cup a\Gamma S)] \\ &= (a\Gamma a \cup a\Gamma a\Gamma S \cup a\Gamma S\Gamma a \cup a\Gamma S\Gamma a\Gamma S] \\ &\subseteq (a\Gamma S\Gamma a\Gamma S].\end{aligned}$$

Hence S is right weakly regular. □

Corollary 3.10. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup. Then the following are equivalent:*

(1) *S is right weakly regular;*
(2) *Every fuzzy right Γ-ideal is idempotent;*
(3) *$f \cap g \subseteq f \circ g$ for each fuzzy right Γ-ideal f and fuzzy Γ-ideal g of S.*

Corollary 3.11. [5] *Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup. A fuzzy subset f of S is a fuzzy right Γ-ideal of S if and only if*

(1) *$f \circ 1 \preceq f$; and*
(2) *if $x \leq y$, then $f(x) \geq f(y)$.*

Corollary 3.12. [6] *An ordered Γ-semigroup S is intra-regular if and only if for each fuzzy right Γ-ideal f and each fuzzy left Γ-ideal g of S, $f \wedge g \preceq g \circ f$.*

Theorem 3.13. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup. Then the following are equivalent:*

(1) *S is both intra-regular and right weakly regular;*
(2) *$f \cap g \cap h \subseteq f \circ g \circ h$ for each fuzzy quasi-Γ-ideal f, each fuzzy right Γ-ideals g and h of S;*
(3) *$f \cap g \cap h \subseteq f \circ g \circ h$ for each fuzzy bi-Γ-ideal f, each fuzzy right Γ-ideals g and h of S;*
(4) *$f \cap g \cap h \subseteq f \circ g \circ h$ for each fuzzy generalized bi-Γ-ideal f, each fuzzy right Γ-ideals g and h of S.*

Proof. (1) ⇒ (4) Let S be both intra-regular and right weakly regular ordered Γ-semigroup. Let f be any fuzzy generalized bi-Γ-ideal and g, h be the fuzzy right Γ-ideals of S. Let $a \in S$. Since S is right weakly regular and intra-regular ordered Γ-semigroup, there exist $x_1, y_1, x_2, y_2 \in S$ and $\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2 \in \Gamma$ such that $a \leq a\alpha_1 x_1 \beta_1 a \gamma_1 y_1$ and $a \leq x_2 \alpha_2 a \beta_2 a \gamma_2 y_2$ which implies that

$$\begin{aligned} a &\leq a\alpha_1 x_1 \beta_1 (x_2 \alpha_2 a \beta_2 a \gamma_2 y_2) \gamma_1 y_1 \\ &= (a\alpha_1 x_1 \beta_1 x_2) \alpha_2 (a \beta_2 a \gamma_2 y_2 \gamma_1 y_1) \end{aligned}$$

i.e., $(a\alpha_1 x_1 \beta_1 x_2, a\beta_2 a \gamma_2 y_2 \gamma_1 y_1) \in A_a$. Therefore

$$\begin{aligned} (f \circ g \circ h)(a) &= \bigvee_{(p,q) \in A_a} \{f(p) \wedge (g \circ h)(q)\} \\ &\geq f(a\alpha_1 x_1 \beta_1 x_2) \wedge (g \circ h)(a\beta_2 a \gamma_2 y_2 \gamma_1 y_1) \\ &\geq f(a) \wedge \{ \bigvee_{(r,s) \in A_{a\beta_2 a \gamma_2 y_2 \gamma_1 y_1}} \{g(r) \wedge h(s)\}\}. \end{aligned}$$

Since $(a, a\gamma_2 y_2 \gamma_1 y_1) \in A_{a\beta_2 a \gamma_2 y_2 \gamma_1 y_1}$,

$$\begin{aligned} \bigvee_{(r,s) \in A_{a\beta_2 a \gamma_2 y_2 \gamma_1 y_1}} \{g(r) \wedge h(s)\} &\geq g(a) \wedge h(a\gamma_2 y_2 \gamma_1 y_1) \\ &\geq g(a) \wedge h(a). \end{aligned}$$

Finally we get $(f \circ g \circ h)(a) \geq f(a) \wedge g(a) \wedge h(a)$. Therefore $f \cap g \cap h \subseteq f \circ g \circ h$.
(4) ⇒ (3) Obvious, as each fuzzy bi-Γ-ideal is fuzzy generalized bi-Γ-ideal.
(3) ⇒ (2) Obvious, as each fuzzy quasi-Γ-ideal is fuzzy bi-Γ-ideal.
(2) ⇒ (1) Let (2) holds i.e., $f \cap g \cap h \subseteq f \circ g \circ h$ for each fuzzy quasi-Γ-ideal f, each fuzzy right Γ-ideals g and h of S. Let g be any fuzzy right Γ-ideal of S. Since fuzzy subset 1 is a fuzzy quasi-ideal of S, by hypothesis, $g = g \cap 1 \cap g \subseteq g \circ 1 \circ g \subseteq g \circ g \subseteq g \circ 1 \subseteq g$. So $g = g \circ g$. Therefore by Theorem 3.9, S is right weakly regular.

