

## Viewing Some Ordinary Differential Equations from the Angle of Derivative Polynomials

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**ABSTRACT.** In the paper, the authors view some ordinary differential equations and their solutions from the angle of (generalized) derivative polynomials and simplify some known identities for the Bernoulli numbers and polynomials, the Frobenius–Euler polynomials, the Euler numbers and polynomials, in terms of the Stirling numbers of the first and second kinds.

**Keywords:** Viewpoint, Ordinary differential equation, Solution, Derivative polynomial, Frobenius–Euler polynomial, Stirling numbers, Bernoulli number, Bernoulli polynomial, Identity.

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### 1. BACKGROUND AND MOTIVATIONS

We first note that a part of this paper is a review of some known results.

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We now recall some results obtained in two different and independent directions and recall a concept in combinatorics about higher derivatives of functions.

**1.1. The first direction.** In [18, p. 1127 and 1131], it was obtained that

$$\frac{1}{(1 - e^{-t})^2} = 1 + \frac{1}{e^t - 1} - \left( \frac{1}{e^t - 1} \right)'$$

and

$$\frac{1}{(1 - e^{-t})^3} = 1 + \frac{1}{e^t - 1} - \frac{3}{2} \left( \frac{1}{e^t - 1} \right)' + \frac{1}{2} \left( \frac{1}{e^t - 1} \right)''.$$

In the preprint [27] and its formally published version [10], the above two identities were generalized inductively and recursively by the following eight identities:

$$\begin{aligned} \left( \frac{1}{e^t - 1} \right)^{(k)} &= \sum_{m=1}^{k+1} \lambda_{k,m} \left( \frac{1}{e^t - 1} \right)^m, \\ \left( \frac{1}{1 - e^{-t}} \right)^{(k)} &= \sum_{m=1}^{k+1} \mu_{k,m} \left( \frac{1}{1 - e^{-t}} \right)^m, \\ \left( \frac{1}{1 - e^{-t}} \right)^{(k)} &= \sum_{m=1}^{k+1} \lambda_{k,m} \left( \frac{1}{e^t - 1} \right)^m, \\ \left( \frac{1}{e^t - 1} \right)^{(k)} &= \sum_{m=1}^{k+1} \mu_{k,m} \left( \frac{1}{1 - e^{-t}} \right)^m, \\ \left( \frac{1}{1 - e^{-t}} \right)^k &= \sum_{m=1}^k a_{k,m-1} \left( \frac{1}{1 - e^{-t}} \right)^{(m-1)}, \\ \left( \frac{1}{e^t - 1} \right)^k &= \sum_{m=1}^k b_{k,m-1} \left( \frac{1}{e^t - 1} \right)^{(m-1)}, \\ \left( \frac{1}{1 - e^{-t}} \right)^k &= 1 + \sum_{m=1}^k a_{k,m-1} \left( \frac{1}{e^t - 1} \right)^{(m-1)}, \\ \left( \frac{1}{e^t - 1} \right)^k &= 1 + \sum_{m=1}^k b_{k,m-1} \left( \frac{1}{1 - e^{-t}} \right)^{(m-1)}, \end{aligned}$$

where

$$\begin{aligned} \lambda_{k,m} &= (-1)^k (m-1)! S(k+1, m), \quad \mu_{k,m} = (-1)^{m-1} (m-1)! S(k+1, m), \\ a_{k,m-1} &= (-1)^{m^2+1} M_{k-m+1}(k, m), \quad b_{k,m-1} = (-1)^{k-m} a_{k,m-1}, \end{aligned}$$

$$M_j(k, i) = \begin{vmatrix} \frac{\binom{k}{i}}{(i-1)!} & S(i+1, i) & S(i+2, i) & \cdots & S(i+j-1, i) \\ \frac{\binom{k}{i+1}}{i!} & S(i+1, i+1) & S(i+2, i+1) & \cdots & S(i+j-1, i+1) \\ \frac{\binom{k}{i+2}}{(i+1)!} & 0 & S(i+2, i+2) & \cdots & S(i+j-1, i+2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\binom{k}{i+j-1}}{(i+j-2)!} & 0 & 0 & \cdots & S(i+j-1, i+j-1) \end{vmatrix}$$

for  $j \in \mathbb{N}$ , and

$$S(k, m) = \frac{1}{m!} \sum_{\ell=1}^m (-1)^{m-\ell} \binom{m}{\ell} \ell^k$$

for  $k \geq m \geq 1$  denote the Stirling numbers of the second kind. In [10, Remark 5.1], it was pointed out that the four functions

$$\frac{1}{e^t - 1}, \quad \frac{1}{1 - e^{-t}}, \quad \frac{1}{1 - e^{-t}}, \quad \text{and} \quad \frac{1}{e^t - 1}$$

are respectively the unique solutions to the linear ordinary equations

$$\sum_{i=1}^k a_{k,i-1} y^{(i-1)} = F_n(t) \quad \text{and} \quad \sum_{i=1}^k (-1)^{i-1} a_{k,i-1} y^{(i-1)} = G_n(t)$$

for all  $k \in \mathbb{N}$  and  $n = 1, 2$ , where

$$F_n(t) = \begin{cases} \frac{1}{(1 - e^{-t})^k} - 1, & n = 1; \\ \frac{1}{(1 - e^{-t})^k}, & n = 2 \end{cases}$$

and

$$G_n(t) = \begin{cases} \frac{1}{(e^t - 1)^k} - 1, & n = 1; \\ \frac{1}{(e^t - 1)^k}, & n = 2. \end{cases}$$

