On the Representation and the Uniform Polynomial Approximation of Polyanalytic Functions of Gevrey Type on the Unit Disk

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ABSTRACT. In this paper we define Gevrey polyanalytic classes of order N on the unit disk D and we obtain for these classes a characteristic expansion into $N-$analytic polynomials on suitable neighborhoods of D. As an application of our main theorem, we perform for the Gevrey polyanalytic classes of order N on the unit disk D, an analogue to E. M. Dyn'kin's theorem. We also derive, for these classes, their characteristic degree of the best uniform approximation on D by $N-$analytic polynomials.

Keywords: Polyanalytic Gevrey class, Degree of polynomial approximation.

2000 Mathematics subject classification: 30D60, 26E05, 41A10.

This modest work is dedicated to the memory of our beloved master Ahmed Intissar (1951-2017), a distinguished Professor, a brilliant Mathematician (PhD at M.I.T, Cambridge), a man with a golden heart.

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1. Introduction

Among the various classes of functions, the class of bianaytic functions, the class of polyanalytic functions and the Gevrey classes occupy a major place in mathematical analysis and in mathematical physics. Given a nonempty open subset U of $\mathbb{R}^2$, a function $f : U \to \mathbb{C}$ is said to be bianalytic if it satisfies, for each $z \in U$, the condition $\left(\frac{\partial}{\partial \bar{z}}\right)^2 F(z) = 0$. Bianalytic functions originates from mechanics where they played a fundamental role in solving the problems of the planar elasticity theory. Their usefulness in mechanics was illustrated by the pioneering works of Kolossoff, Muskhelishvili and their followers ([21]-[23], [38], [39], [47]). By the systematic use of complex variable techniques these authors have greatly simplified and extended the solutions of the problems of the elasticity theory. The class of polyanalytic functions of order N ($N \in \mathbb{N}^*$), is a generalisation of the class of analytic functions and of that of bianalytic functions. The class $H_N(U)$ of polyanalytic functions of order N on U is the set of functions $F : U \to \mathbb{C}$ of class C^N on U such that the following condition holds for each $z \in U$: $\left(\frac{\partial}{\partial \bar{z}}\right)^N F(z) = 0$. The class of polyanalytic functions was studied intensively by the russian school under the supervision of M. B. Balk ([11]). The lines of current research on polyanalytic functions are various : the problem of the best uniform approximation by $N-$analytic polynomials ([30]-[32], [51]), the study of wavelets and Gabor frames ([2]-[5], [9]), the time-frequency analysis ([5], [7], [8]), the sampling and interpolation in function spaces ([1]), the study of coherent states in quantum mechanics ([19], [36]), [37]), the image and signal processing ([6], [7]), etc. Gevrey classes, which are also, but in a completely different way, a generalisation of real analytic functions, were first introduced by Gevrey ([16]). Indeed they are intermediate spaces between the space of real-analytic functions and the space of C^∞ smooth functions. Given $s > 0$, the Gevrey class $G^s(U)$ is defined as the set of all functions $f : U \to \mathbb{C}$, of class C^∞ on U such that there exist a constant $R > 0$ satisfying for every $x \in U$ and $(\alpha_1, \alpha_2) \in \mathbb{N}^2$ the following estimate : $\left|\frac{\partial^{\alpha_1 + \alpha_2} g}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}}(x)\right| \leq R^{\alpha_1 + \alpha_2 + 1}(\alpha_1 + \alpha_2)^{s(\alpha_1 + \alpha_2)}$. The Gevrey classes play an important role in various branches of partial and ordinary differential equations and especially in the analysis of operators whose properties cannot be apprehended by the classical analytic framework. The field of applications of Gevrey classes is very wide : the Gevrey class regularity of the equations of mathematical physics ([20], [25], [26], [48], [49], [53]), the study of singularities in micro-local analysis ([43]-[45]), the Gevrey solvability of differential operators ([12], [44], [45]), the divergent series and singular and the singular differential equations ([34], [41]), the study of dynamical systems ([17], [42]), the evolution partial differential equations ([14], [29], [46]), etc. However despite this great interest devoted to polyanalytic functions and to Gevrey classes there is still, at our knowledge, no interplay between them. Our main goal in this paper is then

to contribute to bridging this gap. In order to achieve this task, we consider the intersection of a Gevrey class on the unit disk D and the class of polyanalytic functions of order N on D and then look for some property which characterizes this class of functions. Indeed we obtain, in the main result of this paper, the complete description of these so-called Gevrey polyanalytic classes of order N by specific expansions into $N-$analytic polynomials on suitable neighborhoods of D. We establish two applications of our main theorem. The first application concerns the proof for the Gevrey polyanalytic classes of order N on the unit disk D of an analogue to the E. M. Dyn'kin's theorem ([13]). Let us recall that this theorem basically says that a function $f : \mathcal{R} \to \mathbb{C}$ of class C^∞ on a region $\mathcal{R}$ of $\mathbb{C}$ belongs to a class $\mathfrak{X}$ of smooth complex valued functions on $\mathcal{R}$ if and only if it has an extension $F : \mathbb{C} \to \mathbb{C}$ of class C^1 on $\mathbb{C}$ so that $\frac{\partial F}{\partial \bar{z}}$ satisfies a growth condition of the form $\left| \frac{\partial F}{\partial \bar{z}} (z) \right| \leq A \mathcal{H}_{\mathfrak{X}} (B \varrho (z, \mathcal{R}))$, $z \in \mathbb{C} \backslash \mathcal{R}$, where $A, B > 0$ are constants, $\varrho (z, \mathcal{R})$ is the euclidean distance from z to $\mathcal{R}$ and $\mathcal{H}_{\mathfrak{X}} : \mathbb{R}^{+*} \to \mathbb{R}$ is a function related to the class $\mathfrak{X}$, depending only on this class and such that $\lim_{t \to 0, t > 0} \mathcal{H}_{\mathfrak{X}} (t) = 0$. $\mathcal{H}_{\mathfrak{X}}$ is then called the weight function of the class $\mathfrak{X}$ while the function F is said to be a pseudoanalytic extension of the function f with respect to the class $\mathfrak{X}$. The second application of our main theorem concerns the construction, for Gevrey polyanalytic classes of order N, of their degree of the best uniform approximation on D by $N-$analytic polynomials.

The paper is structured as follows. In section 2, we state some notations and definitions and prove a fundamental result which is necessary for the proof of the first application of our main result. In section 3, we give the definition of polyanalytic functions of order N and recall their main properties. In section 4, we recall the definition of Gevrey classes and the quantitative version of the closure of these classes under the composition of functions and finally we state the definition of polyanalytic Gevrey classes on an open subset U of the complex plane. In section 5, we state the main result of this paper. The section 6 is devoted to the proof of the main result. Section 7 presents our applications of the main result of the paper. Finally section 8 is an appendix which provides the proofs of some technical estimates which are crucial for the proof of three results : the proof of the direct part for $N = 1$ (proposition 8.1.), the proof of the converse part of corollary 1 (proposition 8.2.), the proof of the converse part of corollary 2 (proposition 8.3.).

2. Preliminary Notes

2.1. **Basic notations.** Let h a function defined on a nonempty subset E of $\mathbb{C}$. We denote by $||h||_{\infty, E}$ the quantity :

$$||h||_{\infty, E} := \sup_{u \in E} |h(u)| \in \mathbb{R}^+ \cup \{+\infty\}$$

Let S be a nonempty subset of $\mathbb{C}$, then we set for each $z \in \mathbb{C}$:

$$\varrho(z, S) := \inf_{u \in S} |z - u|$$

$\varrho(z, S)$ represents the euclidean distance from z to S.

For all $x \in \mathbb{R}$ we set :

$$\lfloor x \rfloor := \max \left(\{p \in \mathbb{Z} : p \leq x\}\right)$$

We set for every $n \in \mathbb{N}$:

$$J(n) := \{p \in \mathbb{N} : 0 \leq p \leq n - 1\}$$

We set for each $\alpha \in \mathbb{N}^n$ and $s \in I(n)$:

$$\alpha! := \prod_{j=1}^{n} \alpha_j!, \quad |\alpha| := \sum_{j=1}^{n} \alpha_j$$

Let $\sigma := (\sigma_1, \sigma_2)$, $\sigma' := (\sigma'_1, \sigma'_2) \in \mathbb{N}^2$. We set :

$$\sigma \preccurlyeq \sigma' \Leftrightarrow (\sigma_1 \leq \sigma'_1 \text{ and } \sigma_2 \leq \sigma'_2)$$

In this case we set :

$$\binom{\sigma'}{\sigma} := \binom{\sigma'_1}{\sigma_1}\binom{\sigma'_2}{\sigma_2}$$

We denote by $d\nu(\zeta)$ the usual Lebesgue measure on $\mathbb{C}$.

