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On the Graded Primal Avoidance Theorem

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Abstract. Let G be an abelian group with identity e. Let R be a G-graded commutative ring and M a graded R-module. In this paper, we generalize the graded primary avoidance theorem for modules to the graded primal avoidance theorem for modules. We also introduce the concept of graded P_L-compactly packed modules and give a number of its properties.

Keywords: Graded primal submodules, Graded primal avoidance, Graded P_L-compactly packed modules.

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1. Introduction and Preliminaries

The graded prime avoidance theorem for modules was introduced and studied by F. Farzalipour and P. Ghiasvand in [5]. The graded primary avoidance theorem for modules was introduced and studied by S.E. Atani and U. Tekir in [3]. Also, graded P-compactly packed modules were introduced and studied by K. Al-Zoubi, I. Jaradat and M. Al-Dolat in [1]. Here, we study the graded primal avoidance theorem for modules. A number of results concerning the graded primal avoidance theorem are given. Also, we introduce the concept of graded P_L-compactly packed modules and give a number of its properties.

Before we state some results, let us introduce some notations and terminologies. Let G be an abelian group with identity e and R be a commutative ring with identity 1_R. Then R is a *G-graded ring* if there exist additive subgroups

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R_g of R such that $R = \bigoplus_{g \in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. The nonzero elements of R_g are said to be *homogeneous* of degree g where the R_g's are additive subgroups of R indexed by the elements $g \in G$. If $x \in R$, then x can be written uniquely as $\sum_{g \in G} x_g$, where x_g is the homogeneous component of x in R_g. Moreover, $h(R) = \bigcup_{g \in G} R_g$. Let I be an ideal of R. Then I is called a *graded ideal* of a G-graded ring R if $I = \bigoplus_{g \in G}(I \bigcap R_g)$. Thus, if $x \in I$, then $x = \sum_{g \in G} x_g$ with $x_g \in I$. An ideal of a G-graded ring need not be G-graded (see [7, 8].)

Let R be a G-graded ring and M an R-module. We say that M is a *G-graded R-module* (or *graded R-module*) if there exists a family of subgroups $\{M_g\}_{g \in G}$ of M such that $M = \bigoplus\limits_{g \in G} M_g$ (as abelian groups) and $R_g M_h \subseteq M_{gh}$ for all $g, h \in G$. Here, $R_g M_h$ denotes the additive subgroup of M consisting of all finite sums of elements $r_g s_h$ with $r_g \in R_g$ and $s_h \in M_h$. Also, we write $h(M) = \bigcup\limits_{g \in G} M_g$ and the elements of $h(M)$ are said to be *homogeneous*. Let $M = \bigoplus\limits_{g \in G} M_g$ be a graded R-module and N a submodule of M. Then N is called a *graded submodule* of M if $N = \bigoplus\limits_{g \in G} N_g$ where $N_g = N \cap M_g$ for $g \in G$. In this case, N_g is called the *g-component* of N (see [7, 8].)