In [10, Remark 5.2], it was concluded that

$$\begin{pmatrix} \frac{1}{e^t - 1} \\ \left(\frac{1}{e^t - 1}\right)' \\ \left(\frac{1}{e^t - 1}\right)'' \\ \vdots \\ \left(\frac{1}{e^t - 1}\right)^{(i-1)} \\ \left(\frac{1}{e^t - 1}\right)^{(i)} \end{pmatrix} = \begin{pmatrix} \lambda_{0,1} & 0 & \cdots & 0 & 0 \\ \lambda_{1,1} & \lambda_{1,2} & \cdots & 0 & 0 \\ \lambda_{2,1} & \lambda_{2,2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda_{k-2,1} & \lambda_{k-2,2} & \cdots & \lambda_{k-2,k-1} & 0 \\ \lambda_{k-1,1} & \lambda_{k-1,2} & \cdots & \lambda_{k-1,k-1} & \lambda_{k-1,k} \end{pmatrix} \begin{pmatrix} \frac{1}{e^t - 1} \\ \frac{1}{(e^t - 1)^2} \\ \frac{1}{(e^t - 1)^3} \\ \vdots \\ \frac{1}{(e^t - 1)^i} \\ \frac{1}{(e^t - 1)^{i+1}} \end{pmatrix}$$

and

$$\begin{pmatrix} \frac{1}{e^t-1} \\ \frac{1}{(e^t-1)^2} \\ \frac{1}{(e^t-1)^3} \\ \vdots \\ \frac{1}{(e^t-1)^k} \end{pmatrix} = (-1)^{(k-1)k/2} \begin{pmatrix} a_{1,0} & 0 & 0 & \cdots & 0 \\ a_{2,0} & a_{2,1} & 0 & \cdots & 0 \\ a_{3,0} & a_{3,1} & a_{3,2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{k,0} & a_{k,1} & a_{k,2} & \cdots & a_{k,k-1} \end{pmatrix} \begin{pmatrix} \frac{1}{e^t-1} \\ \left(\frac{1}{e^t-1}\right)' \\ \left(\frac{1}{e^t-1}\right)'' \\ \vdots \\ \left(\frac{1}{e^t-1}\right)^{(k-1)} \end{pmatrix}.$$

All of the above results in [10, 27] were established by induction.

In [78, Theorems 3.1 and 3.2], by the Faà di Bruno formula and combinatorial techniques, the above identities in [10, 27] were generalized and unified as follows: For  $\alpha, \lambda \in \mathbb{R}$  and  $n \in \mathbb{N}$ , we have

$$\left(\frac{1}{1-\lambda e^{\alpha t}}\right)^{(n)} = (-1)^n \alpha^n \sum_{k=1}^{n+1} (-1)^{k-1} (k-1)! S(n+1, k) \left(\frac{1}{1-\lambda e^{\alpha t}}\right)^k \quad (1.1)$$

and

$$\left(\frac{1}{1-\lambda e^{\alpha t}}\right)^n = \frac{(-1)^{n-1}}{(n-1)!} \sum_{k=1}^n \frac{(-1)^{k-1}}{\alpha^{k-1}} s(n, k) \left(\frac{1}{1-\lambda e^{\alpha t}}\right)^{(k-1)}, \quad (1.2)$$

where  $s(n, k)$  for  $n \geq k \geq 1$  denote the Stirling numbers of the first kind.

In [9, Theorem 2.1], the above identities (1.1) and (1.2) were inductively proved, without using the Faà di Bruno formula and related techniques, once again and were rewritten as follows: For real constants  $\lambda \neq 0$  and  $\alpha \neq 0$  and for  $k \in \mathbb{N}$ , when  $\lambda > 0$  and  $t \neq -\frac{\ln \lambda}{\alpha}$  or when  $\lambda < 0$  and  $t \in \mathbb{R}$ , we have

$$\frac{d^k}{dt^k} \left(\frac{1}{\lambda e^{\alpha t} - 1}\right) = (-1)^k \alpha^k \sum_{m=1}^{k+1} (m-1)! S(k+1, m) \left(\frac{1}{\lambda e^{\alpha t} - 1}\right)^m \quad (1.3)$$

and

$$\left(\frac{1}{\lambda e^{\alpha t} - 1}\right)^k = \frac{1}{(k-1)!} \sum_{m=1}^k \frac{(-1)^{m-1}}{\alpha^{m-1}} s(k, m) \frac{d^{m-1}}{dt^{m-1}} \left(\frac{1}{\lambda e^{\alpha t} - 1}\right). \quad (1.4)$$

In the papers [1, 72], there are some new results on higher derivatives of the function  $\frac{1}{e^x-1}$ .