Let $\zeta \in \mathbb{C}$ and $r > 0$. We denote by $\Delta(\zeta, r)$ $\left(\text{resp. } \overline{\Delta}(\zeta, r)\right)$ the usual open (resp. closed) disk of center ζ and radius r. $\Gamma(\zeta, r)$ denotes the usual circle of center ζ and radius r. $D := \Delta(0, 1)$ (resp. $\overline{D} := \overline{\Delta}(0, 1)$) is called the open (resp.closed) unit disk of the complex plane. It is then clear that :

$$\varrho(z, D) = \begin{cases} 0 \text{ if } z \in D \\ |z| - 1 \text{ else} \end{cases}$$

For all $r, R > 0$ and $n \in \mathbb{N}^*$ we set :

$$D_R := \Delta(0, 1 + R), \quad \overline{D}_R := \overline{\Delta}(0, 1 + R), \quad D_{k,R,n} := D_{Rn^{\frac{-1}{k}}}$$

For each $m \in \mathbb{N}^* \backslash \{1\}$ we denote by $\mathcal{L}_m$ the function defined on the set $\mathcal{U}_m := \left\{(s_1, ..., s_{m-1}) \in \mathbb{R}^{m-1} : 1 < s_1 < ... < s_{m-1}\right\}$ by the formula :

$$\mathcal{L}_m(s_1, ..., s_{m-1})$$
$$: \ = \prod_{j=1}^{m-1} \left(\frac{s_j^2 + 1}{s_j^2 - 1}\right) \sum_{p=1}^{m-1} \left(s_p^{m-1} \prod_{j \neq p} \left(\frac{s_j^2 + 1}{|s_j^2 - s_p^2|}\right)\right)$$

For each real numbers r and r_0 such that $r > r_0 > 0$ we set :

$$\mathcal{J}_m(r_0, r)$$
$$: \ = \mathcal{L}_m \left(1 + \frac{r - r_0}{mr_0}, 1 + 2\left(\frac{r - r_0}{mr_0}\right) ..., 1 + (m - 1)\left(\frac{r - r_0}{mr_0}\right)\right)$$

Proposition 2.1. *The following estimate holds for each* $m \in \mathbb{N}^*$ *and* $\varepsilon > 0$:

$$\mathcal{J}_m \left(1 + \frac{\varepsilon}{2}, 1 + \varepsilon\right) \leq (m-1) \left(\frac{5m}{2}\right)^{2m-2} \left(1 + \frac{2}{\varepsilon}\right)^{2m-2} \tag{2.1}$$

Proof. Indeed we have :

$$\mathcal{J}_m \left(1 + \frac{\varepsilon}{2}, 1 + \varepsilon\right)$$

$$= \mathcal{L}_m \left(1 + \frac{1}{m}\left(\frac{\varepsilon}{2+\varepsilon}\right), 1 + \frac{2}{m}\left(\frac{\varepsilon}{2+\varepsilon}\right), ..., 1 + \left(\frac{m-1}{m}\right)\left(\frac{\varepsilon}{2+\varepsilon}\right)\right)$$

$$= \prod_{j=1}^{m-1} \left(\frac{\left(1 + \frac{j}{m}\left(\frac{\varepsilon}{2+\varepsilon}\right)\right)^2 + 1}{\left(1 + \frac{j}{m}\left(\frac{\varepsilon}{2+\varepsilon}\right)\right)^2 - 1}\right) \sum_{p=1}^{m-1} \left(\begin{array}{c}\left(1 + \frac{p}{m}\left(\frac{\varepsilon}{2+\varepsilon}\right)\right)^{m-1} \cdot \\ \cdot \prod_{j\neq p}\left(\frac{\left(1 + \frac{j}{m}\left(\frac{\varepsilon}{2+\varepsilon}\right)\right)^2 + 1}{\left|\left(1 + \frac{j}{m}\left(\frac{\varepsilon}{2+\varepsilon}\right)\right)^2 - \left(1 + \frac{p}{m}\left(\frac{\varepsilon}{2+\varepsilon}\right)\right)^2\right|}\right)\end{array}\right)$$

$$\leq \prod_{j=1}^{m-1} \left(\frac{5m\left(\frac{2+\varepsilon}{\varepsilon}\right)}{j\left(2 + \frac{j}{m}\left(\frac{\varepsilon}{2+\varepsilon}\right)\right)}\right) \sum_{p=1}^{m-1} \left(2^{m-1} \prod_{j\neq p}\left(\frac{5m\left(\frac{2+\varepsilon}{\varepsilon}\right)}{|j-p|\left(2 + \frac{(j+p)}{m}\left(\frac{\varepsilon}{2+\varepsilon}\right)\right)}\right)\right)$$

$$\leq (m-1)\left(\frac{5m}{2}\right)^{2m-2}\left(1 + \frac{2}{\varepsilon}\right)^{2m-2}$$

Thence we achieve the proof of the proposition. $\qquad\square$

From now on $N \in \mathbb{N}^*$ and $k > 0$ are arbitrary but fixed real numbers.

2.2. Some function spaces and differential operators. $C\left(\overline{D}\right)$ represents the set of continuous complex valued functions on $\overline{D}$ while $C_0^\infty (\mathbb{C})$ denotes the set of complex valued functions defined and of class C^∞ on $\mathbb{C}$ and of compact support.

We denote by $\frac{\partial}{\partial \overline{z}}$ the well-known Cauchy-Riemann operator differential operator defined by the formula :

$$\frac{\partial}{\partial \overline{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

while $\frac{\partial}{\partial z}$ is the differential operator whose definition is :

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

For each $\alpha := (\alpha_1, \alpha_2) \in \mathbb{N}^2$, we denote by D^α the differential operator :

$$D^\alpha := \frac{\partial^{|\alpha|}}{\partial x^{\alpha_1}\partial y^{\alpha_2}}$$

while $\frac{\partial^{|\alpha|}}{\partial z^{\alpha_1}\partial \overline{z}^{\alpha_2}}$ is the differential operator defined by :

$$\frac{\partial^{|\alpha|}}{\partial z^{\alpha_1}\partial \overline{z}^{\alpha_2}} := \left(\frac{\partial}{\partial z}\right)^{\alpha_1}\left(\frac{\partial}{\partial \overline{z}}\right)^{\alpha_2}$$

The following proposition will play a fundamental role in the proof of the first application of our main result.

Proposition 2.2. *For every* $\varphi \in C_0^\infty (\mathbb{C})$ *and* $N \in \mathbb{N}^*$ *the following relation holds :*

$$\varphi (z) = \iint_{\mathbb{C}} \frac{\left(\overline{z} - \overline{\zeta}\right)^{N-1}}{\pi (N-1)! (z-\zeta)} \left(\frac{\partial}{\partial \overline{z}}\right)^N \varphi (\zeta) \, d\nu (\zeta)$$

Proof. It is well-known ([52]; page 126, exercise 11-4) that the function :

$$\mathcal{V}_N : (x,y) \mapsto \frac{(x-iy)^{N-1}}{\pi . (N-1)! (x+iy)}$$

is a fundamental solution of the differential operator $\left(\frac{\partial}{\partial \overline{z}}\right)^N$. Let us then write $\mathcal{V}_N (x,y)$ in the form :

$$\mathcal{V}_N (z) := \frac{\overline{z}^{N-1}}{\pi . (N-1)! z}$$

where $z := x + iy$. Consequently for each $\varphi \in C_0^\infty (\mathbb{C})$ and $N \in \mathbb{N}^*$ the following formula holds for all $z \in \mathbb{C}$:

$$
\begin{aligned}
\varphi (z) &= \iint_{\mathbb{C}} \mathcal{V}_N (z-\zeta) \left(\frac{\partial}{\partial \overline{z}}\right)^N \varphi (\zeta) \, d\nu (\zeta) \\
&= \iint_{\mathbb{C}} \frac{\left(\overline{z} - \overline{\zeta}\right)^{N-1}}{\pi (N-1)! (z-\zeta)} \left(\frac{\partial}{\partial \overline{z}}\right)^N \varphi (\zeta) \, d\nu (\zeta)
\end{aligned}
$$

The proof of the proposition is then complete. $\qquad\square$

3. Polyanalytic Functions of Order N : Definition and Main Properties

Let U be a nonempty open subset of $\mathbb{C}$. The set $H_N(U)$ of polyanalytic functions of order N on U is the set of functions $F : U \to \mathbb{C}$ of class C^N on U such that :

$$(\forall z \in U) : \left(\frac{\partial}{\partial \overline{z}}\right)^N F(z) = 0$$

Then $H_1(U)$ is the set of holomorphic functions on U, while $H_2(U)$ is the set of bianalytic functions on U. It is well known ([11], pages 10 and 11) that a function $F : U \to \mathbb{C}$ is polyanalytic of order N if and only if it is of the form :

$$(\forall z \in U) : F(z) = \sum_{p=0}^{N-1} F_p(z)\overline{z}^p \tag{3.1}$$

where $F_0, ..., F_{N-1}$ are holomorphic on U. We can prove by an easy induction on N that the representation (3.1) of F is unique. For every $p \in J(N)$, the function F_p is called the holomorphic component of order p of F and labelled

by the notation $F_p := \mathcal{K}_p(F)$. It follows also from the formula (3.1) that every function $f \in H_N(U)$ is of class C^∞ on U. We denote by Π_N the vector space of complex polynomial functions P of the form :

$$z \in \mathbb{C} \mapsto P(z) := \sum_{p=0}^{N-1} Q_p(z)\bar{z}^p,$$

where $Q_0,,...,Q_{N-1}$ are holomorphic polynomials. The members of Π_N are called N-analytic polynomials. The degree of $P \neq 0$ is then the integer :

$$d^\circ(P) := \max_{0 \leq p \leq N-1,\ Q_p \neq 0} \deg(Q_p)$$

where $\deg(Q_p)$ denotes the usual degree of $Q_p \in \mathbb{C}[X]$. We will set $d^\circ(0) = -\infty$. With the convention that

$$(\forall n \in \mathbb{N}) : -\infty \leq n$$

$\Pi_{N,n}$ is, for each $n \in \mathbb{N}$, the vector subspace of Π_N defined by :

$$\Pi_{N,n} := \{P \in \Pi_N : d^\circ(P) \leq n\}$$

For each continuous function $f : \overline{D} \to \mathbb{C}$, the $N-$approximating number of order n is :

$$\mathcal{E}_{N,n}(f) := \inf_{P \in \Pi_{N,n}} \|f - P\|_{\infty,\overline{D}}$$

Let $\mathfrak{X}$ be a nonempty subset of $C(\overline{D})$. A degree of the best uniform $N-$polynomial approximation of functions of $\mathfrak{X}$ is a set $\mathcal{R} := \{\mathfrak{m}_\mu : \mu \in \Lambda\}$ of functions $\mathfrak{m}_\mu : \mathbb{R}^+ \to \mathbb{R}^+$ (Λ a nonempty set) such that :

$$\left(\forall f \in C(\overline{D})\right) : [(f \in \mathfrak{X}) \Leftrightarrow ((\exists \mu \in \Lambda)(\forall n \in \mathbb{N}) : \mathcal{E}_{N,n}(f) \leq \mathfrak{m}_\mu(n))]$$

The following results, whose proofs can be found in ([11], pages 21-25 and 27), will play a fundamental role in the current paper.

