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Nearly Rational Frobenius Groups

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Abstract. In this paper, we study the structure of finite Frobenius groups whose non-rational or non-real irreducible characters are linear.

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1. Introduction

Throughout this paper all groups are finite. A group in which all irreducible characters are rational is called rational group. In [4], Darafsheh and Sharifi characterize all rational Frobenius groups. In [9], Norooz-Abadian and Sharifi find the structure of Frobenius $\mathbb{Q}_1'$-groups, that is, non-rational Frobenius groups whose non-rational irreducible characters are linear. In this paper, we provide a different short proof of the following theorem:

Theorem 1.1 (Main Theorem of [9]). *If G is a Frobenius $\mathbb{Q}_1'$-group, then one of the following situations occurs:*

1. *$G \cong E(p^n) : Z_t$, where p is an odd prime, $n \geq 1$, and $t \geq 1$ is even,*
2. *$G \cong G' : Z_t$, where G' is a rational 2-group and $t \geq 1$ is odd,*
3. *$G \cong E(p^n) : H$, where H is a metacyclic group of order $2^m q$, for some Fermat prime q and $n, m \geq 1$.*

Furthermore, in Corollary 2.1, we find the structure of Frobenius groups whose non-real irreducible characters are linear. We summarize our notations.

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$cl_G(a)$ denotes the conjugacy class of a in G, $\phi(n)$ denotes the Euler function that counts the positive integers less than n that are relatively prime to n, $Re(G)$ denotes the set of all the real elements of G, $Ra(G)$ denotes the set of all the rational elements of G, $Lin(G)$ denotes the set of all the linear characters of G, and $\pi(G)$ denotes the set of prime numbers dividing the order of G.

2. Proof of Main Theorem

Let G be a $\mathbb{Q}'_1$-group, then every non-rational irreducible character of G is linear. Moreover, we know that each non-rational linear character is non-real, hence G is a non-real group whose non-real irreducible characters are linear. In [3], Chillag and Mann studied the structure of such groups. Our new proof is on the basis of Theorem 1.4 of [3].

Proof. Assume that G is a Frobenius $\mathbb{Q}'_1$-group with a Frobenius kernel K and complement H. By Theorem 13.8 of [7], $Lin(G) = Lin(H)$, H is a $\mathbb{Q}'_1$-group, and $K \subseteq G'$. By Theorem 1.4 of [3], G has a normal π-complement N in which $\pi = \pi(|G : Re(G)|)$ and $Re(G) = \{x \in G | x^2 \in G'\} = Ra(G)$. Furthermore, G is one of the following cases.

`Case (1):` $U \cong G/N$ is abelian with a cyclic Hall $2'$-subgroup Q and $G'Q$ is a Frobenius group with nilpotent kernel G'. Moreover, $Re(G) = N(U \cap Re(G))$ in which $U \cap Re(G) = Re(U)$.

By Exercise 3 of [2, p. 272], if K is a proper subgroup of G', then G' is a Frobenius group, contradicting the fact that G' is nilpotent. Therefore $K = G'$ and H is abelian.

Suppose, first, that U is of odd order, then $U = Q$, $Re(G) = N = G'$, and $G = G'Q$. By Lemma 6.3 of [6], $Re(G) = Re(G')$ and G' is real (rational) which implies G' is a rational 2-group and (2) follows.

Now, assume that U has an even order and $U = SQ$ where S is an abelian Sylow 2-subgroup of G. Moreover, since $G' \subseteq N \subseteq Re(G)$ and N is of odd order, we can conclude that $G' = N$ and H is of even order. By Lemma 3 of [2, p. 273], we know that every Sylow subgroup of H is either cyclic or a generalized quaternion group, thus S and H are cyclic. On the other hand, by Theorem 13.3 of [7], G' is abelian of odd order and if $\langle x \rangle$ is a subgroup of order p^n of G' where p is odd, we have

$$\left| \frac{N_G(\langle x \rangle)}{C_G(\langle x \rangle)} \right| = \phi(p^n) = p^{n-1}(p-1) \text{ divides } |H| \tag{2.1}$$

and since $(|H|, |K|) = 1$, G' is an elementary abelian p-group and claim (1) follows.

`Case (2):` $U \cong G/N$ is a non-real 2-group in which every non-real irreducible character is linear.

Since N is a normal 2-complement of G, then by Theorem 2.5 of [8] every non-linear irreducible character of G has even degree. Additionally, since $K' \neq K$ and $\chi(1) = |H|$ in which $\chi = \lambda^G \in Irr(G)$ for some non-trivial linear character λ of K, we can obtain that H is of even order and every non-linear irreducible character of H has even degree and hence H has a normal 2-complement, that is, $H = SO$ where S is a Sylow 2-subgroup of H. And, we know that $S \cong H/O$ is either cyclic or a generalized quaternion $\mathbb{Q}_1'$-group, thus S is cyclic. Moreover, by Theorem 6 of [2, p. 277] and Theorem 3.6 of [5], $Z(H)$ is a non-trivial elementary abelian 2-group. It follows that, by Theorem 18.3 of [10], $O = H'$ is cyclic and H is a metacyclic group. Therefore by Theorem 2 of [1], $|H| = 2^m q$ where $|H'| = q$ for some odd prime number q. Consider $\langle x \rangle = H' \subseteq G'$, we can write that

$$\left| \frac{N_H(\langle x \rangle)}{C_H(\langle x \rangle)} \right| = \phi(q) = (q-1) \text{ divides } |S| = 2^m \tag{2.2}$$

and so q is a Fermat prime number. On the other hand, since H is of even order then K is abelian and using an equation similar to (2.1), we can show that K is an elementary abelian p-group, as claimed. □

Corollary 2.1. *Let G be a finite Frobenius group whose non-real irreducible characters are linear. Then one of the following situations occurs:*

(1) *$G \cong G' : Z_t$, where G' is abelian of odd order and $t \geq 1$ is even,*
(2) *$G \cong G' : Z_t$, where G' is a real 2-group and $t \geq 1$ is odd,*
(3) *$G \cong K : H$, where K is an abelian subgroup of G' and H is a metacyclic group of even order.*

Example 2.2. Let G be a Frobenius group that is isomorphic to $\mathbb{Z}_{13} : (\mathbb{Z}_3 : \mathbb{Z}_4)$. We can easily check that G satisfies the properties of situation (3) of Corollary 2.1.

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