**1.2. The second direction.** In [13, 14], it was obtained inductively and recursively that

(1) the function  $F(t) = \frac{1}{u-e^t}$  is a solution of the nonlinear differential equation

$$F^n(t) = \frac{n}{u^{n-1}} \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \sum_{\ell_1+\ell_2+\cdots+\ell_{k+1}=n} \frac{1}{\ell_1 \ell_2 \cdots \ell_{k+1}} F^{(k)}(t), \quad (1.5)$$

where  $u \in \mathbb{C} \setminus \{1\}$  and  $n \in \mathbb{N}$ ;

(2) for  $m \in \mathbb{N}$  and  $n \geq 0$ , we have

$$H_n^{(m)}(u) = m \left( \frac{u-1}{u} \right)^{m-1} \sum_{k=0}^{m-1} \frac{1}{(k+1)!} \sum_{\ell_1 + \ell_2 + \dots + \ell_{k+1} = m} \frac{H_{n+k}(u)}{\ell_1 \ell_2 \dots \ell_{k+1}},$$

where  $H_n(u)$  for  $n \geq 0$  are called the  $n$ th Frobenius–Euler numbers which can be generated by

$$\frac{1-u}{e^t - u} = \sum_{n=0}^{\infty} H_n(u) \frac{t^n}{n!}, \quad u \neq 1;$$

(3) for  $m \in \mathbb{N}$  and  $n \geq 0$ , we have

$$\begin{aligned} & \sum_{\ell_1 + \ell_2 + \dots + \ell_m = n} \binom{n}{\ell_1, \ell_2, \dots, \ell_m} H_{\ell_1}(u) H_{\ell_2}(u) \dots H_{\ell_m}(u) \\ &= m \left( \frac{u-1}{u} \right)^{m-1} \sum_{k=0}^{m-1} \frac{1}{(k+1)!} \sum_{\ell_1 + \ell_2 + \dots + \ell_{k+1} = m} \frac{H_{n+k}(u)}{\ell_1 \ell_2 \dots \ell_{k+1}}; \end{aligned}$$

(4) for  $m \in \mathbb{N}$  and  $n \geq 0$ , we have

$$\begin{aligned} H_n^{(m)}(x|u) &= m \left( \frac{u-1}{u} \right)^{m-1} \sum_{k=0}^{m-1} \frac{1}{(k+1)!} \\ &\quad \times \sum_{\ell_1 + \ell_2 + \dots + \ell_{k+1} = m} \frac{1}{\ell_1 \ell_2 \dots \ell_{k+1}} \sum_{q=0}^n \binom{n}{q} H_{q+k}(u) x^{n-q}, \end{aligned}$$

where  $H_n(x|u)$  for  $n \geq 0$  are called the Frobenius–Euler polynomials which can be generated by

$$\frac{1-u}{e^t - u} e^{xt} = \sum_{n=0}^{\infty} H_n(x|u) \frac{t^n}{n!}, \quad u \neq 1.$$

Influenced by the papers [13, 14], the authors in [11] derived inductively and recursively several formulas for the Bernoulli polynomials of the  $r$ th order

$$\left( \frac{t}{e^t - 1} \right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x) \frac{t^n}{n!}$$

in terms of the Bernoulli numbers  $B_n = B_n^{(1)}(0)$ .

In [70, Theorem 2.1], it was procured inductively and recursively that the function  $F(t) = \frac{1}{e^t + 1}$  is a solution of the nonlinear differential equation

$$F^n(t) = n \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \sum_{\ell_1 + \ell_2 + \dots + \ell_{k+1} = n} \frac{1}{\ell_1 \ell_2 \dots \ell_{k+1}} F^{(k)}(t), \quad n \in \mathbb{N}. \quad (1.6)$$

By virtue of this, as did in [11], the authors of [70] presented several formulas for the Euler numbers and polynomials of the  $r$ th order  $E_n^{(r)}(0)$  and  $E_n^{(r)}(x)$

defined by

$$\left(\frac{2}{e^t + 1}\right)^r e^{xt} = \sum_{n=0}^{\infty} E_n^{(r)}(x) \frac{t^n}{n!}$$

in terms of the so-called Euler numbers  $E_n = E_n^{(1)}(0)$  which are different from the classical and traditional Euler numbers  $E_n = 2^n E(\frac{1}{2})$ .

In [12], it was acquired inductively and recursively that the function  $F(t) = \frac{1}{\ln(1+t)}$  is a solution of the nonlinear differential equation

$$F^{(n)}(t) = \frac{(-1)^n (n-1)!}{(1+t)^n} \sum_{j=2}^{n+1} (j-1)! H_{n-1, j-2} F^j(t) \quad (1.7)$$

for  $n \in \mathbb{N}$ , where

$$H_{n,0} = 1, \quad H_{n,1} = \sum_{k=1}^n \frac{1}{k}, \quad H_{0, j-1} = 0, \quad H_{n, j} = \sum_{k=1}^n \frac{H_{k-1, j-1}}{k}$$

for  $2 \leq j \leq n$ . Thereafter, the authors in [12] established an identity involving the higher order Bernoulli numbers of the second kind  $b_n^{(k)}$  defined by

$$\left[\frac{t}{\ln(1+t)}\right]^k = \sum_{n=0}^{\infty} b_n^{(k)} \frac{t^n}{n!}.$$