Theorem 3.1. *Let* $\gamma_p := \Gamma(\zeta, r_p)$ *be* N *circles where* $0 < r_0 < r_1 < ... < r_{N-1} < r$ *and* $f \in H_N(\Delta(\zeta, r))$.

1. *Maximum modulus principle for polyanalytic functions of order* N *on* $\Delta(\zeta, r)$:
a) *The following estimates hold*

$$\|f\|_{\infty,\overline{\Delta}(\zeta,r_0)} \leq \mathcal{L}_N\left(\frac{r_1}{r_0}, ..., \frac{r_{N-1}}{r_0}\right) \max_{0 \leq p \leq N-1} \|f\|_{\infty,\gamma_p} \tag{3.2}$$

b) *If we assume that :*

$$\|f\|_{\infty,\Delta(\zeta,r)\setminus\overline{\Delta}(\zeta,r_0)} < +\infty$$

then the following estimates hold :

$$\|f\|_{\infty,\overline{\Delta}(\zeta,r_0)} \leq \mathcal{J}_N(r_0,r)\|f\|_{\infty,\Delta(\zeta,r)\setminus\overline{\Delta}(\zeta,r_0)} \tag{3.3}$$

$$\left(\forall p \in J\left(N\right)\right): \left\|\mathcal{K}_p\left(f\right)\right\|_{\infty, \overline{\Delta}(\zeta, r_0)} \leq \frac{\mathcal{J}_N\left(r_0, r\right) \left\|f\right\|_{\infty, \Delta(\zeta, r) \setminus \overline{\Delta}(\zeta, r_0)}}{r_0^p} \qquad (3.4)$$

2. *Weierstrass theorem for polyanalytic functions of order on* $\Delta(\zeta, r)$:

Let $(f_n)_{n \geq 1}$ *be a sequence of polyanalytic functions of order* N *on* $\Delta(\zeta, r)$ *which is uniformly convergent on every compact subset of* $\Delta(\zeta, r)$ *to a function* f . *Then* $f \in H_N\left(\Delta\left(\zeta, r\right)\right)$ *and for each* $(p, q) \in \mathbb{N}^2$ *the sequence* $\left(\frac{\partial^{p+q} f_n}{\partial z^p \partial \overline{z}^q}\right)_{n \geq 1}$ *is uniformly convergent on every compact subset of* $\Delta(\zeta, r)$ *to the function* $\frac{\partial^{p+q} f}{\partial z^p \partial \overline{z}^q}$.

By means of the theorem 3.1. we prove easily the following result.

Proposition 3.2. *Let* $(f_n)_{n \geq 1}$ *be a sequence of polyanalytic functions of order* N *on an open disk* $\Delta(\zeta, r)$. *Let us assume that the sequence* $(f_n)_{n \geq 1}$ *is uniformly convergent on every compact subset of* $\Delta(\zeta, r)$ *to the function* f . *Then* $f \in H_N\left(\Delta\left(\zeta, r\right)\right)$ *and, for every* $p \in J\left(N - 1\right)$, *the sequence of functions* $(\mathcal{K}_p\left(f_n\right))_{n \geq 1}$ *is uniformly convergent on every compact subset of* $\Delta(\zeta, r)$ *to the function* $\mathcal{K}_p\left(f\right)$.

The following result plays a crucial role in the proof of the second application of the main result of this paper.

Proposition 3.3. Bernstein-Walsh inequality for N-analytic polynomials on the unit disk :

For each $P \in \Pi_{N,n}$ *and* $z \in \mathbb{C} \setminus D$, *the following inequality holds* :

$$\left|P\left(z\right)\right| \leq \left(2^{N+1} - 1\right) \mathcal{J}_N\left(\frac{1}{2}, 1\right) \left\|P\right\|_{\infty, \overline{D}} \left|z\right|^{n+N-1}$$

Proof. We have for each $j \in J\left(N\right)$:

$$\left\|\mathcal{K}_j\left(P\right)\right\|_{\infty, \overline{\Delta}_{\frac{1}{2}}} \leq 2^j \mathcal{J}_N\left(\frac{1}{2}, 1\right) \left\|P\right\|_{\infty, \overline{D}}$$

On the other hand, by virtue of the well known Bernstein-Walsh inequality ([10]), we have for every $z \in \mathbb{C} \setminus D$ and $j \in J\left(N - 1\right)$:

$$\left|\mathcal{K}_j\left(P\right)\left(z\right)\right| \leq 2^n \left\|\mathcal{K}_j\left(P\right)\right\|_{\infty, \overline{\Delta}_{\frac{1}{2}}} \left|z\right|^n$$

It follows that :

$$\begin{aligned}
\left|P\left(z\right)\right| &\leq \sum_{j=0}^{N-1} \left|\mathcal{K}_j\left(P\right)\left(z\right)\right| \left|z\right|^j \\
&\leq \sum_{j=0}^{N-1} 2^j \mathcal{J}_N\left(\frac{1}{2}, 1\right) \left\|P\right\|_{\infty, \overline{D}} \left|z\right|^{n+j} \\
&\leq \left(2^{N+1} - 1\right) \mathcal{J}_N\left(\frac{1}{2}, 1\right) \left\|P\right\|_{\infty, \overline{D}} \left|z\right|^{n+N-1}
\end{aligned}$$

Thence we achieve the proof of the proposition. $\qquad \square$

4. Gevrey Classes and Gevrey Polyanalytic Classes of Order N

Definition 4.1. Let $k, s > 0$ and $N \in \mathbb{N}^*$ be given fixed numbers. *Let U be a* nonempty subset of $\mathbb{C}$ and I an interval of $\mathbb{R}$.

(1) *The Gevrey class $G^s(U)$ is the set of functions $f : U \to \mathbb{C}$ of class C^∞ on U such that :*

$$\left(\forall \alpha \in \mathbb{N}^2\right) : \|D^\alpha f\|_{\infty, U} \leq B_0^{|\alpha|+1} |\alpha|^{s|\alpha|}$$

$B_0 > 0$ being a constant, with the convention that $0^0 = 1$.

(2) *The Gevrey class $G^s(I)$ is the set of functions $f : I \to \mathbb{C}$ of class C^∞ on I such that :*

$$(\forall n \in \mathbb{N}) : \left\|f^{(n)}\right\|_{\infty, I} \leq B_1^{n+1} n^{sn}$$

$B_1 > 0$ being a constant.

(3) *The Gevrey polyanalytic class of order N on U, $H_N^k(U)$, is the set of functions $f \in H_N(U)$ such that :*

$$\left(\forall \alpha \in \mathbb{N}^2\right) : \|D^\alpha f\|_{\infty, U} \leq B_2^{|\alpha|+1} |\alpha|^{\left(1 + \frac{1}{k}\right)|\alpha|}$$

$B_2 > 0$ being a constant. It follows that :

$$H_N^k(U) = H_N(U) \cap G^{1 + \frac{1}{k}}(U)$$

Remark 4.2. We prove easily by direct computations that $H_N^k(U)$ is the set of functions $f \in H_N(U)$ such that :

$$\left(\forall (n, m) \in \mathbb{N}^2\right) : \left\|\frac{\partial^{n+m} f}{\partial z^n \partial \overline{z}^m}\right\|_{\infty, U} \leq B_3^{n+m+1} (n+m)^{\left(1 + \frac{1}{k}\right)(n+m)}$$

$B_3 > 0$ being a real constant.

Remark 4.3. We prove, by an easy induction on $N \geq 1$, that the following equivalence holds for each $f \in H_N(U)$:

$$f \in H_N^k(U) \Leftrightarrow \left[(\forall p \in J(N)) : \mathcal{K}_p(f) \in H_1^k(U)\right]$$

A slight refinement of the proof by D. Figueirinhas ([15], theorem 2. 5. pages $11 - 13$) of the closure of Gevrey classes under composition provide us with the following result which is essential for the proof of direct part of our main result.

Theorem 4.4. *Let I be an interval of $\mathbb{R}$ and $f \in G^s(I)$, $s > 1$. Let U be an open set of $\mathbb{C}$ and $g : U \to \mathbb{C}$ a function of class C^∞ on J such that $f(I) \subset U$ and $g \in G^s(U)$. The function $h := g \circ f$ belongs also to $G^s(I)$ and if we assume that :*

$$\left\|f^{(n)}\right\|_{\infty, I} \leq c_1 d_1^n n^{sn}, \ n \in \mathbb{N}$$

$$\|D^\alpha g\|_{\infty, U} \leq c_2 d_2^{|\alpha|} |\alpha|^{s|\alpha|}, \ \alpha \in \mathbb{N}^2$$

$(c_1,\ d_1,\ c_2,\ d_2 > 0$ *being constants) then we will have :*

$$\left\|h^{(n)}\right\|_{\infty,I} \leq c_3 d_3^n n^{sn},\ n \in \mathbb{N}$$

where :

$$c_3 = \frac{ec_1 c_2 d_2}{1 + ec_1 d_2},\ d_3 = d_1\left(1 + ec_1 d_2\right)$$

5. STATEMENT OF THE MAIN RESULT

Our main result, in this paper, is the following.

Theorem 5.1.