Next, since each fuzzy left Γ-ideal f is fuzzy quasi-Γ-ideal and the fuzzy subset 1 is fuzzy right Γ-ideal of S, we have, $f \cap g = 1 \cap f \cap g \subseteq 1 \circ f \circ g \subseteq f \circ g$. Hence by Corollary 3.12, S is intra-regular. □

Corollary 3.14. [6] *Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup f, g fuzzy subsets of S, and $a \in S$. Then the following are equivalent:*

(1) $(f \circ g)(a) \neq 0$*;*
(2) *There exists $(x, y) \in A_a$ such that $f(x) \neq 0$ and $g(y) \neq 0$.*

Corollary 3.15. [6] *Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup, f a fuzzy subsets of S and $a \in S$. Then the following are equivalent:*

(1) $(f \circ 1)(a) \neq 0$*;*
(2) *There exists $(x, y) \in A_a$ such that $f(x) \neq 0$.*

Corollary 3.16. [6] *Let $(S,\Gamma,\leq)$ be an ordered Γ-semigroup, g a fuzzy subsets of S and $a\in S$. Then the following are equivalent:*

(1) $(1\circ g)(a)\neq 0$;
(2) *There exists $(x,y)\in A_a$ such that $g(y)\neq 0$.*

Theorem 3.17. *Let $(S,\Gamma,\leq)$ be an ordered Γ-semigroup. Then S is right weakly regular if and only if for each fuzzy subset f of S, $f\preceq f\circ 1\circ f\circ 1$.*

Proof. Let f be a fuzzy subset of S and $a\in S$. Since S is right weakly regular, there exist $x,y\in S$ and $\alpha,\beta,\gamma\in\Gamma$ such that $a\leq a\alpha x\beta a\gamma y$ i.e., $(a\alpha x, a\gamma y)\in A_a$. Now

$$\begin{aligned}(f\circ 1\circ f\circ 1)(a) &= \bigvee_{(p,q)\in A_a}\{(f\circ 1)(p)\wedge(f\circ 1)(q)\}\\ &\geq (f\circ 1)(a\alpha x)\wedge(f\circ 1)(a\gamma y).\end{aligned}$$

Since $(a,x)\in A_{a\alpha x}$ and $(a,y)\in A_{a\gamma y}$, we have

$$\begin{aligned}&(f\circ 1)(a\alpha x)\wedge(f\circ 1)(a\gamma y)\\ &=\{\bigvee_{(u,v)\in A_{a\alpha x}}\{f(u)\wedge 1(v)\}\}\wedge\{\bigvee_{(r,s)\in A_{a\gamma y}}\{f(r)\wedge 1(s)\}\}\\ &\geq (f(a)\wedge 1(x))\wedge(f(a)\wedge 1(y))\\ &= f(a)\wedge f(a)=f(a).\end{aligned}$$

Therefore $f\preceq f\circ 1\circ f\circ 1$.

Conversely suppose that $a\in S$. Since f_a is a fuzzy subset of S, by hypothesis, $1=f_a(a)\leq(f_a\circ 1\circ f_a\circ 1)(a)\leq 1$. So $(f_a\circ 1\circ f_a\circ 1)(a)\neq 0$. By Corollary 3.14, there exists $(x,y)\in A_a$ such that $(f_a\circ 1)(x)\neq 0$ and $(f_a\circ 1)(y)\neq 0$. By Corollary 3.15, there exists $(u,v)\in A_x$ and $(r,s)\in A_y$ such that $f_a(u)\neq 0$ and $f_a(r)\neq 0$. This implies that $a=u=r$. As $(x,y)\in A_a,(u,v)\in A_x$ and $(r,s)\in A_y$, we have $a\leq x\alpha y, x\leq u\beta v$ and $y\leq r\gamma s$ for some $\alpha,\beta,\gamma\in\Gamma$. Therefore $a\leq u\beta v\alpha r\gamma s$. Since $a=u=r$, $a\leq a\beta v\alpha a\gamma s$. Hence S is right weakly regular. □