In [17], it was procured inductively and recursively that

(1) the function  $F(t) = \frac{1}{(1+\lambda t)^{1/\lambda+1}}$  is a solution of the nonlinear differential equation

$$F^{(n)}(t) = \frac{(-1)^n}{(1+\lambda t)^n} \sum_{i=1}^{n+1} (-1)^{i-1} a_i(n, \lambda) F^i(t) \quad (1.8)$$

for  $n \in \mathbb{N}$ , where

$$\begin{aligned} a_i(n, \lambda) = & (i-1)! \lambda^{n-i+1} \sum_{m_{i-1}=0}^{n-i+1} \sum_{m_{i-2}=0}^{n-m_{i-1}-i+1} \cdots \sum_{m_1=0}^{n-m_{i-1}-\cdots-m_2-i+1} \binom{n}{n-m_{i-1} + \frac{i}{\lambda}}_{m_{i-1}} \binom{n-m_{i-1}-m_{i-2}-1 + \frac{i-1}{\lambda}}{m_{i-2}} \cdots \binom{n-m_{i-1}}{-\cdots-m_1-i+2 + \frac{2}{\lambda}}_{m_1} \left(\frac{1}{\lambda}\right)_{n-m_{i-1}-m_{i-2}-\cdots-m_1-i+1}, \quad (1.9) \end{aligned}$$

(2) the function  $F(t) = \frac{1}{(1+\lambda t)^{1/\lambda-1}}$  is a solution of the nonlinear differential equation

$$F^{(n)}(t) = \frac{(-1)^n}{(1+\lambda t)^n} \sum_{i=1}^{n+1} a_i(n, \lambda) F^i(t) \quad (1.10)$$

for  $n \in \mathbb{N}$ .

Hereafter, the authors in [17] gave some new identities involving degenerate Euler numbers and polynomials.

For  $x \in \mathbb{R}$ , let

$$\langle x \rangle_n = \begin{cases} x(x-1) \cdots (x-n+1), & n \geq 1 \\ 1, & n = 0 \end{cases}$$

denotes the falling factorial of  $x$ . In [7, Theorem 2.1] and its formally published version [15, Theorem 2.1], it was presented inductively and recursively, among other things, that the function  $F(t) = \frac{(1+t)^x}{2+t}$  is a solution of the linear differential equations

$$F^{(n)}(t) = \left[ \sum_{i=0}^n a_i(n, x) \frac{(2+t)^{i-n}}{(1+t)^i} \right] F(t)$$

for  $n \geq 0$ , where  $a_0(n, x) = (-1)^n n!$  and

$$\begin{aligned} a_j(n, x) &= (-1)^{n-j} (n-j)! \langle x \rangle_j \\ &\times \sum_{i_{j-1}=0}^{n-j} \sum_{i_{j-2}=0}^{n-j-i_{j-1}} \cdots \sum_{i_1=0}^{n-j-i_{j-1}-\cdots-i_2} (n-i_{j-1}-\cdots-i_1-j+1) \end{aligned} \quad (1.11)$$

for  $1 \leq j \leq n$ .

In [16], it was demonstrated inductively and recursively that

- (1) the function  $F(t) = \frac{1}{t+c}$  is a solution of the nonlinear differential equation  $F^{(n)}(t) = (-1)^n n! F^{n+1}(t)$  for  $n \in \mathbb{N}$ ,
- (2) the function  $G(t) = \frac{1}{e^t+1}$  is a solution of the nonlinear differential equation

$$G^{(n)}(t) = (-1)^n \sum_{k=1}^{n+1} (-1)^{k-1} a_{n,k} G^k(t), \quad (1.12)$$

where

$$\begin{aligned} a_{n,k} &= (k-1)! \sum_{m_1=0}^{n-k+2} \sum_{m_2=0}^{n-k+2-m_1} \cdots \sum_{m_{k-2}=0}^{n-k+2-m_1-\cdots-m_{k-3}} k^{m_1} \\ &\times (k-1)^{m_2} \cdots 3^{m_{k-2}} (2^{n-m_1-m_2-\cdots-m_{k-2}-k+3} - 1). \end{aligned} \quad (1.13)$$

Hereafter, the authors gave two identities for the so-called Changhee polynomials of the  $r$ th order  $\text{Ch}_n^{(r)}(x)$  defined by

$$\left( \frac{2}{t+2} \right)^r (1+t)^x = \sum_{n=0}^{\infty} \text{Ch}_n^{(r)}(x) \frac{t^n}{n!}.$$

**1.3. Derivative polynomials.** Suppose  $f$  is a function whose derivative is a polynomial in  $f$ , that is,  $f'(x) = P(f(x))$  for some polynomial  $P$ . Then all the higher order derivatives of  $f$  are also polynomials in  $f$ , so we have a sequence of polynomials  $P_n$  defined by  $f^{(n)}(x) = P_n(f(x))$  for  $n \geq 0$ . As usual, we call  $P_n(u)$  the derivative polynomials of  $f$ . For more information and results on derivative polynomials, please refer to [1, 2, 3, 4, 6, 43, 77] and closely related references therein.

Now we more accurately introduce a new notion below. If there exists a sequence of polynomials  $P_n(u) = \sum_{k=0}^n a_k u^k$  for  $a_k \in \mathbb{C}$  and a sequence of functions  $h_{n,k}(x)$  for  $n, k \geq 0$  such that  $f^{(n)}(x) = \sum_{k=0}^{n+q} a_k h_{n,k}(x) f^k(x)$  for  $n, q \geq 0$ , then we call  $P_{n+q}(u)$  the generalized derivative polynomials of  $f(x)$  with respect to  $h_{n,k}(x)$ .

## 2. ALTERNATIVE VIEWPOINTS AND DERIVATIVE POLYNOMIALS

Now we are in a position to discuss the above conclusions from alternative viewpoints and (generalized) derivative polynomials.