(1) *Let $F \in H_N^k(D)$. Then there exist constants $C > 0$, $R > 0$ and $\delta \in]0,1[$ and a sequence $(P_n)_{n\in\mathbb{N}^*}$ of $N-$analytic polynomials such that :*

$$\begin{cases} (\forall n \in \mathbb{N}^*) : ||P_n||_{\infty,D_{k,R,n}} \leq C\delta^n \\ (\forall z \in D) : \sum_{n=1}^{+\infty} P_n(z) = F(z) \\ (\forall n \in \mathbb{N}^*) : d^\circ\left(P_n\right) \leq n^{\frac{k+1}{k}} \end{cases}$$

(2) *Conversly, let $(f_n)_{n\in\mathbb{N}^*}$ be a sequence of $N-$analytic polynomials such that :*

$$(\forall n \in \mathbb{N}^*) : ||f_n||_{\infty,D_{k,R,n}} \leq C\delta^n$$

for some constants $C > 0$, $R > 0$ and $\delta \in]0,1[$. Then the function series $\sum f_n$ converges uniformly on D to a function $f \in H_N^k(D)$.

6. PROOF OF THE MAIN RESULT

6.1. **Proof of the direct part of the main result.**

6.1.1. *Proof of the direct part for N=1.*

Lemma 6.1. *Let*

$$\begin{aligned} F : \quad D \quad &\to \quad \mathbb{C} \\ z \quad &\mapsto \quad \sum_{p=0}^{+\infty} a_p z^p \end{aligned}$$

be a function which belongs to $H_1^k(D)$. Then there exists constants $\mathcal{P}_1, \mathcal{P}_2 > 0$ such that :

$$(\forall p \in \mathbb{N}) : |a_p| \leq \mathcal{P}_1 \exp\left(-\mathcal{P}_2 p^{\frac{k}{k+1}}\right)$$

Proof. Let us associate to every $t \in [0,1[$ the function :

$$\begin{aligned} \varphi_t : \quad [0,2\pi] \quad &\to \quad \mathbb{C} \\ \theta \quad &\mapsto \quad F\left(te^{i\theta}\right) \end{aligned}$$

Assume that $F \in H_1^k(D)$, then there exists $\mathcal{P}_0 \geq 1$ such that :

$$(\forall n \in \mathbb{N}) : ||F^{(n)}||_{\infty,D} \leq \mathcal{P}_0^{n+1} n^{(1+\frac{1}{k})n}$$

Direct computations based on the result of theorem 4. 4. prove that :

$$(\forall t \in [0,1[) \, (\forall n \in \mathbb{N}^*) \quad :$$
$$||\varphi_t^{(n)}||_{\infty,[0,2\pi]} \leq (1 + e\mathcal{P}_0)^n \, n^{n(1+\frac{1}{k})}$$

On the other hand for every $n \in \mathbb{N}^*$ and $t \in \,]0,1[$, the function $\varphi_t^{(n)}$ is a $2\pi-$periodic function whose Fourier coefficient of order $p \in \mathbb{N}^*$ is :

$$i^n p^n t^p a_p = \frac{1}{2\pi} \int\limits_0^{2\pi} \varphi_t^{(n)}(\theta) e^{-ip\theta} d\theta$$

It follows that :

$$\left(\forall \, (p,n) \in (\mathbb{N}^*)^2 \right) (\forall \, t \in \,]0,1[) \quad :$$
$$|a_p| \leq \frac{1}{p^n t^p}(1 + e\mathcal{P}_0)^n n^{n(1+\frac{1}{k})}$$

Thence we have :

$$(\forall p \in \mathbb{N}^*) : |a_p| \leq 2\mathcal{P}_0 \inf_{n \in \mathbb{N}^*} n^{n(1+\frac{1}{k})} \left(\frac{1 + e\mathcal{P}_0}{p} \right)^n$$

But straightforward computations show that the following relation holds for all $p \in \mathbb{N}^*$:

$$\inf_{n \in \mathbb{N}^*} n^{n(1+\frac{1}{k})} \left(\frac{1 + e\mathcal{P}_0}{p} \right)^n$$
$$= \min \left(\left(\left(\frac{1 + e\mathcal{P}_0}{p} \right)^{\frac{k}{k+1}} r_p \right)^{(1+\frac{1}{k})r_p} , \left(\left(\frac{1 + e\mathcal{P}_0}{p} \right)^{\frac{k}{k+1}} (r_p + 1) \right)^{(1+\frac{1}{k})(r_p+1)} \right)$$

where $r_p := \left\lfloor e^{-1} (1 + e\mathcal{P}_0)^{-\frac{k}{k+1}} p^{\frac{k}{k+1}} \right\rfloor$. Consequently there exists two constants $\mathcal{P}_1, \mathcal{P}_2 > 0$ such that :

$$(\forall p \in \mathbb{N}^*) : \inf_{n \in \mathbb{N}^*} \frac{n^{n(1+\frac{1}{k})}}{\left(\frac{p}{1+e\mathcal{P}_0} \right)^n} \leq \frac{\mathcal{P}_1}{1 + e\mathcal{P}_0} \exp \left(-\mathcal{P}_2 p^{\frac{k}{k+1}} \right)$$

It follows that :

$$(\forall p \in \mathbb{N}^*) : |a_p| \leq \mathcal{P}_1 \exp \left(-\mathcal{P}_2 p^{\frac{k}{k+1}} \right)$$

Thence we achieve the proof of the proposition.   □

End of the proof of the direct part for $N = 1$.

Let us set for every $z \in \mathbb{C}$ and $n \in \mathbb{N}$:

$$P_n(z) := \sum_{2^{-\left(\frac{k+1}{k}\right)} n^{\frac{k+1}{k}} \leq p < 2^{-\left(\frac{k+1}{k}\right)}(n+1)^{\frac{k+1}{k}}} a_p z^p$$

Then $(P_n)_{n \in \mathbb{N}^*}$ is a sequence of $1-$analytic polynomials such that :

$$(\forall n \in \mathbb{N}^*) : d^\circ (P_n) \leq n^{\frac{k+1}{k}}$$

Furthermore the following inequality holds for each $n \in \mathbb{N}^*$ and $z \in D_{k, \frac{\mathcal{P}_2}{4}, n}$:

$$|P_n(z)|$$

$$\leq \sum_{2^{-\left(\frac{k+1}{k}\right)} n^{\frac{k+1}{k}} \leq p < 2^{-\left(\frac{k+1}{k}\right)} (n+1)^{\frac{k+1}{k}}} |a_p| |z|^p$$

$$\leq \sum_{2^{-\left(\frac{k+1}{k}\right)} n^{\frac{k+1}{k}} \leq p < 2^{-\left(\frac{k+1}{k}\right)} (n+1)^{\frac{k+1}{k}}} \mathcal{P}_1 \exp\left(-\mathcal{P}_2 p^{\frac{k}{k+1}}\right) \cdot$$

$$\cdot \left(1 + \frac{\mathcal{P}_2}{4} n^{-\frac{1}{k}}\right)^p$$

$$\leq \mathcal{P}_1 \exp\left(-\frac{\mathcal{P}_2}{2} n\right) \left(2^{-\left(\frac{k+1}{k}\right)} (n+1)^{\frac{k+1}{k}} - 2^{-\left(\frac{k+1}{k}\right)} n^{\frac{k+1}{k}} + 1\right) \cdot$$

$$\cdot \left(1 + \frac{\mathcal{P}_2}{4} n^{-\frac{1}{k}}\right)^{n^{\frac{k+1}{k}}}$$

$$\leq 2^{\frac{k+1}{k}} \mathcal{P}_1 \exp\left(-\frac{\mathcal{P}_2}{2} n\right) \left((n+1)^{\frac{k+1}{k}} - n^{\frac{k+1}{k}} + 1\right) \left(1 + \frac{\mathcal{P}_2}{4} n^{-\frac{1}{k}}\right)^{n^{\frac{k+1}{k}}}$$

But we will prove in the proposition 8.1. in the appendix below, that there exists a constant $\mathcal{P}_3 > 0$ such that

$$(\forall n \in \mathbb{N}) : 2^{\frac{k+1}{k}} \mathcal{P}_1 \exp\left(-\frac{\mathcal{P}_2}{2} n\right) \left((n+1)^{\frac{k+1}{k}} - n^{\frac{k+1}{k}} + 1\right) \cdot$$

$$\cdot \left(1 + \frac{\mathcal{P}_2}{4} n^{-\frac{1}{k}}\right)^{n^{\frac{k+1}{k}}}$$

$$\leq \mathcal{P}_3 \left(e^{-\frac{\mathcal{P}_2}{8}}\right)^n$$

Consequently the following relation holds :

$$(\forall n \in \mathbb{N}) : \|P_n\|_{\infty, D_{k, \frac{\mathcal{P}_2}{4}, n}} \leq \mathcal{P}_3 \left(e^{-\frac{\mathcal{P}_2}{8}}\right)^n$$

But $e^{-\frac{\mathcal{P}_2}{8}} \in \,]0, 1[$ thence we achieve the proof of the direct part of the main result for $N = 1$.