Let R be a G-graded ring and $S \subseteq h(R)$ be a multiplicatively closed subset of R. Then the ring of fraction $S^{-1}R$ is a graded ring which is called the graded ring of fractions and denoted by R_S. Indeed, $R_S = \bigoplus\limits_{g \in G} (R_S)_g$ where $(R_S)_g = \{r/s : r \in h(R), s \in S \text{ } and \text{ } g = (\deg s)^{-1}(\deg r)\}$. Let M be a graded module over a G-graded ring R and $S \subseteq h(R)$ be a multiplicatively closed subset of R. The module of fractions $S^{-1}M$ over a graded ring $S^{-1}R$ is a graded module $M_S = \bigoplus\limits_{g \in G} (M_S)_g$ where $(M_S)_g = \{m/s : m \in h(M), s \in S \text{ } and$ $g = (\deg s)^{-1}(\deg m)\}$. We write $h(R_S) = \bigcup\limits_{g \in G} (R_S)_g$ and $h(M_S) = \bigcup\limits_{g \in G} (M_S)_g$. Consider the graded homomorphism $\eta : M \to M_S$ defined by $\eta(m) = m/1$. For any graded submodule N of M, the submodule of M_S generated by $\eta(N)$ is denoted by N_S. Similar to non graded case, one can prove that $N_S = \{\beta \in M_S : \beta = m/s \text{ for } m \in N \text{ and } s \in S\}$ and that $N_S \neq S^{-1}M$ if and only if $S \cap (N :_R M) = \phi$. If K is a graded submodule of an R_S-module M_S, then $K \cap M$ will denote the graded submodule $\eta^{-1}(K)$ of M. A graded R-module M is called *graded finitely generated* if there exist $x_{g_1}, x_{g_2}, ..., x_{g_n} \in h(M)$ such that $M = Rx_{g_1} + \cdots + Rx_{g_n}$. A graded R-module M is called *graded cyclic* if $M = Rm_g$ where $m_g \in h(M)$. For more details, one can refer to [7, 8]. A proper graded ideal P of R is said to be *a graded prime ideal* if whenever $r, s \in h(R)$ with $rs \in P$, then either $r \in P$ or $s \in P$ (see [9].) It is known that a graded prime ideal is not a prime ideal (see [9, Example 1.6].) A proper graded submodule P of a graded R-module M is said to be *a graded primary* *submodule* if whenever $r \in h(R)$ and $m \in h(M)$ with $rm \in P$, then either $m \in P$ or $r^k \in (P :_R M)$ for some positive integer k (see [3].)

2. The Graded Primal Avoidance Theorem

Let R be a G-graded ring, M a graded R-module and N a graded submodule of M. An element $a \in h(R)$ is called *homogeneous prime to* N if $am \in N$, with $m \in h(M)$, implies that $m \in N$. An element $a = \sum_{g \in G} a_g \in R$ is called *G-prime to* N if at least one homogeneous component a_g of a is homogeneous prime to N. Denote by $g(N)$ the set of all homogeneous elements of R that are not homogeneous prime to N and by $G(N)$ the set of all elements of R that are not G-prime to N. N is called a *graded primal submodule* of M if $N \neq M$ and $P = G(N)$ is an ideal of R. By [4, Theorem 1.4], this ideal is always a graded prime ideal, called the adjoint graded prime ideal of N. In this case we also say that N is a graded P-primal submodule of M. The graded primal and primal submodules are different concepts (see[4].) Recall that the concept of graded primal submodules is a generalization of the concept of graded primary submodules. For more details, one can refer to [4].

The following Lemma is known (see [2, Lemma 1.2] and [6, Lemma 1.1 and Lemma 1.2]) and we write it her for the sake of references.

Lemma 2.1. *Let R be a G-graded ring and M a graded R-module. Then the following hold:*

(i) *If I and J are graded ideals of R, then $I + J$ and $I \cap J$ are graded ideals.*
(ii) *If N is a graded submodule of M, $r \in h(R)$, $x \in h(M)$ and I is a graded ideal of R, then Rx, IN and rN are graded submodules of M.*
(iii) *If N and K are graded submodules of M, then $(N :_R M) = \{r \in R : rM \subseteq N\}$ is a graded ideal of R.*
(iv) *Let $\{N_\lambda\}$ be a collection of graded submodules of M. Then $\sum_\lambda N_\lambda$ and $\bigcap_\lambda N_\lambda$ are graded submodules of M.*

Proposition 2.2. *Let R be a G-graded ring, M be a graded R-module, N and K be two proper graded submodules of M and J be a graded ideal of R. If $JK \subseteq N$, then either $K \subseteq N$ or $J \subseteq G(N)$.*

Proof. Assume that $JK \subseteq N$ and $K \nsubseteq N$. Then there exists $m \in K \bigcap h(M) - N$. Let $x \in J$. Then $xm \in JK \subseteq N$ and $m \notin N$. Hence, $x \in G(N)$, because N is a graded submodule of M. Thus $J \subseteq G(N)$.