Corollary 3.18. [5] *Let $(S,\Gamma,\leq)$ be an ordered Γ-semigroup. A fuzzy subset f of S is a fuzzy left Γ-ideal of S if and only if*

(1) $1\circ f\preceq f$;
(2) *if $x\leq y$, then $f(x)\geq f(y)$.*

Theorem 3.19. *Let $(S,\Gamma,\leq)$ be an ordered Γ-semigroup. A fuzzy subset f of S is a fuzzy interior Γ-ideal of S if and only if*

(1) $1\circ f\circ 1\preceq f$ *and*
(2) *if $x\leq y$, then $f(x)\geq f(y)$.*

Proof. Let f be any fuzzy interior Γ-ideal of an ordered Γ-semigroup S, and let $a \in S$. If $A_a = \phi$, then $(1 \circ f \circ 1)(a) = 0 \leq f(a)$. On the other hand

$$(1 \circ f \circ 1)(a) = \bigvee_{(y,z) \in A_a} \{(1 \circ f)(y) \wedge 1(z)\} = \bigvee_{(y,z) \in A_a} (1 \circ f)(y).$$

It is sufficient to show that for each $(y, z) \in A_a$,

$$\bigvee_{(y,z) \in A_a} (1 \circ f)(y) \leq f(a).$$

So take any $(y, z) \in A_a$. If $A_y = \phi$, then equation (1) holds. So let $A_y \neq \phi$. Then

$$(1 \circ f)(y) = \bigvee_{(x,w) \in A_y} \{1(x) \wedge f(w)\} = \bigvee_{(x,w) \in A_y} f(w).$$

Now we need to show only that $f(w) \leq f(a)$ for each $(x, w) \in A_y$. So take any $(x, w) \in A_y$. Then $y \leq x\alpha w$ for some $\alpha \in \Gamma$. As $(y, z) \in A_a, a \leq y\beta z$ for some $\beta \in \Gamma$. Therefore $a \leq x\alpha w\beta z$. Since f is an interior Γ-ideal of S, $f(a) \geq f(x\alpha w\beta z) \geq f(w)$. Hence $f(w) \leq f(a)$ for each $(x, w) \in A_y$.

Conversely assume that $x, a, y \in S$ and $\alpha, \beta \in \Gamma$. Since $(1 \circ f \circ 1) \preceq f$, $(1 \circ f \circ 1)(x\alpha a\beta y) \leq f(x\alpha a\beta y)$. Since $(x\alpha a, y) \in A_{x\alpha a\beta y}$,

$$(1 \circ f \circ 1)(x\alpha a\beta y) = \bigvee_{(p,q) \in A_{x\alpha a\beta y}} \{(1 \circ f)(p) \wedge 1(q)\} \geq (1 \circ f)(x\alpha a) \wedge 1(y) = (1 \circ f)(x\alpha a).$$

Since $(x, a) \in A_{x\alpha a}$,

$$(1 \circ f)(x\alpha a) = \bigvee_{(r,s) \in A_{x\alpha a}} \{1(r) \wedge f(s)\} \geq 1(x) \wedge f(a) = f(a).$$

Thus $f(x\alpha a\beta y) \geq f(a)$. Hence f is a fuzzy interior Γ-ideal of S. □

Theorem 3.20. *Let $(S, \Gamma, \leq)$ be a right weakly regular ordered Γ-semigroup. Then the following holds:*

(1) *Each fuzzy interior Γ-ideal of S is idempotent;*
(2) *Each fuzzy Γ-ideal of S is idempotent;*
(3) *Each fuzzy right Γ-ideal of S is idempotent.*

Proof. (1). Let S be a right weakly regular ordered Γ-semigroup and f be any fuzzy interior Γ-ideal of S. Take any $a \in S$. Since S is right weakly regular, there exist $x, y \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that $a \leq a\alpha x\beta a\gamma y \leq (a\alpha x\beta a\gamma y)\alpha x\beta a\gamma y = ((a\alpha x)\beta a\gamma y)\alpha(x\beta a\gamma y)$ i.e., $((a\alpha x)\beta a\gamma y, x\beta a\gamma y) \in A_a$.