□

6.1.2. *Proof of the direct part in the general case* $N \geq 2$. Let $f \in H_N^k(D)$. Then there exist, thanks to the remark 3 above, a functions $g_0, ..., g_N$ belonging to $H_1^k(D)$ such that :

$$(\forall z \in D) : f(z) = \sum_{p=0}^{N-1} g_p(z) \bar{z}^p$$

Hence there exist for each $p \in J(N)$ a triple $(C_p, R_p, \delta_p) \in \mathbb{R}^{+*} \times \mathbb{R}^{+*} \times \,]0, 1[$ of constants and a sequence $(Q_{n,p})_{n \in \mathbb{N}^*}$ of holomorphic polynomials such that

:

$$\begin{cases} (\forall n \in \mathbb{N}^*) : d^\circ\left(Q_{n,p}\right) \le n^{\frac{k+1}{k}} \\ (\forall n \in \mathbb{N}) : \|Q_{n,p}\|_{\infty, D_{k,R_p,n}} \le C_p \delta_p^n \\ (\forall z \in D) : \sum_{n=1}^{+\infty} Q_{n,p}(z) = g_p(z) \end{cases}$$

Let us set $R := \min\left(R_0, ..., R_{N-1}\right), \delta := \max\left(\delta_0, ..., \delta_{N-1}\right), C := \max\left(C_0, ..., C_{N-1}\right),$ $Q_n\left(z\right) := \sum_{p=0}^{N-1} Q_{n,p}\left(z\right) \overline{z}^p.$ Then $\delta \in \left]0, 1\right[$ and Q_n is for each $n \in \mathbb{N}$ a $N-$analytic polynomial such that :

$$\begin{cases} (\forall n \in \mathbb{N}^*) : d^\circ\left(Q_n\right) \le n^{\frac{k+1}{k}} \\ (\forall n \in \mathbb{N}) : \|Q_n\|_{\infty, D_{k,R,n}} \le NC\left(1+R\right)^{N-1} \delta^n \end{cases}$$

Furthermore we have for each $z \in D$:

$$\begin{aligned} f\left(z\right) &= \sum_{p=0}^{N} g_p\left(z\right) \overline{z}^p \\ &= \sum_{n=0}^{+\infty} Q_n\left(z\right) \end{aligned}$$

Thence we achieve the proof of the direct part of the main result.

$\square$

6.2. **Proof of the converse part of the main result.** Let $A > 0$ and for each $n \in \mathbb{N}$, a function $f_n \in H_1(D_{k,A,n})$ such that :

$$\begin{cases} (\forall n \in \mathbb{N}^*) : f_n \in H_N(D_{k,A,n}) \\ (\forall n \in \mathbb{N}^*) : \|f_n\|_{\infty, D_{k,A,n}} \le C\rho^n \end{cases}$$

where $C > 0$, $\rho \in \left]0, 1\right[$ are constant. Without loss of the generality we can also assume that $A < 1$. It follows, by virtue of the Weierstrass theorem for polyanalytic functions of order N, that the function series $\sum f_n|_D$ converges uniformly to a function $f \in H_N(D)$. For every $n \in \mathbb{N}^*$ and $p \in J\left(N\right)$, let $a_{p,n}$ be the holomorphic component of order p of f_n. Let a_p be the holomorphic component of order p of f. By virtue of the remark, for each $p \in J\left(N\right)$ the function series $\sum a_{p,n}$ is uniformly convergent on every compact of D to the function a_p. Then the inequalities (3.4) and (2.1) entail that the following

estimate holds for each $(p, n) \in \mathbb{N}^2$:

$$
\begin{aligned}
\|a_{p,n}\|_{\infty, D_{k, \frac{A}{2}, n}} &\leq \mathcal{J}_N\left(1 + \frac{A}{2} n^{-\frac{1}{k}}, 1 + A n^{-\frac{1}{k}}\right) C \rho^n \\
&\leq C\left(N-1\right)\left(\frac{5}{2} N\right)^{2N-2}\left(1 + \frac{2}{A} n^{\frac{1}{k}}\right)^{2N-2} \rho^n \\
&\leq C\left(N-1\right)\left(\frac{10}{A} N\right)^{2N-2} n^{\frac{2N-2}{k}} \rho^n \\
&\leq C\left(N-1\right)\left(\frac{10}{A} N\right)^{2N-2}\left(\sup_{t \geq 0} t^{\frac{2N-2}{k}} \sqrt{\rho}^t\right) \sqrt{\rho}^n
\end{aligned}
$$

But direct computations prove that :

$$
(\forall l \in \mathbb{N}) : \sup_{t \geq 0}\left(t^{\frac{l}{k}} \sqrt{\rho}^t\right) = \left(\left(\frac{2}{ek \ln\left(\frac{1}{\rho}\right)}\right)^{\frac{1}{k}}\right)^l l^{\frac{l}{k}} \tag{6.1}
$$

It follows that :

$$
\|a_{p,n}\|_{\infty, D_{k, \frac{A}{2}, n}} \leq \mathcal{Q} \sqrt{\rho}^n
$$

where :

$$
\mathcal{Q} := C\left(N-1\right)\left(\frac{10}{A} N\right)^{2N-2}(2N-2)^{\frac{2N-2}{k}}\left(\left(\frac{2}{ek \ln\left(\frac{1}{\rho}\right)}\right)^{\frac{1}{k}}\right)^{2N-2}
$$

Thence, in view of the Cauchy's inequalities for holomorphic functions, we can write :

$$
\begin{aligned}
(\forall n \in \mathbb{N}^*) \quad : \quad & \left\|a_{p,n}^{(l)}\right\|_{\infty, D} \\
&\leq \mathcal{Q} l!\left(\frac{2}{A}\right)^l n^{\frac{l}{k}} e^{-\ln\left(\frac{1}{\sqrt[4]{\rho}}\right) n} \sqrt[4]{\rho}^n \\
&\leq \mathcal{Q}\left(\frac{2}{A}\left(\frac{1}{ek \ln\left(\frac{1}{\rho}\right)}\right)^{\frac{1}{k}}\right)^l l^{(1+\frac{1}{k})l} \sqrt[4]{\rho}^n
\end{aligned}
$$

Thence the following estimates hold :

$$
(\forall (l, n) \in \mathbb{N} \times \mathbb{N}^*) : \|a_{p,n}^{(l)}\|_{\infty, D} \leq \mathcal{P}_4^{l+1} l^{(1+\frac{1}{k})l} \sqrt[4]{\rho}^n \tag{6.2}
$$

where $\mathcal{P}_4 := \max\left(\mathcal{Q}, \mathcal{Q}\left(\frac{2}{A}\left(\frac{1}{ek \ln\left(\frac{1}{\rho}\right)}\right)^{\frac{1}{k}}\right)^l, 1\right)$. Consequently the function series $\sum a_{p,n}^{(l)}$ is uniformly convergent on each compact subset of D, and we

obtain the following estimate :

$$(\forall l \in \mathbb{N}) : ||a_p^{(l)}||_{\infty,D} \leq \frac{\mathcal{P}_4}{1 - \sqrt[4]{\rho}} \mathcal{P}_4^l l^{(1+\frac{1}{k})l}$$

Consequently :

$$(\forall p \in J(N)) : a_p \in H_1^k(D)$$

Let us consider the function $f \in H_N(D)$ defined by the formula :

$$(\forall z \in D) : f(z) := \sum_{j=0}^{N-1} a_p(z)\overline{z}^p$$

Then we have for each $(l, m) \in \mathbb{N}^2$:

$$(\forall z \in D) \quad :$$

$$\left| \frac{\partial^{l+m} f}{\partial z^l \partial \overline{z}^m}(z) \right| \leq \sum_{j=0}^{N-1} p! \left| a_p^{(l)}(z) \right|$$

$$\leq \frac{N! N \mathcal{P}_4}{1 - \sqrt[4]{\rho}} \mathcal{P}_4^l l^{(1+\frac{1}{k})l}$$

$$\leq \frac{N! N \mathcal{P}_4}{1 - \sqrt[4]{\rho}} \mathcal{P}_4^{l+m} (l+m)^{(1+\frac{1}{k})(l+m)}$$

It follows that $f \in H_N^k(D)$.

The proof of the converse part of the main result is then achieved.

$\square$

Remark 6.2. It follows easily from the relation (6.1) that if $(f_n)_{n \in \mathbb{N}^*}$ is a sequence of $N-$analytic polynomials such that

$$(\forall n \in \mathbb{N}^*) : ||f_n||_{\infty, D_{k,A,n}} \leq C \rho^n$$

for some constants $C > 0$, $A > 0$ and $\rho \in]0, 1[$ then there exist some constants $\mathcal{P} \geq 1$ such that the following estimates hold for each $(n, \alpha) \in \mathbb{N}^* \times \mathbb{N}^2$:

$$\|D^\alpha f_n\|_{\infty, D_{k, \frac{A}{3}, n}} \leq \mathcal{P}^{|\alpha|+1} |\alpha|^{(1+\frac{1}{k})|\alpha|} \sqrt[4]{\rho}^n$$

7. Applications

7.1. **E. M. Dyn'kin's theorem for the Gevrey class $H_N^k(D)$.**

Corollary 7.1. *Let $f \in H_N(D)$.*

1. *If $f \in H_N^k(D)$ then there exists a function $F : \mathbb{C} \to \mathbb{C}$ of class C^∞ on $\mathbb{C}$ with compact support such that :*

$$\begin{cases} F|_D = f \\ (\forall z \in \mathbb{C} \setminus \overline{D}) : \left| \left(\frac{\partial}{\partial \overline{z}}\right)^N F(z) \right| \leq C_1 \exp\left[-C_2 (|z| - 1)^{-k}\right] \end{cases}$$

where C_1, $C_2 > 0$ are constants.

    2. Conversly, if there exists a function $F : \mathcal{U} \to \mathbb{C}$ of class C^∞ on an open neighborhood $\mathcal{U}$ of the closed unit disk $\overline{D}$ such that :

$$\begin{cases} F|_D = f \\ \left(\forall z \in \mathcal{U} \setminus \overline{D}\right) : \left|\left(\frac{\partial}{\partial \overline{z}}\right)^N F(z)\right| \leq C_1 \exp\left[-C_2\left(|z|-1\right)^{-k}\right] \end{cases}$$

where C_1, $C_2 > 0$ are constants then $f \in H_N^k(D)$.