□

Lemma 2.3. *Let R be a G-graded ring and M a graded R-module. If U and V are graded submodules of M, then $(U \bigcap V :_R M) = (U :_R M) \bigcap (V :_R M)$.*

Proof. The proof is straightforward. □

Let $N_1, N_2, \dots N_n$ be graded submodules of a graded R-module M. We call a covering $N \subseteq N_1 \cup N_2 \cup \cdots \cup N_n$ *efficient* if N is not contained in the union of any $n-1$ of the graded submodules $N_1, N_2, \dots N_n$. We say that $N = N_1 \cup N_2 \cup \cdots \cup N_n$ is an *efficient union*, if non of the N_k may be excluded. Any covering of a union of graded submodules can be reduced to an efficient one, called an *efficient reduction*, by deleting any unnecessary terms, (see [3].)

Theorem 2.4. *Let R be a G-graded ring, M a graded R-module and N a graded submodule of M. Let $N \subseteq P_1 \bigcup P_2 \bigcup \cdots \bigcup P_n$ be an efficient covering of graded submodules of M where $n > 2$. Then $N \bigcap (\bigcap_{i \neq k} P_i) \subseteq P_k$ for all $k \in \{1, 2, \dots, n\}$.*

Proof. By Lemma 2.1(iv), $N \cap P_i$ is a graded submodule of M for $i = 1, \dots, n$. Hence, by assumption, $N = (N \bigcap P_1) \bigcup (N \bigcap P_2) \bigcup \cdots \bigcup (N \bigcap P_n)$ is an efficient union. By [3, Lemma 2.3], we conclude that $N \bigcap (\bigcap_{i \neq k} P_i) = \bigcap_{i \neq k} (N \bigcap P_i) = \bigcap_{i=1}^{n} (N \bigcap P_i) \subseteq N \bigcap P_k \subseteq P_k$. □

Theorem 2.5. *Let R be a G-graded ring, M a graded R-module and N a graded submodule of M. Let $N \subseteq P_1 \bigcup P_2 \bigcup \cdots \bigcup P_n$ be an efficient covering of graded submodules of M where $n > 2$, then for all $k \in \{1, 2, \dots, n\}$, $\bigcap_{i \neq k} (P_i :_R M) \subseteq G(P_k)$.*

Proof. For $k \in \{1, 2, \dots, n\}$, put $J_k = \bigcap_{i \neq k} (P_i :_R M)$. By Lemma 2.3 , $J_k = (\bigcap_{i \neq k} P_i :_R M)$. Then $J_k M \subseteq \bigcap_{i \neq k} P_i$ and so $J_k N \subseteq \bigcap_{i \neq k} P_i$. Hence $J_k N \subseteq N \bigcap \left(\bigcap_{i \neq k} P_i \right)$. By Theorem 2.4, $J_k N \subseteq N \bigcap \left(\bigcap_{i \neq k} P_i \right) \subseteq P_k$. Note that $N \nsubseteq P_k$ by assumption. Thus, by Proposition 2.2 $J_k = \bigcap_{i \neq k} (P_i :_R M) \subseteq G(P_k)$. □

Corollary 2.6. *Let R be a G-graded ring, M a graded R-module and N a graded submodule of M. Let $P_1, P_2, \dots, P_n$ be graded submodules of M such that $N \subseteq P_1 \bigcup P_2 \bigcup \cdots \bigcup P_n$. Assume that $\bigcap_{k \neq i} (P_i :_R M) \nsubseteq G(P_k)$ for all $k = 1, 2, \dots, n$ except possibly for at most two of the k's. Then $N \subseteq P_k$ for some $k \in \{1, 2, \dots, n\}$.*

Proof. We may assume that the covering is efficient without loss of generality. Then $n \neq 2$. By Theorem 2.5, $n \leq 2$. Thus $n = 1$ and hence $N \subseteq P_k$ for some $k \in \{1, 2, \dots, n\}$. □

Lemma 2.7. *Let R be a G-graded ring and M a graded R-module. Let $P_1, P_2, \dots, P_n$ be graded submodules of M. If for $1 \leq k \leq n$, P_k is a graded primal submodule of M, then the following are equivalent:*

(i) $\bigcap_{k \neq i} (P_i :_R M) \nsubseteq G(P_k)$.