Now

$$\begin{aligned}(f\circ f)(a) &= \bigvee_{(p,q)\in A_a}\{f(p)\wedge f(q)\}\\ &\geq f((a\alpha x)\beta a\gamma y)\wedge f(x\beta a\gamma y)\\ &\geq f(a)\wedge f(a)=f(a).\end{aligned}$$

This implies $f\subseteq f\circ f$. Again, as S is right weakly regular, by Theorem 3.17 $f\preceq f\circ 1\circ f\circ 1$. Therefore $f\circ f\subseteq f\circ 1\circ f\circ 1\circ f = f\circ(1\circ f\circ 1)\circ f$. As f is a fuzzy interior Γ-ideal, $f\circ(1\circ f\circ 1)\circ f\subseteq f\circ f\circ f\subseteq 1\circ f\circ 1\subseteq f$. Thus $f\circ f\subseteq f$. Hence $f=f\circ f$.
(2). Obvious, as each fuzzy Γ-ideal is a fuzzy interior Γ-ideal of S.
(3). Follows by Theorem 3.9. □

Proposition 3.21. *Let $(S,\Gamma,\leq)$ be a right weakly regular ordered Γ-semigroup, f a fuzzy right Γ-ideal and g a fuzzy Γ-ideal of S. Then $f\circ g$ is a fuzzy quasi-Γ-ideal of S.*

Proof. Let $(S,\Gamma,\leq)$ be a right weakly regular ordered Γ-semigroup, f a fuzzy right Γ-ideal and g a fuzzy Γ-ideal of S. Then, by Theorem 3.4, $f\wedge g = f\circ g$. Since $f\wedge g\preceq f,g$, $(f\wedge g)\circ 1\preceq f\circ 1, g\circ 1$. As f is a fuzzy right Γ-ideal, $f\circ 1\preceq f$. Therefore $(f\wedge g)\circ 1\preceq f$. Again $1\circ(f\wedge g)\preceq 1\circ f, 1\circ g$. As g is fuzzy Γ-ideal, $1\circ g\preceq g$. Hence $1\circ(f\wedge g)\preceq g$. Therefore $1\circ(f\wedge g)\preceq g$. Thus $((f\wedge g)\circ 1)\wedge(1\circ(f\wedge g))\preceq f\wedge g$.

Next, take any $x,y\in S$ such that $x\leq y$. Then we have to show that $(f\wedge g)(x)\geq(f\circ g)(y)$. Since f and g are fuzzy right Γ-ideal and fuzzy Γ-ideal of S, $f(x)\geq f(y)$ and $g(x)\geq g(y)$. Now

$$\begin{aligned}(f\wedge g)(x) &= \min\{f(x),g(x)\}\\ &\geq \min\{f(y),g(y)\}=(f\wedge g)(y).\end{aligned}$$

Therefore $f\circ g$ is fuzzy quasi-Γ-ideal of S. □

Definition 3.22. An ordered Γ-semigroup S is called semisimple if for each $a\in S$, there exist $x,y,z\in S$ and $\alpha,\beta,\gamma,\delta\in\Gamma$ such that $a\leq x\alpha a\beta y\gamma a\delta z$.

Corollary 3.23. [2] *Every fuzzy interior Γ-ideal of a semisimple ordered Γ-semigroup S is a fuzzy Γ-ideal.*

Theorem 3.24. *Let $(S,\Gamma,\leq)$ be an ordered Γ-semigroup. Then the following are equivalent:*

(1) *S is semisimple;*
(2) *Every fuzzy Γ-ideal of S is idempotent;*
(3) *every fuzzy interior Γ-ideal of S is idempotent;*
(4) *$f\wedge g=f\circ g$, for every fuzzy Γ-ideal f and g of S;*
(5) *$f\wedge g=f\circ g$, for every fuzzy Γ-ideal f and every fuzzy interior Γ-ideal g of S;*

(6) *$f \wedge g = f \circ g$, for every fuzzy interior f and every fuzzy Γ-ideal g of S;*
(7) *$f \wedge g = f \circ g$, for every fuzzy interior Γ-ideal f and g of S;*
(8) *$f \circ g = g \circ f$ and $f \circ f = f$ for each fuzzy Γ-ideal f and g of S;*
(9) *$f \circ g = g \circ f$ and $f \circ f = f$ for each fuzzy interior Γ-ideal f and g of S.*

Proof. (1) $\Rightarrow$ (7) Assume that (1) holds. Let f and g be any fuzzy interior Γ-ideals of S. By Corollary 3.23, f and g are fuzzy Γ-ideals of S. So $f \circ g \subseteq f \circ 1 \subseteq f$ and $f \circ g \subseteq 1 \circ g \subseteq g$. Therefore $f \circ g \subseteq f \wedge g$.