Proof. 1. Assume that $f \in H_N^k(D)$ then, according to theorem 5. 1., there exists constants $A \in]0,1[$, $C > 0$, $\rho \in]0,1[$ and a sequence of $N-$polynomial functions $(f_n)_{n \in \mathbb{N}}$ such that :

$$\begin{cases} (\forall n \in \mathbb{N}) : \|f_n\|_{\infty, D_{k,A,n}} \leq C\rho^n \\ (\forall x \in D) : \sum_{n=0}^{+\infty} f_n(x) = f(x) \end{cases}$$

On the other hand there exist ([50], lemma 3.3., page 77) for each $n \in \mathbb{N}$ a function $h_n : \mathbb{C} \longrightarrow [0,1]$ and a family of positive constants $(L_\nu)_{\nu \in \mathbb{N}^2}$ such that :

$$\begin{cases} \left(\forall z \in D_{k,\frac{A}{4},n}\right) : h_n(z) = 1 \\ \left(\forall z \in \mathbb{C} \setminus D_{k,\frac{A}{3},n}\right) : h_n(z) = 0 \\ \left(\forall \nu \in \mathbb{N}^2\right)\left(\forall z \in \mathbb{R}^2\right) : |D^\nu h_n(z)| \leq L_\nu n^{\frac{|\nu|}{k}} \end{cases}$$

We set for each $p \in \mathbb{N}$:

$$\mathcal{M}_p := \max_{|\nu| \leq p} L_\nu$$

We denote by F_n the function defined by :

$$\begin{cases} (\forall z \in D_{k,A,n}) : F_n(z) = h_n(z)f_n(z) \\ (\forall z \in \mathbb{C} \setminus D_{k,A,n}) : F_n(z) = 0 \end{cases}$$

The function F_n is of class C^∞ on $\mathbb{C}$ and satisfies the condition :

$$F_n|_{D_{k,\frac{A}{4},n}} = f_n|_{D_{k,\frac{A}{4},n}}$$

Since :

$$(\forall n \in \mathbb{N}) : \|F_n\|_{\infty,\mathbb{C}} \leq C\rho^n$$

it follows that the function series $\sum F_n$ is uniformly convergent on $\mathbb{C}$ to a continuous function F with compact support contained in D_A. Furthermore we have for all $z \in D$:

$$\begin{aligned} F(z) &= \sum_{n=0}^{+\infty} F_n(z) \\ &= \sum_{n=0}^{+\infty} f_n(z) \\ &= f(z) \end{aligned}$$

Thence F is an extension to $\mathbb{C}$ of f. Let $n \in \mathbb{N}$, $\alpha \in \mathbb{N}^2$ and $z \in \mathbb{C}$. If $z \in \mathbb{C} \setminus D_{k, \frac{A}{3}, n}$ then we will have :

$$D^\alpha F_n(z) = 0$$

Now if $z \in D_{k, \frac{A}{3}, n}$ then we have :

$$|D^\alpha F_n(z)|$$
$$\leq \sum_{\beta \preccurlyeq \alpha} \binom{\alpha}{\beta} |D^\beta h_n(z)| \, |D^{\alpha - \beta} f_n(z)|$$

But there exists, thanks to remark 4.2., a constant $\mathcal{P}_5 \geq 1$ such that the following estimate holds for each $v \in \mathbb{N}^2$:

$$\| D^v f_n \|_{\infty, \, D_{k, \frac{A}{3}, n}} \leq \mathcal{P}_5^{|v|+1} \, |v|^{\left(1 + \frac{1}{k}\right)|v|} \, \sqrt[4]{\rho}^{\,n}$$

Hence we can write :

$$|D^\alpha F_n(z)|$$
$$\leq \sum_{\beta \preccurlyeq \alpha} \binom{\alpha}{\beta} \mathcal{M}_{|\alpha|} n^{\frac{|\beta|}{k}} |D^{\alpha - \beta} f_n(z)|$$
$$\leq \sum_{\beta \preccurlyeq \alpha} \mathcal{M}_{|\alpha|} \binom{\alpha}{\beta} n^{\frac{|\beta|}{k}} \mathcal{P}_5^{|\alpha - \beta| + 1} \, |\alpha - \beta|^{\left(1 + \frac{1}{k}\right)|\alpha - \beta|} \, \sqrt[4]{\rho}^{\,n}$$
$$\leq \mathcal{P}_5 \mathcal{M}_{|\alpha|} \left(2\mathcal{P}_5\right)^{|\alpha|} |\alpha|^{\left(1 + \frac{1}{k}\right)|\alpha|} n^{\frac{|\alpha|}{k}} \sqrt[4]{\rho}^{\,n}$$

It follows that the function series $\sum D^\alpha F_n(z)$ is for all $\alpha \in \mathbb{N}^2$ normally convergent on $\mathbb{C}$. Hence the function $F = \sum_{n=1}^\infty F_n$ is of class C^∞ on $\mathbb{C}$ and we have for each $z \in \mathbb{C} \setminus D$ and $\alpha \in \mathbb{N}^2$:

$$\begin{cases} D^\alpha F(z) = \sum_{n=1}^{+\infty} D^\alpha F_n(z) \\ \left(\frac{\partial}{\partial \bar{z}}\right)^N F(z) = \sum_{n=1}^{+\infty} \left(\frac{\partial}{\partial \bar{z}}\right)^N F_n(z) \end{cases}$$

On the other hand we have for each $n \in \mathbb{N}^*$:

$$\left(\frac{\partial}{\partial \bar{z}}\right)^N F_n(z) = 0 \text{ if } \varrho(z, D) \geq \frac{A}{3} n^{-\frac{1}{k}} \text{ or } \varrho(z, D) < \frac{A}{4} n^{-\frac{1}{k}}$$

But if $\frac{A}{4} n^{-\frac{1}{k}} \leq \varrho(z, D) \leq \frac{A}{3} n^{-\frac{1}{k}}$ then we have the following estimates :

$$\left| \left(\frac{\partial}{\partial \bar{z}}\right)^N F_n(z) \right| \leq \sum_{p=0}^N \frac{\binom{N}{p}}{2^N} \left| D^{(p, N-p)} F_n(z) \right|$$
$$\leq \mathcal{P}_5 \mathcal{M}_N \left(2\mathcal{P}_5\right)^N N^{\left(1 + \frac{1}{k}\right)N} n^{\frac{N}{k}} \sqrt[4]{\rho}^{\,n}$$
$$\leq \mathcal{P}_6 e^{\frac{\ln(\rho)}{8} n} \sup_{t \geq 0} t^{\frac{N}{k}} e^{-\frac{\ln\left(\frac{1}{\rho}\right)}{8} t}$$

where $\mathcal{P}_6 := \mathcal{P}_5 \mathcal{M}_N \left(2\mathcal{P}_5\right)^N N^{\left(1+\frac{1}{k}\right)N}$. The relation (6.1) entails then that the following relation holds :

$$\left|\left(\frac{\partial}{\partial \overline{z}}\right)^N F_n(z)\right| \leq \mathcal{P}_6 \left(\frac{8N}{ek \ln\left(\frac{1}{\rho}\right)}\right)^{\frac{N}{k}} e^{\frac{\ln(\rho)}{8}n}$$

Let us then set $\mathcal{P}_7 := \dfrac{\mathcal{P}_6 \left(\frac{8N}{ek \ln\left(\frac{1}{\rho}\right)}\right)^{\frac{N}{k}}}{1-e^{\frac{\ln(\rho)}{8}}}$ and $\mathcal{P}_8 := \frac{1}{4}\left(\frac{A}{4}\right)^k \ln\left(\frac{1}{\rho}\right)$ .Thence we have for every $z \in \mathbb{C} \setminus \overline{D}$ the following estimate :

$$\begin{aligned}
\left|\left(\frac{\partial}{\partial \overline{z}}\right)^N F(z)\right| &\leq& \left(1-e^{\frac{\ln(\rho)}{8}}\right)\mathcal{P}_7 \sum_{\frac{A}{4}n^{-\frac{1}{k}} \leq \varrho(z,D) \leq \frac{A}{3}n^{-\frac{1}{k}}} e^{\frac{\ln(\rho)}{4}n} \\
&\leq& \left(1-e^{\frac{\ln(\rho)}{8}}\right)\mathcal{P}_7 \sum_{\left(\frac{A}{4\varrho(z,D)}\right)^k \leq n} e^{\frac{\ln(\rho)}{4}n} \\
&\leq& \mathcal{P}_7 \exp\left(-\mathcal{P}_8 \varrho(z,D)^{-k}\right) \\
&\leq& \mathcal{P}_7 \exp\left(-\mathcal{P}_8 \left(|z|-1\right)^{-k}\right)
\end{aligned}$$

2. Let $f \in H_N(D)$. Assume that there exists a function $F : \mathcal{U} \to \mathbb{C}$ of class C^∞ on a neighborhood $\mathcal{U}$ of the closed unit disk $\overline{D}$ such that :

$$\begin{cases} F|_D = f \\ \left(\forall z \in \mathcal{U} \setminus \overline{D}\right) : \left|\left(\frac{\partial}{\partial \overline{z}}\right)^N F(z)\right| \leq \mathcal{P}_9 \exp\left(-\mathcal{P}_{10}\left(|z|-1\right)^{-k}\right) \end{cases}$$

where $\mathcal{P}_9$, $\mathcal{P}_{10} > 0$ are constants. Let $R_0 > 0$ be such that the disk D_{1+R_0} is contained in $\mathcal{U}$. There exist ([50]) a function $\Phi : \mathbb{C} \longrightarrow [0,1]$ of class C^∞ on $\mathbb{C}$ such that :

$$\begin{cases} \left(\forall z \in \overline{D}_{1+\frac{R_0}{3}}\right) : \Phi(z) = 1 \\ \left(\forall z \in \mathbb{C} \setminus \overline{D}_{1+\frac{2R_0}{3}}\right) : \Phi(z) = 0 \end{cases}$$

We denote by $\widetilde{F}$ the function defined by :

$$\begin{cases} \left(\forall z \in \overline{D}_{1+\frac{2R_0}{3}}\right) : \widetilde{F}(z) = \Phi(z)F(z) \\ \left(\forall z \in \mathbb{C} \setminus \overline{D}_{1+\frac{2R_0}{3}}\right) : \widetilde{F}(z) = 0 \end{cases}$$