**2.1.** Almost all the above linear or nonlinear ordinary differential equations and their solutions mentioned in last section can be alternatively regarded as problems of (generalized) derivative polynomials.

**2.2.** Taking  $\lambda = -1$  and  $\alpha = 1$  in (1.3) and simplifying yield

$$\frac{d^k}{dt^k} \left( \frac{1}{e^t + 1} \right) = (-1)^k \sum_{m=1}^{k+1} (-1)^{m-1} (m-1)! S(k+1, m) \left( \frac{1}{e^t + 1} \right)^m.$$

Comparing this with (1.12) reveals that the coefficients  $a_{n,k}$  in (1.13) is just equal to  $(k-1)! S(n+1, k)$ . In other words,

$$\begin{aligned} S(n+1, k) &= \sum_{m_1=0}^{n-k+2} \sum_{m_2=0}^{n-k+2-m_1} \cdots \sum_{m_{k-2}=0}^{n-k+2-m_1-\cdots-m_{k-3}} k^{m_1} (k-1)^{m_2} \cdots 3^{m_{k-2}} \\ &\quad \times (2^{n-m_1-m_2-\cdots-m_{k-2}-k+3} - 1). \end{aligned}$$

We can also say that generalized derivative polynomials of the function  $\frac{1}{e^t+1}$  is

$$P_{k+1}(x) = (-1)^k \sum_{m=1}^{k+1} (-1)^{m-1} (m-1)! S(k+1, m) x^m$$

with respect to  $h_{k,m}(t) \equiv 1$  for  $k \geq 1$  and  $1 \leq m \leq k+1$ . It is much straightforward and simple to see that the derivative polynomials of the function  $\frac{1}{t+c}$  is  $P_n(x) = (-1)^n n! x^{n+1}$ . Consequently, we find an alternative viewpoint to examine the results in [16].

**2.3.** By Leibniz's theorem for differentiation of a product, we obtain

$$\begin{aligned} \frac{d^n}{dt^n} \left[ \frac{(1+t)^x}{2+t} \right] &= \sum_{k=0}^n \binom{n}{k} \frac{d^k [(1+t)^x]}{dt^k} \frac{d^{n-k}}{dt^{n-k}} \left( \frac{1}{2+t} \right) \\ &= \sum_{k=0}^n \binom{n}{k} \langle x \rangle_k (1+t)^{x-k} \frac{(-1)^{n-k} (n-k)!}{(2+t)^{n-k+1}} \\ &= \sum_{k=0}^n \binom{n}{k} \langle x \rangle_k \frac{(-1)^{n-k} (n-k)!}{(1+t)^k (2+t)^{n-k}} \frac{(1+t)^x}{2+t}. \end{aligned}$$

This implies that the coefficients  $a_j(n, x)$  in (1.11) can be simplified explicitly as

$$a_k(n, x) = (-1)^{n-k} (n-k)! \binom{n}{k} \langle x \rangle_k = (-1)^{n-k} \frac{n!}{k!} \langle x \rangle_k, \quad 0 \leq k \leq n.$$

Moreover, it follows that

$$\binom{n}{k} = \sum_{i_{k-1}=0}^{n-k} \sum_{i_{k-2}=0}^{n-k-i_{k-1}} \cdots \sum_{i_1=0}^{n-k-i_{k-1}-\cdots-i_2} (n - i_{k-1} - \cdots - i_1 - k + 1).$$

In other words, generalized derivative polynomials of the function  $\frac{(1+t)^x}{2+t}$  is  $P_{1+0}(u) = u$  with respect to

$$h_{n,1}(t) = \sum_{k=0}^n \binom{n}{k} \langle x \rangle_k \frac{(-1)^{n-k} (n-k)!}{(1+t)^k (2+t)^{n-k}}, \quad n \geq 0.$$

**2.4.** In combinatorial analysis, the Faà di Bruno formula plays an important role and can be described in terms of the Bell polynomials of the second kind

$$B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \sum_{\substack{1 \leq i \leq n, \ell_i \in \{0\} \cup \mathbb{N} \\ \sum_{i=1}^n i \ell_i = n \\ \sum_{i=1}^n \ell_i = k}} \frac{n!}{\prod_{i=1}^{n-k+1} \ell_i!} \prod_{i=1}^{n-k+1} \binom{x_i}{i!}^{\ell_i}$$

for  $n \geq k \geq 0$ , see [5, p. 134, Theorem A], by

$$\frac{d^n}{dt^n} [f \circ h(t)] = \sum_{k=0}^n f^{(k)}(h(t)) B_{n,k}(h'(t), h''(t), \dots, h^{(n-k+1)}(t)) \quad (2.1)$$

for  $n \geq 0$ , see [5, p. 139, Theorem C]. Replacing  $f(u)$  and  $u = h(t)$  in (2.1) respectively by  $\frac{1}{u \pm 1}$  and  $(1 + \lambda t)^{1/\lambda}$  yields

$$f^{(k)}(u) = \frac{(-1)^k k!}{(u \pm 1)^{k+1}}, \quad h^{(k)}(t) = \left[ \prod_{\ell=0}^{k-1} (1 - \ell \lambda) \right] (1 + \lambda t)^{1/\lambda - k},$$