(ii) $(P_i :_R M) \nsubseteq G(P_k)$ *whenever* $i \neq k$.

Proof. $(i) \Rightarrow (ii)$ Clear.

$(ii) \Rightarrow (i)$ Assume that $(P_i :_R M) \nsubseteq G(P_k)$ whenever $i \neq k$. Since P_k is a graded primal submodule of M, by [4, Theorem 1.4] $G(P_k)$ is a graded prime ideal of R. Thus, $\bigcap_{k\neq i}(P_i :_R M) \nsubseteq G(P_k)$.

□

The following theorem is a generalization of the graded primary Avoidance Theorem for modules that was proved in [3].

Theorem 2.8. *[The Graded Primal Avoidance Theorem] Let R be a G-graded ring, M a graded R-module and N a graded submodule of M. Let $P_1, P_2, \dots, P_n$ be graded submodules of M such that $N \subseteq P_1 \bigcup P_2 \bigcup \cdots \bigcup P_n$. Assume that at most two of the P_k's are not graded primal and $(P_i :_R M) \nsubseteq G(P_k)$ whenever $i \neq k$, then $N \subseteq P_k$ for some $k \in \{1, 2, \dots, n\}$.*

Proof. This follows from corollary 2.6 and Lemma 2.7. □

3. Graded P_L-Compactly Packed Modules

Let N be a graded submodule of a graded R-module M. If $r = \sum_{h\in G} r_h \in R$ and $x = \sum_{g\in G} x_g \in G(N)$, then for all $g \in G$ there exists $m_\lambda \in h(M) - N$ with $x_g m_\lambda \in N$ and hence $r_h x_g m_\lambda \in N$. Since $m_\lambda \notin N$, $r_h x_g \in g(N) \subseteq G(N)$. Hence $rx \in G(N)$. Thus to prove $G(N)$ is an ideal of R we only prove that $G(N)$ is closed under the addition.

Definition 3.1. Let R be a G-graded ring and M a graded R-module.

(i) A proper graded submodule N of M is called *graded irreducible* if N cannot be expressed as the intersection of two strictly larger graded submodules of M.

(ii) A proper graded submodule N of M is called *graded strongly irreducible if for any family* $\{P_\alpha\}_{\alpha\in\Delta}$ of graded submodules of M with $N = \bigcap_{\alpha\in\Delta} P_\alpha$, $N = P_\beta$ for some $\beta \in \Delta$.

Lemma 3.2. *Let R be a G-graded ring and M a graded R-module. Then every graded irreducible submodule of M is graded primal.*

Proof. Let N be a graded irreducible submodule of M. Let $x, y \in G(N)$. Then there exist $m, m' \in h(M) - N$ such that $xm \in N$ and $ym' \in N$. Let $U = N + Rm$ and $V = N + Rm'$. By Lemma 2.1, U and V are graded submodules of M. Since N is graded irreducible $N \subsetneq U \bigcap V$. Then there exists $s_h \in h(M)$ such that $s_h \in U \bigcap V - N$. Since $xs_h \in x(N + Rm) = xN + Rxm \subseteq N$ and $ys_h \in y(N + Rm') = yN + Rym' \subseteq N$, we conclude that $(x + y)s_h = xs_h + ys_h \in N$ while $s_h \notin N$. Thus $x+y$ is not homogeneous prime to N. Hence $x+y \in G(N)$. Thus N is graded primal. □

Theorem 3.3. *Let R be a G-graded ring, M a graded R-module and N a proper graded submodule of M. If $m_g \in h(M) - N$, then there exists a graded strongly irreducible submodule that contains N and does not contain m_g.*