Then it is clear that the function $\widetilde{F}$ is an extension of the function f to $\mathbb{C}$, and that $\widetilde{F}$ is of class C^∞ on $\mathbb{C}$ with compact support contained in $\overline{D}_{1+\frac{2R_0}{3}}$ and that the following estimate holds :

$$\left(\forall z \in \mathbb{C} \setminus \overline{D}\right) : \left|\left(\frac{\partial}{\partial \overline{z}}\right)^N \widetilde{F}(z)\right| \leq \mathcal{P}_{11} \exp\left(-\mathcal{P}_{12}\left(|z|-1\right)^{-k}\right)$$

with some convenient constants $\mathcal{P}_{11}$, $\mathcal{P}_{12} > 0$. Thanks to proposition 2.1., the following relations hold for every $z \in \mathbb{C}$:

$$
\begin{aligned}
\widetilde{F}(z) &= \iint_{\mathbb{C}} \frac{\left(\overline{z} - \overline{\zeta}\right)^{N-1}}{\pi(N-1)!(z-\zeta)} \left(\frac{\partial}{\partial \overline{z}}\right)^N \widetilde{F}(\zeta)\, d\nu(\zeta) \\
&= \iint_{D_{1+\frac{2R_0}{3}} \setminus \overline{D}} \frac{\left(\overline{z} - \overline{\zeta}\right)^{N-1}}{\pi(N-1)!(z-\zeta)} \left(\frac{\partial}{\partial \overline{z}}\right)^N \widetilde{F}(\zeta)\, d\nu(\zeta)
\end{aligned}
$$

Let us denote by Ψ the function :

$$
\begin{aligned}
\Psi: \quad \mathbb{C} \times \left(D_{1+R_0} \setminus \overline{D}\right) &\rightarrow \mathbb{C} \\
(z, \zeta) &\mapsto \frac{\left(\overline{z} - \overline{\zeta}\right)^{N-1}}{\pi(N-1)!(z-\zeta)} \left(\frac{\partial}{\partial \overline{z}}\right)^N \widetilde{F}(\zeta)
\end{aligned}
$$

Then we have the following estimates for each $(l, m) \in \mathbb{N}^2$:

$$
\sup_{(z,\zeta)\in D\times D_{1+\frac{2R_0}{3}},\, \zeta\notin\overline{D}\cup\{z\}} \left| \frac{\partial^{l+m}\Psi}{\partial z^l \partial \overline{z}^m}(z,\zeta) \right|
$$

$$
\leq \sup_{(z,\zeta)\in D\times D_{1+\frac{2R_0}{3}},\, \zeta\notin\overline{D}\cup\{z\}} \max\left(|z-\zeta|,1\right)^{N-1} \frac{\left|\left(\frac{\partial}{\partial \overline{z}}\right)^N \widetilde{F}(\zeta)\right| l!}{\pi |z-\zeta|^{l+1}}
$$

$$
\leq \pi^{-1}\mathcal{P}_{11}\left(2\left(1+\frac{2R_0}{3}\right)\right)^{N-1} \cdot
$$

$$
\cdot \sup_{(z,\zeta)\in D\times D_{1+\frac{2R_0}{3}},\, \zeta\notin\overline{D}\cup\{z\}} \frac{\exp\left(-\mathcal{P}_{12}\left(|\zeta|-1\right)^{-k}\right) l!}{|z-\zeta|^{l+1}}
$$

But there exists, according to proposition 8.2. in the appendix below, a constant $\mathcal{P}_{13} > 0$ such that :

$$
\sup_{(z,\zeta)\in D\times D_{1+\frac{2R_0}{3}},\, \zeta\notin\overline{D}\cup\{z\}} \frac{\exp\left(-\mathcal{P}_{12}\left(|\zeta|-1\right)^{-k}\right) l!}{|z-\zeta|^{l+1}}
$$

$$
\leq \frac{1}{\pi^{-1}\mathcal{P}_{11}\left(2\left(1+\frac{2R_0}{3}\right)\right)^{N-1}} \mathcal{P}_{13}^{l+1} l^{\left(1+\frac{1}{k}\right)l}
$$

Consequently the following condition holds :

$$
\left(\forall\, (l,m) \in \mathbb{N}^2\right): \quad \sup_{(z,\zeta)\in D\times D_{1+\frac{2R_0}{3}},\, \zeta\notin\overline{D}\cup\{z\}} \left| \frac{\partial^{l+m}\Psi}{\partial z^l \partial \overline{z}^m}(z,\zeta) \right| \leq \mathcal{P}_{13}^{l+1} l^{\left(1+\frac{1}{k}\right)l}
$$

It follows that, when applying the differential operator $\mathcal{L}_{lm} := \frac{\partial^{l+m}}{\partial z^l \partial \overline{z}^m}$ to F, we can interchange $\mathcal{L}_{lm}$ and the integral sign $\displaystyle\iint\limits_{D_{1+\frac{2R_0}{3}} \setminus \overline{D}}$ to write for each $z \in D$:

$$
\begin{aligned}
\frac{\partial^{l+m} f}{\partial z^l \partial \overline{z}^m}(z) &= \frac{\partial^{l+m} \widetilde{F}}{\partial z^l \partial \overline{z}^m}(z) \\
&= \iint\limits_{D_{1+\frac{2R_0}{3}} \setminus \overline{D}} \frac{\partial^{l+m} \Psi}{\partial z^l \partial \overline{z}^m}(z, \zeta)\, d\nu(\zeta)
\end{aligned}
$$

Hence we can write for each $z \in D$:

$$
\left| \frac{\partial^{l+m} \widetilde{F}}{\partial z^l \partial \overline{z}^m}(z) \right| \leq \frac{4\pi R_0 (R_0 + 3)}{9} \mathcal{P}_{13}^{l+m+1} (l+m)^{(1+\frac{1}{k})(l+m)}
$$

Consequently $f \in H_N^k(D)$.

     The proof of the corollary is complete. $\hfill\square$

7.2. **Degree of the best uniform N-polynomial approximation of functions of the Gevrey class $H_N^k(D)$.**

Corollary 7.2. *For every $k > 0$, the set of the functions*

$$
\begin{aligned}
\Theta_{\alpha,\beta} : \quad \mathbb{R}^+ &\to \qquad\qquad \mathbb{R}^+ \\
t &\mapsto \quad \alpha \exp\left(-\beta t^{\frac{k}{k+1}} \right)
\end{aligned}
$$

where (α, β) runs over $\mathbb{R}^{+*} \times \mathbb{R}^{+*}$, is a degree of the best uniform $N-$polynomial approximation of the Gevrey class $H_N^k(D)$.

Proof. 1. Let $f \in H_N^k(D)$. According to the main result of this paper there exists a sequence $(P_n)_{n \in \mathbb{N}^*}$ of $N-$analytic polynomials such that :

$$
\begin{cases}
(\forall n \in \mathbb{N}^*) : \|P_n\|_{\infty, D_{k,R,n}} \leq C\delta^n \\
(\forall z \in D) : \sum\limits_{n=1}^{+\infty} P_n(z) = F(z) \\
(\forall n \in \mathbb{N}^*) : d^\circ(P_n) \leq n^{\frac{k+1}{k}}
\end{cases}
$$

where $C > 0$ and $\delta \in]0, 1[$ are constants. Let us denote for each $n \in \mathbb{N}^*$ by Q_n the finite sum $Q_n := \sum\limits_{j^{\frac{k+1}{k}} \leq n} P_j$. Then $Q_n \in \Pi_{N,n}$ and the following inequalities

hold :

$$
\begin{aligned}
\mathcal{E}_{N,n}\left(f\right) &\leq \|f - Q_n\|_{\infty,D} \\
&\leq \sum_{j^{\frac{k+1}{k}} > n} \|P_j\|_{\infty,D} \\
&\leq C \sum_{j^{\frac{k+1}{k}} > n} \delta^j \\
&\leq \frac{C}{1-\delta} \delta^{n^{\frac{k}{k+1}}}
\end{aligned}
$$

Consequently we have for each $n \in \mathbb{N}$:

$$
\mathcal{E}_{N,n}\left(f\right) \leq \mathcal{P}_{14} \exp\left(-\mathcal{P}_{15} n^{\frac{k}{k+1}}\right)
$$

where $\mathcal{P}_{14} := \max\left(\frac{C}{1-\delta}, \mathcal{E}_{N,0}\left(f\right)\right) > 0$ and $\mathcal{P}_{15} := \ln\left(\frac{1}{\delta}\right) > 0$.

2. Conversly, let $f \in C\left(\overline{D}\right)$ which fullfiles the following condition :

$$
(\forall n \in \mathbb{N}) : \mathcal{E}_{N,n}\left(f\right) \leq \mathcal{P}_{16} \exp\left(-\mathcal{P}_{17} n^{\frac{k}{k+1}}\right)
$$

where $\mathcal{P}_{16}, \mathcal{P}_{17} > 0$ are constants. Then there exists for each $n \in \mathbb{N}$, a function $W_n \in \Pi_{N,n}$ such that :

$$
\|f - W_n\|_{\infty,D} \leq \mathcal{P}_{16} \exp\left(-\mathcal{P}_{17} n^{\frac{k}{k+1}}\right)
$$

We denote for each $n \in \mathbb{N}^*$, by Y_n the finite sum $\displaystyle\sum_{n^{\frac{k+1}{k}} \leq j < (n+1)^{\frac{k+1}{k}}} \left(W_j - W_{j-1}\right)$.