and

$$\left[ \frac{1}{(1 + \lambda t)^{1/\lambda} \pm 1} \right]^{(n)} = \sum_{k=0}^n \frac{(-1)^k k!}{[(1 + \lambda t)^{1/\lambda} \pm 1]^{k+1}} B_{n,k} \left( (1 + \lambda t)^{1/\lambda - 1}, \right.$$

$$\begin{aligned}
& (1-\lambda)(1+\lambda t)^{1/\lambda-k}, \dots, \left[ \prod_{\ell=0}^{n-k} (1-\ell\lambda) \right] (1+\lambda t)^{1/\lambda-n+k-1} \\
&= \frac{1}{(1+\lambda t)^n} \sum_{k=0}^n \frac{(-1)^k k! (1+\lambda t)^{k/\lambda}}{[(1+\lambda t)^{1/\lambda} \pm 1]^{k+1}} B_{n,k} \left( 1, 1-\lambda, \dots, \prod_{\ell=0}^{n-k} (1-\ell\lambda) \right),
\end{aligned}$$

where we used in the above line the formula

$$B_{n,k}(abx_1, ab^2x_2, \dots, ab^{n-k+1}x_{n-k+1}) = a^k b^n B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) \quad (2.2)$$

listed in [5, p. 135].

In [5, p. 133], it was listed that

$$\frac{1}{k!} \left( \sum_{m=1}^{\infty} x_m \frac{t^m}{m!} \right)^k = \sum_{n=k}^{\infty} B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) \frac{t^n}{n!}$$

for  $k \geq 0$ . Therefore, we obtain that

$$\begin{aligned}
& \sum_{n=k}^{\infty} B_{n,k} \left( 1, 1-\lambda, \dots, \prod_{\ell=0}^{n-k} (1-\ell\lambda) \right) \frac{t^n}{n!} = \frac{1}{k!} \left( \sum_{m=1}^{\infty} \prod_{\ell=0}^{m-1} (1-\ell\lambda) \frac{t^m}{m!} \right)^k \\
&= \frac{1}{k!} [(\lambda t + 1)^{1/\lambda} - 1]^k = \frac{1}{k!} \sum_{\ell=0}^k (-1)^{k-\ell} \binom{k}{\ell} (\lambda t + 1)^{\ell/\lambda}.
\end{aligned}$$

Accordingly, it follows that

$$\begin{aligned}
& B_{n,k} \left( 1, 1-\lambda, (1-\lambda)(1-2\lambda), \dots, \prod_{\ell=0}^{n-k} (1-\ell\lambda) \right) \\
&= \frac{1}{k!} \lim_{t \rightarrow 0} \sum_{\ell=0}^k (-1)^{k-\ell} \binom{k}{\ell} [(\lambda t + 1)^{\ell/\lambda}]^{(n)} \\
&= \frac{1}{k!} \lim_{t \rightarrow 0} \sum_{\ell=0}^k (-1)^{k-\ell} \binom{k}{\ell} \prod_{q=0}^{n-1} (\ell - q\lambda) (1+\lambda t)^{\ell/\lambda-n} \\
&= \frac{(-1)^k}{k!} \sum_{\ell=0}^k (-1)^{\ell} \binom{k}{\ell} \prod_{q=0}^{n-1} (\ell - q\lambda).
\end{aligned}$$

In a word, we derive that

$$\left[ \frac{1}{(1+\lambda t)^{1/\lambda} \pm 1} \right]^{(n)} = \sum_{k=0}^n \left[ \sum_{\ell=0}^k (-1)^{\ell} \binom{k}{\ell} \prod_{q=0}^{n-1} (\ell - q\lambda) \right] \frac{(1+\lambda t)^{k/\lambda-n}}{[(1+\lambda t)^{1/\lambda} \pm 1]^{k+1}} \quad (2.3)$$

for  $n \geq 0$ . This means that the functions  $\frac{1}{(1+\lambda t)^{1/\lambda} \pm 1}$  have the same generalized derivative polynomials

$$P_{n+1}(u) = \sum_{k=1}^{n+1} \left[ \sum_{\ell=0}^{k-1} (-1)^{\ell} \binom{k-1}{\ell} \prod_{q=0}^{n-1} (\ell - q\lambda) \right] u^k$$

with respect to  $h_{n,k}(t) = (1 + \lambda t)^{(k-1)/\lambda - n}$  for  $n \geq 0$  and  $1 \leq k \leq n + 1$ .

By the way, the form of the equation (2.3) is different from the ones in (1.8) and (1.10). The coefficients in the brackets on the right-hand side of the equation (2.3) are more nice, more explicit, easier to compute than the coefficients  $a_i(n, \lambda)$  defined by (1.9) in (1.8) and (1.10).