Next take any $a \in S$. Since S is semisimple, there exist $x, y, z \in S$ and $\alpha, \beta, \gamma, \delta \in \Gamma$ such that $a \leq x\alpha a\beta y\gamma a\delta z$ i.e., $(x\alpha a\beta y, a\delta z) \in A_a$. Hence

$$\begin{aligned}(f \circ g)(a) &= \bigvee_{(p,q)\in A_a} \{f(p) \wedge g(q)\} \\ &\geq f(x\alpha a\beta y) \wedge g(a\delta z) \\ &\geq f(a) \wedge g(a) = (f \cap g)(a).\end{aligned}$$

Therefore $f \wedge g \subseteq f \circ g$. Hence $f \wedge g = f \circ g$
(7) $\Rightarrow$ (6) Obvious, as every fuzzy Γ-ideal is fuzzy interior Γ-ideal.
(6) $\Rightarrow$ (4) Obvious, as every fuzzy Γ-ideal is fuzzy interior Γ-ideal.
(7) $\Rightarrow$ (5) $\Rightarrow$ (4) $\Rightarrow$ (2).
(7) $\Rightarrow$ (3) $\Rightarrow$ (2) and (7) $\Rightarrow$ (9) $\Rightarrow$ (8) $\Rightarrow$ (2).
(2) $\Rightarrow$ (1) Assume that (2) holds. Take any $a \in S$. Let $J(a) = (a \cup a\Gamma S \cup S\Gamma a \cup S\Gamma a\Gamma S]$, the Γ-ideal generated by a. Then, by Lemma 2.11, $f_{J(a)}$ is a fuzzy Γ-ideal of S. So $f_{(J(a)\Gamma J(a)]}(a) = (f_{(J(a)]}(a) \circ f_{(J(a)]}(a)) = f_{(J(a)]}(a) = 1$. Implying that $a \in (J(a)\Gamma J(a)]$. Therefore

$$\begin{aligned}a &\in ((a \cup a\Gamma S \cup S\Gamma a \cup S\Gamma a\Gamma S]\Gamma(a \cup a\Gamma S \cup S\Gamma a \cup S\Gamma a\Gamma S]] \\ &= ((a \cup a\Gamma S \cup S\Gamma a \cup S\Gamma a\Gamma S)\Gamma(a \cup a\Gamma S \cup S\Gamma a \cup S\Gamma a\Gamma S)] \\ &= (a\Gamma a \cup a\Gamma a\Gamma S \cup a\Gamma S\Gamma a\Gamma S \cup a\Gamma S\Gamma a \cup a\Gamma S\Gamma a\Gamma S \cup a\Gamma S\Gamma S\Gamma a\Gamma S \\ &\cup S\Gamma a\Gamma a\Gamma S \cup S\Gamma a\Gamma S\Gamma a \cup S\Gamma a\Gamma S\Gamma a\Gamma S \cup S\Gamma a\Gamma S\Gamma a \cup S\Gamma a\Gamma S\Gamma a\Gamma S \\ &\cup S\Gamma a\Gamma S\Gamma S\Gamma \cup S\Gamma a\Gamma S\Gamma S\Gamma a\Gamma S] \\ &\subseteq (S\Gamma a\Gamma S\Gamma a\Gamma S].\end{aligned}$$

Hence S is semisimple. □

Conclusion: In ordered semigroups, the class of regular ordered semigroups play an important role in studying the structures of ordered semigroups, for instance, Kehayopulu and Tsingelis [7] studied the class of regular orddered semigroups by employing fuzzy ideals of ordered semigroups. Thereafter many authors such as [2, 5, 6, 13] had studied the class of regular ordered Γ-semigroups. In this paper, we enhance the understanding of right weakly regular class of ordered Γ-semigroups through its fuzzy Γ-ideals, fuzzy bi-Γ-ideals, fuzzy generalized bi-Γ-ideals and fuzzy interior Γ-ideals which will motivate the researchers towards this new class in ordered Γ-semigroups. Since ordered Γ-semigroups are the generalizations of ordered semigroups, all the results of this paper hold true for ordered semigroups by simply taking the set Γ as any singleton set which is the main application of the results of this paper and a reasonable justification of the new notions introduced.

Acknowledgments

The authors wish to thank sincerely the referees for their valuable comments and suggestions.

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