Then $Y_n \in \Pi_N$ and we obtain, for every $z \in D_{k,\frac{\beta}{3},n}$, by virtue of the proposition 3.3., the following estimates :

$$
\begin{aligned}
|Y_n\left(z\right)| &\leq \sum_{n^{\frac{k+1}{k}} \leq j < (n+1)^{\frac{k+1}{k}}} |W_j(z) - W_{j-1}(z)| \\
&\leq \sum_{n^{\frac{k+1}{k}} \leq j < (n+1)^{\frac{k+1}{k}}} \left(2^{N+1} - 1\right) \mathcal{J}_N\left(\frac{1}{2}, 1\right) \cdot \\
&\quad \cdot \mathcal{P}_{16}\left(\exp\left(-\mathcal{P}_{17} j^{\frac{k}{k+1}}\right) + \exp\left(-\mathcal{P}_{17}\left(j-1\right)^{\frac{k}{k+1}}\right)\right) |z|^{j+N} \\
&\leq \mathcal{P}_{16}\left(2^{N+1} - 1\right) \mathcal{J}_N\left(\frac{1}{2}, 1\right) \exp\left(-\mathcal{P}_{17} n\right) \cdot \\
&\quad \cdot \sum_{n^{\frac{k+1}{k}} \leq j < (n+1)^{\frac{k+1}{k}}} \left(1 + \exp\left(\mathcal{P}_{17}\left(j^{\frac{k}{k+1}} - (j-1)^{\frac{k}{k+1}}\right)\right)\right) \cdot \\
&\quad \cdot \left(1 + \frac{\mathcal{P}_{17}}{3} n^{-\frac{1}{k}}\right)^{j+N}
\end{aligned}
$$

But, according to the proposition 8.3. in the appendix below, we have the following asymptotic estimates :

$$
\exp\left(-\mathcal{P}_{17}n\right) \sum_{n \leq j^{\frac{k}{k+1}} < n+1} \left(1 + \exp\left(\mathcal{P}_{17}\left(j^{\frac{k}{k+1}} - (j-1)^{\frac{k}{k+1}}\right)\right)\right) \cdot
$$
$$
\cdot \left(1 + \frac{\mathcal{P}_{17}}{3}n^{-\frac{1}{k}}\right)^{j+N}
$$
$$
= \underset{n \to +\infty}{O}\left(\left(e^{-\frac{\mathcal{P}_{17}}{4}}\right)^n\right)
$$

Consequently there exist a constant $\mathcal{P}_{18} > 0$ such that the following estimate holds for each $n \in \mathbb{N}$:

$$
\|Y_n\|_{\infty, D_{k, \frac{\beta}{3}, n}} \leq \mathcal{P}_{18}\left(e^{-\frac{\mathcal{P}_{17}}{4}}\right)^n
$$

Since the function series $\sum Y_n$ is uniformly convergent on D to the function f it follows, thanks to theorem 5. 1., that $f \in H_N^k(D)$.

The proof of the corollary is then achieved. $\qquad\square$

8. Appendix

Proposition 8.1. *For each constant $\mathcal{B} > 0$, there exists a constant $\mathcal{D}_1 > 0$ such that the following estimate holds* :

$$
(\forall n \in \mathbb{N}) \quad : \quad e^{-\mathcal{B}n}\left((n+1)^{\frac{k+1}{k}} - n^{\frac{k+1}{k}} + 1\right) \cdot
$$
$$
\cdot \left(1 + \frac{\mathcal{B}}{2}n^{-\frac{1}{k}}\right)^{n^{\frac{k+1}{k}}}
$$
$$
\leq \quad \mathcal{D}_1\left(e^{-\frac{\mathcal{B}}{4}}\right)^n
$$

Proof. The following inequalities hold for each $n \in \mathbb{N}$:

$$
e^{-\mathcal{B}n}\left((n+1)^{\frac{k+1}{k}} - n^{\frac{k+1}{k}} + 1\right)\left(1 + \frac{\mathcal{B}}{2}n^{-\frac{1}{k}}\right)^{n^{\frac{k+1}{k}}}
$$
$$
\leq \quad e^{-\mathcal{B}n}e^{\frac{\mathcal{B}}{2}n}\left((n+1)^{\frac{k+1}{k}} - n^{\frac{k+1}{k}} + 1\right)
$$
$$
\leq \quad \left(\frac{k+1}{k}\right)e^{-\frac{\mathcal{B}}{2}n}\left(n^{\frac{1}{k}} + 1\right)
$$
$$
\leq \quad \left(\frac{k+1}{k}\right)e^{-\frac{\mathcal{B}}{4}n}\left(n^{\frac{1}{k}} + 1\right)e^{-\frac{\mathcal{B}}{4}n}
$$

Consequently there exist a constant $\mathcal{D}_1 > 0$ such that :

$$(\forall n \in \mathbb{N}) \quad : \quad e^{-\mathcal{B}n}\left((n+1)^{\frac{k+1}{k}} - n^{\frac{k+1}{k}} + 1\right) \cdot$$

$$\cdot \left(1 + \frac{\mathcal{B}}{2} n^{-\frac{1}{k}}\right)^{n^{\frac{k+1}{k}}}$$

$$\leq \quad \mathcal{D}_1 \left(e^{-\frac{\mathcal{B}}{4}}\right)^n$$

Thence we achieve the proof of the proposition.   □

Proposition 8.2. *For each constants* $R, \mathcal{B} > 0$, *there exists a constant* $\mathcal{D}_2 > 0$ *such that the following estimate holds* :

$$(\forall l \in \mathbb{N}) \quad : \quad \sup_{(z,\zeta)\in D \times D_{1+\frac{2R}{3}},\, \zeta \notin \overline{D} \cup \{z\}} \left(\frac{\exp\left(-\mathcal{B}\left(|\zeta| - 1\right)^{-k}\right) l!}{|z - \zeta|^{l+1}}\right)$$

$$\leq \quad \mathcal{D}_2^{l+1} l^{(1+\frac{1}{k})l}$$

Proof. For each constants $R, \mathcal{B} > 0$, we have for every $l \in \mathbb{N}$:

$$\sup_{(z,\zeta)\in D \times D_{1+\frac{2R}{3}},\, \zeta \notin \overline{D} \cup \{z\}} \left(\frac{\exp\left(-\mathcal{B}\left(|\zeta| - 1\right)^{-k}\right) l!}{|z - \zeta|^{l+1}}\right)$$

$$\leq \quad l! \sup_{(z,\zeta)\in D \times D_{1+\frac{2R}{3}},\, \zeta \notin \overline{D} \cup \{z\}} \left(\frac{\exp\left(-\mathcal{B}\left|z - \zeta\right|^{-k}\right)}{|z - \zeta|^{l+1}}\right)$$

$$\leq \quad \mathcal{B}^{-\frac{l+1}{k}} l! \sup_{t \geq 0}\left(t^{\frac{l+1}{k}} e^{-t}\right)$$

But, according to the relation (6.1), we can write :

$$\mathcal{B}^{-\frac{(l+1)}{k}} \sup_{t \geq 0}\left(t^{\frac{l+1}{k}} e^{-t}\right)$$

$$\leq \quad \left(\frac{\mathcal{B}}{k}\right)^{\frac{1}{k}}\left(\left(\frac{k}{\mathcal{B}}\right)^{\frac{1}{k}}\right)^l l^{\frac{l}{k}}$$

Let us set $\mathcal{D}_2 := \max\left(\left(\frac{\mathcal{B}}{k}\right)^{\frac{1}{k}}, 1\right)$. Then we have the following relation :

$$(\forall l \in \mathbb{N}) \quad : \quad \sup_{(z,\zeta)\in D \times D_{1+\frac{2R}{3}},\, \zeta \notin \overline{D} \cup \{z\}} \left(\frac{\exp\left(-\mathcal{B}\left(|\zeta| - 1\right)^{-k}\right) l!}{|z - \zeta|^{l+1}}\right)$$

$$\leq \quad \mathcal{D}_2^{l+1} l^{(1+\frac{1}{k})l}$$

Thence we achieve the proof of the proposition.   □

Proposition 8.3. *For each constant $\mathcal{B} > 0$, the following asymptotic estimate holds* :

$$e^{-\mathcal{B}n} \sum_{n^{\frac{k+1}{k}} \leq j < (n+1)^{\frac{k+1}{k}}} \left(1 + \exp\left(\mathcal{B}\left(j^{\frac{k}{k+1}} - (j-1)^{\frac{k}{k+1}}\right)\right)\right) \cdot$$

$$\cdot \left(1 + \frac{\mathcal{B}}{3} n^{-\frac{1}{k}}\right)^{j+N}$$

$$= \underset{n \to +\infty}{O} \left(\left(e^{-\frac{\mathcal{B}}{4}}\right)^n\right)$$

Proof. The following asymptotic relations hold for each constant $\mathcal{B} > 0$:

$$e^{-\mathcal{B}n} \sum_{n^{\frac{k+1}{k}} \leq j < (n+1)^{\frac{k+1}{k}}} \left(1 + \exp\left(\mathcal{B}\left(j^{\frac{k}{k+1}} - (j-1)^{\frac{k}{k+1}}\right)\right)\right) \left(1 + \frac{\mathcal{B}}{3} n^{-\frac{1}{k}}\right)^{j+N}$$

$$= \underset{n \to +\infty}{O} \left(e^{-\mathcal{B}n} \exp\left(\left(1 + \underset{n \to +\infty}{o}(1)\right) \frac{\mathcal{B}}{3} n\right) \left(\sum_{n^{\frac{k+1}{k}} \leq j < (n+1)^{\frac{k+1}{k}}} 1\right)\right)$$

$$= \underset{n \to +\infty}{O} \left(e^{\frac{2\mathcal{B}}{3}n} e^{-\mathcal{B}n} \left((n+1)^{\frac{k+1}{k}} - n^{\frac{k+1}{k}}\right)\right)$$

$$= \underset{n \to +\infty}{O} \left(n^{\frac{1}{k}} e^{-\frac{\mathcal{B}}{3}n}\right)$$

$$= \underset{n \to +\infty}{O} \left(\left(e^{-\frac{\mathcal{B}}{4}}\right)^n\right)$$

Thence we achieve the proof of the proposition. $\qquad\qquad\square$

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