Proof. (Using Zorn's Lemma). Let $\mathcal{F}$ be the set of all graded submodules of M that contain N and not containing m_g. Since $N \in \mathcal{F}$, $\mathcal{F} \neq \phi$. Let $\{P_i\}_{i\in I}$ be a chain in $\mathcal{F}$. It is clear that $\bigcup_{i\in I} P_i$ is an upper bound of $\{P_i\}_{i\in I}$. Then by Zorn's Lemma $\mathcal{F}$ contains a maximal element K. We claim that K is graded strongly irreducible. Let $\{U_\alpha\}_{\alpha\in\Delta}$ be a family of graded submodules such that $K = \bigcap_{\alpha\in\Delta} U_\alpha$. Assume that $K \neq U_\alpha$ for all $\alpha \in \Delta$. Then $m_g \in U_\alpha$ for all $\alpha \in \Delta$ and hence $m_g \in \bigcap_{\alpha\in\Delta} U_\alpha = K$, which is a contradiction. □

Corollary 3.4. *Let R be a G-graded ring, M a graded R-module and K and U be two proper graded submodules of M. Then $K \subseteq U$ if and only if every graded strongly irreducible submodule of M containing U also contains K.*

Proof. ($\Rightarrow$)Clear.

($\Leftarrow$) Assume that every graded strongly irreducible submodule of M containing U also contains K and $K \nsubseteq U$. Then there exists $n_g \in K \bigcap h(M) - U$. By Theorem 3.3, there exists a graded strongly irreducible submodule L such that $U \subseteq L$ and $n_g \notin L$. So $K \nsubseteq L$, which is a contradiction. Thus $K \subseteq U$. □

The following definition is a generalization of the concept of graded P-compactly packed modules (see [1, Definition 2.1].)

Definition 3.5. Let R be a G-graded ring, M a graded R-module and N a proper graded submodule of M. N is called *graded P_L-compactly packed* if whenever N is contained in the union of a family of graded primal submodules of M, N is contained in one of the graded primal submodules of the family. M is called *graded P_L-compactly packed* if every proper graded submodule of M is graded P_L-compactly packed.

Theorem 3.6. *Let R be a G-graded ring and M a graded R-module. Then the following statements are equivalent:*

(i) *M is a graded P_L-compactly packed module.*

(ii) *For each proper graded submodule N of M, if $\{P_\alpha\}_{\alpha\in\Delta}$ is a family of graded irreducible submodules of M and $N \subseteq \bigcup_{\alpha\in\Delta} P_\alpha$, then $N \subseteq P_\beta$ for some $\beta \in \Delta$.*

(iii) *For each proper graded submodule N of M, if $\{P_\alpha\}_{\alpha\in\Delta}$ is a family of graded strongly irreducible submodules of M and $N \subseteq \bigcup_{\alpha\in\Delta} P_\alpha$, then $N \subseteq P_\beta$ for some $\beta \in \Delta$.*

(iv) *Every proper graded submodule of M is graded cyclic.*

(v) *For each proper graded submodule N of M, if $\{P_\alpha\}_{\alpha\in\Delta}$ is a family of graded submodules of M and $N \subseteq \bigcup_{\alpha\in\Delta} P_\alpha$, then $N \subseteq P_\beta$ for some $\beta \in \Delta$.*

Proof. $(i) \Rightarrow (ii) \Rightarrow (iii)$ Clear.

$(iii) \Rightarrow (iv)$ Assume that (iii) holds and let N be a proper graded submodule of M. It is clear that $Rn_g \subseteq N$ for each $n_g \in N \bigcap h(M)$. Suppose that $N \nsubseteq Rn_g$

for each $n_g \in N \bigcap h(M)$. By Corollary 3.4, for each $n_g \in N \bigcap h(M)$ there exists a graded strongly irreducible submodule K_{n_g} such that $Rn_g \subseteq K_{n_g}$ and $N \nsubseteq K_{n_g}$. Hence $N = \bigcup_{n_g \in N} Rn_g \subseteq \bigcup_{n_g \in N} K_{n_g}$, which is a contradiction.

$(iv) \Rightarrow (v)$ Assume that (iv) holds and let N be a proper graded submodule of M. Let $\{P_\alpha\}_{\alpha \in \Delta}$ be a family of graded submodules of M such that $N \subseteq \bigcup_{\alpha \in \Delta} P_\alpha$. By (iv), there exists $n_g \in N \bigcap h(M)$ such that $N = Rn_g$, then $n_g \in \bigcup_{\alpha \in \Delta} P_\alpha$ and hence $n_g \in P_\beta$ for some $\beta \in \Delta$. Hence $N = Rn_g \subseteq P_\beta$.