**2.5.** In [22, 28], it was obtained that

$$\left[ \frac{1}{\ln(1+t)} \right]^{(m)} = \frac{1}{(1+t)^m} \sum_{i=0}^m (-1)^i i! \frac{s(m, i)}{[\ln(1+t)]^{i+1}}, \quad m \geq 0$$

and

$$\left( \frac{1}{\ln x} \right)^{(n)} = \frac{(-1)^n}{x^n} \sum_{i=2}^{n+1} \frac{a_{n,i}}{(\ln x)^i}, \quad n \in \mathbb{N},$$

where  $a_{n,2} = (n-1)!$  and

$$a_{n,i} = (i-1)!(n-1)! \sum_{\ell_1=1}^{n-1} \frac{1}{\ell_1} \sum_{\ell_2=1}^{\ell_1-1} \frac{1}{\ell_2} \cdots \sum_{\ell_{i-3}=1}^{\ell_{i-4}-1} \frac{1}{\ell_{i-3}} \sum_{\ell_{i-2}=1}^{\ell_{i-3}-1} \frac{1}{\ell_{i-2}}$$

for  $n+1 \geq i \geq 3$ . Comparing this with (1.7) and rearranging lead to

$$H_{n-1, i-1} = \sum_{\ell_1=1}^{n-1} \frac{1}{\ell_1} \sum_{\ell_2=1}^{\ell_1-1} \frac{1}{\ell_2} \cdots \sum_{\ell_{i-3}=1}^{\ell_{i-4}-1} \frac{1}{\ell_{i-3}} \sum_{\ell_{i-2}=1}^{\ell_{i-3}-1} \frac{1}{\ell_{i-2}} = (-1)^{n+i} \frac{s(n, i)}{(n-1)!}$$

for  $3 \leq i \leq n$ . This connects (1.7) with the Stirling numbers of the first kind  $s(n, i)$  and generalized derivative polynomials: generalized derivative polynomials of  $\frac{1}{\ln(1+t)}$  is

$$P_{m+1}(u) = \sum_{i=1}^{m+1} (-1)^{i-1} (i-1)! s(m, i-1) u^i$$

with respect to  $h_{m,i} \equiv \frac{1}{(1+t)^m}$  for  $m \geq 0$ . Hence, we are viewing the paper [12] from a different angle.

**2.6.** Letting  $\lambda = -1$  and  $\alpha = 1$  in (1.4) results in

$$\left( \frac{1}{e^t + 1} \right)^k = \frac{1}{(k-1)!} \sum_{m=1}^k (-1)^{m+k} s(k, m) \frac{d^{m-1}}{dt^{m-1}} \left( \frac{1}{e^t + 1} \right). \quad (2.4)$$

Comparing this with (1.6) gives

$$s(n, k+1) = (-1)^{n+k+1} \frac{n!}{(k+1)!} \sum_{\sum_{q=1}^{k+1} \ell_q = n} \prod_{q=1}^{k+1} \frac{1}{\ell_q}, \quad n \in \mathbb{N}. \quad (2.5)$$

As a result, Theorem 2.1 in [70] can be simplified as (2.4) and has something to do with the Stirling numbers of the first kind  $s(n, k)$ .

**2.7.** Setting  $\lambda = \frac{1}{u}$  and  $\alpha = 1$  in (1.4) produces

$$\left(\frac{1}{u - e^t}\right)^k = \frac{(-1)^{k-1}}{(k-1)!} \frac{1}{u^{k-1}} \sum_{m=0}^{k-1} (-1)^m s(k, m+1) \frac{d^m}{dt^m} \left(\frac{1}{u - e^t}\right)$$

for  $k \in \mathbb{N}$ . Comparing this with (1.5) recovers the formula (2.5) which is an alternative expression for the Stirling numbers of the first kind  $s(n, k)$ . Therefore, all the above-mentioned results in [13, 14] can be restated simply in terms of the Stirling numbers of the first kind  $s(n, k)$ .

**2.8.** In [9, 10, 78], some formulas for computing the Bernoulli numbers, the Euler polynomials, the Apostol–Bernoulli numbers, the Eulerian polynomials, and the Fubini numbers in terms of the Stirling numbers of the second kind were established. These formulas are more concise, simpler, more meaningful than those in [11, 12, 13, 14, 16, 70], as showed above. Due to limitation on the length of this paper, we do not elaborate in further details here.

**2.9.** In [25, 43, 46, 62, 68, 72, 77], there are more information and new conclusions about (generalized) derivative polynomials, explicit formulas for the Bernoulli numbers and polynomials, for the Euler numbers and polynomials, for higher derivatives of some elementary functions, properties of the functions  $\frac{\pm 1}{e^{\pm t} - 1}$ , and the like. Due to limitation on the length of this paper, we do not elaborate in further details yet.

**2.10.** It is common knowledge that mathematicians should try to represent, explain, or interpret a new mathematical quantities in terms of some known and popular quantities in mathematics. Once a new mathematical quantity were established connections or relations with some famous or important quantities, it would be more meaningful and significant in mathematics.

### 3. REMARKS

Finally we would like to give several remarks.

*Remark 3.1.* From the derivation of the equation (2.3), we can conclude that

$$\begin{aligned} B_{n,k} \left( 1, 1 - \lambda, (1 - \lambda)(1 - 2\lambda), \dots, \prod_{\ell=0}^{n-k} (1 - \ell\lambda) \right) \\ = \frac{(-1)^k}{k!} \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \prod_{q=0}^{n-1} (\ell - q\lambda) \end{aligned} \quad (3.1)$$

for  $\lambda \in \mathbb{C}$  and  $n \geq k \geq 0$ . This formula can be used to derive many known and new special values of the Bell polynomials of the second kind  $B_{n,k}$  such as,

when taking  $\lambda = 0, -1, 2, \frac{1}{2}, -2, -\frac{1}{2}$  in (3.1) respectively,

$$B_{n,k}(1, 1, 1, \dots, 1) = \frac{(-1)^k}{k!} \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \ell^n = S(n, k), \quad (3.2)$$

$$\begin{aligned} B_{n,k}(1!, 2!, 3!, \dots, (n-k+1)!) &= \frac{(-1)^k}{k!} \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \prod_{q=0}^{n-1} (\ell+q) \\ &= \binom{n}{k} \binom{n-1}{k-1} (n-k)! \\ &= L(n, k), \end{aligned} \quad (3.3)$$

$$B_{n,k}((-1)!!, 1!!, 3!!, \dots, (2(n-k)-1)!!) = \frac{(-1)^k}{k!} \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \prod_{q=0}^{n-1} (\ell-2q), \quad (3.4)$$

$$\begin{aligned} B_{n,k}\left(1, \frac{1}{2}, 0, \dots, 0\right) &= \frac{(-1)^k}{k!} \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \prod_{q=0}^{n-1} \left(\ell - \frac{q}{2}\right) \\ &= \frac{(n-k)!}{4^{n-k}} \binom{n}{k} \binom{k}{n-k}, \end{aligned} \quad (3.5)$$

$$B_{n,k}(1!!, 3!!, 5!!, \dots, (2(n-k)+1)!!) = \frac{(-1)^k}{k!} \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \prod_{q=0}^{n-1} (\ell+2q), \quad (3.6)$$

and

$$B_{n,k}\left(1, \frac{3}{2}, 3, \frac{15}{2}, \dots, \frac{(n-k+2)!}{2^{n-k+1}}\right) = \frac{(-1)^k}{k!} \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \prod_{q=0}^{n-1} \left(\ell + \frac{q}{2}\right) \quad (3.7)$$

for  $n \geq k \geq 0$ , where  $L(n, k)$  for  $n \geq k \geq 0$  denote the Lah numbers. The identities (3.2) and (3.3) can be found in [5, p. 135]. The identity (3.3) can also be found in [26, Theorem 1]. The identity (3.4) recovers the main results [62, Theorems 1 and 2]. By virtue of the formula (2.2), the identity (3.5) is a special case of

$$B_{n,k}(x, 1, 0, \dots, 0) = \frac{(n-k)!}{2^{n-k}} \binom{n}{k} \binom{k}{n-k} x^{2k-n}$$

which was established alternatively in [46, Theorem 4.1], [62, Eq. (2.8)], and [68, Section 3]. The identities (3.6) and (3.7) are, to the best of our knowledge, new.

The formula (3.1) has an equivalence

$$B_{n,k}(\langle \alpha \rangle_1, \langle \alpha \rangle_2, \dots, \langle \alpha \rangle_{n-k+1}) = \frac{(-1)^k}{k!} \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \langle \alpha \ell \rangle_n, \quad \alpha \in \mathbb{C}$$

which was presented in [58, Theorems 2.1 and 4.1]. The identity (3.1) has been cited and applied in the papers [38, 40, 41, 47, 50, 53, 57, 58].

In recent years, there have been some literature such as [8, 19, 20, 21, 30, 32, 44, 45, 50, 54, 59, 60, 62, 63, 71, 73, 74, 75, 76] devoting to deep and intrinsic investigations and extensive applications of the Bell polynomials of the second kind  $B_{n,k}(x_1, \dots, x_{n-k+1})$ .

*Remark 3.2.* By the way, the formulas (3.4) and (3.5) in [28, p. 325, Corollary 3.1] should be slightly corrected as

$$\left[ \frac{x}{\ln(1+x)} \right]^{(i)} = \frac{(-1)^i}{(1+x)^i} \sum_{k=2}^{i+1} \frac{xa_{i,k} - i(1+x)a_{i-1,k}}{[\ln(1+x)]^k}, \quad i \geq 2,$$

$$\left[ \frac{x}{\ln(1+x)} \right]^{(i)} = \frac{1}{(1+x)^i} \sum_{k=1}^i \frac{(-1)^k k! [xs(i,k) + i(1+x)s(i-1,k)]}{[\ln(1+x)]^{k+1}}, \quad i \geq 2,$$

or

$$\left[ \frac{x}{\ln(1+x)} \right]^{(i)} = \frac{1}{(1+x)^i} \sum_{k=0}^i \frac{(-1)^k k! [xs(i,k) + i(1+x)s(i-1,k)]}{[\ln(1+x)]^{k+1}}, \quad i \geq 1,$$

where  $a_{n,2} = (n-1)!$  for  $n \in \mathbb{N}$ ,

$$a_{n,i} = (i-1)!(n-1)! \sum_{\ell_1=1}^{n-1} \frac{1}{\ell_1} \sum_{\ell_2=1}^{\ell_1-1} \frac{1}{\ell_2} \cdots \sum_{\ell_{i-3}=1}^{\ell_{i-4}-1} \frac{1}{\ell_{i-3}} \sum_{\ell_{i-2}=1}^{\ell_{i-3}-1} \frac{1}{\ell_{i-2}}$$

for  $n+1 \geq i \geq 3$ , and

$$a_{n,i} = (-1)^{n+i-1} (i-1)! s(n, i-1)$$

for  $2 \leq i \leq n+1$ .

*Remark 3.3.* This paper has two preprints [49, 50] and is the first one in a series of articles including [9, 10, 23, 24, 29, 31, 33, 34, 35, 36, 37, 38, 39, 42, 45, 48, 51, 52, 53, 55, 56, 57, 61, 64, 65, 66, 67, 69, 79].

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