$(v) \Rightarrow (i)$ Clear. □

Recall that a graded R-module M is said to be *with graded primary decomposition* if each of its proper graded submodules is an intersection, possibly infinite, of graded primary submodules of M (see [1].)

Every graded primary submodule is graded primal by [4, Theorem 1.6], so every graded P_L-compactly packed module is a graded P-compactly packed module.

By combining [1, Theorem 2.6] and Theorem 3.6, we have the following corollary.

Corollary 3.7. *Let R be a G-graded ring and M a graded R-module with graded primary decomposition. If M is a graded P-compactly packed module, then M is a graded P_L-compactly packed module.*

Recall that a graded R-module M is called *a graded Noetherian module* if it satisfies the ascending chain condition on graded submodules of M (see [8].)

Corollary 3.8. *Let R be a G-graded ring and M a graded R-module such that every graded finitely generated submodule of M is graded cyclic. If M is graded Noetherian, then M is a graded P_L-compactly packed module.*

Proof. Let N be a proper graded submodule of M. Since M is graded Noetherian, N is a graded finitely generated submodule and hence it's graded cyclic. By Theorem 3.6, M is a graded P_L-compactly packed module. □

Recall that a proper graded submodule N of a graded R-module M is said to be *a graded maximal submodule* if there is no graded submodule K of M such that $N \subsetneqq K \subsetneqq M$, (see [8].)

Theorem 3.9. *Let R be a G-graded ring and M a graded R-module. If M is a graded P_L-compactly packed module which has at least one graded maximal submodule, then M is graded Noetherian.*

Proof. Let $P_1 \subseteq P_2 \subseteq P_3 \subseteq \cdots$ be an ascending chain of graded submodules of M. If $P_i = M$ for some i, then the result follows immediately, so assume that none of P_i's is M and let $N = \bigcup_{i=1}^{\infty} P_i$. We claim that N is a proper graded submodule of M. Assume on contrary that $N = M$ and let K be a graded maximal submodule of M. Then $K \subseteq \bigcup_{i=1}^{\infty} P_i$. Since M is a graded P_L-compactly packed module, by Theorem 3.6 $K \subseteq P_n$ for some n. Thus $K = P_n$ and hence P_n is graded maximal. Hence $P_n = P_i$ for all $i \geq n$ it follows that $P_n = \bigcup_{i=1}^{\infty} P_i = M$, which is impossible. Thus N is a proper graded submodule of M. Since M is a graded P_L-compactly packed module, by Theorem 3.6 $N \subseteq P_m$ for some m and hence $P_m = P_i$ for all $i \geq m$. Therefore M is graded Noetherian. □

Theorem 3.10. *Let R be a G-graded ring and M a graded R-module and $S \subseteq h(R)$ a multiplicatively closed subset of R. If M is a graded P_L-compactly packed R-module, then M_S is a graded P_L-compactly packed R_S-module.*

Proof. Let N be a proper graded submodule of M_S and let $\{P_\alpha\}_{\alpha\in\Delta}$ be a family of graded primal submodules of M_S such that $N \subseteq \bigcup_{\alpha\in\Delta} P_\alpha$. Hence $N \cap M \subseteq \left(\bigcup_{\alpha\in\Delta} P_\alpha\right) \bigcap M$ and so $N \bigcap M \subseteq \bigcup_{\alpha\in\Delta}(P_\alpha \bigcap M)$. By [4, Proposition 2.4.], $P_\alpha \bigcap M$ is a graded primal submodule of M for all $\alpha \in \Delta$. Since M is a graded P_L-compactly packed R-module, there exists $\beta \in \Delta$ such that $N \cap M \subseteq P_\beta \bigcap M$. Hence $(N \bigcap M)_S \subseteq (P_\beta \bigcap M)_S$. By [4, Proposition 2.4], $N \subseteq P_\beta$. Therefore M_S is a graded P_L-compactly packed R_S-module. □

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