

## Application of Tau Approach for Solving Integro-Differential Equations with a Weakly Singular Kernel

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**ABSTRACT.** In this work, the convection-diffusion integro-differential equation with a weakly singular kernel is discussed. The Legendre spectral tau method is introduced for finding the unknown function. The proposed method is based on expanding the approximate solution as the elements of a shifted Legendre polynomials. We reduce the problem to a set of algebraic equations by using operational matrices. Also the convergence analysis for shifted Legendre polynomials and error estimation for tau method have been discussed and approved with the exact solution. Finally, several numerical examples are given to demonstrate the high accuracy of the method.

**Keywords:** Shifted Legendre tau method, Weakly singular kernel, Integro-differential equation, Convection-diffusion equation.

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## 1. INTRODUCTION

It is well known that the spectral methods are very useful for the numerical solution of many kinds of differential and integral equations especially with smooth solutions [6, 7, 12, 23]. Spectral methods have been studied intensively in the last two decades because of their good approximation properties.

An operational technique for the numerical solution of nonlinear ordinary differential equations based on the tau method [18], is presented by Ortiz and Samara [19]. This method, through which the spectral methods can be described as special case, as shown in [20, 21, 10], has found extensive application for numerical solution of many operator equations in recent years. Many authors have been used this method for solving various types of equations [19, 20, 21, 10, 13, 14, 15, 8, 11]. Authors of [1] used this method for the system of nonlinear volterra integro-differential equations.

In this Letter, we consider the Legendre spectral tau method for solving the convection-diffusion integro-differential equation with a weakly singular kernel. Many phenomena in various fields of biology, physics, engineering including space and time variables, are modeled by partial differential equations. The convection-diffusion equation describes the flow of heat, particles, or other physical quantities in situations where there is both diffusion and convection. The term diffusion means the movement of a substance down a concentration gradient, whereas convection means the movement of molecules within fluids.

In this paper the following convection-diffusion integro-differential equation with a weakly singular kernel is considered [24]

$$u_t(x, t) + au_x(x, t) - bu_{xx}(x, t) = \int_0^t K(t-s)u(x, s)ds + f(x, t) \quad (1.1)$$

where,  $a$  and  $b$  are considered to be positive constants quantifying the advection (convection) and diffusion processes, respectively. The integral term is called memory term, the kernel is a weakly singular kernel.

$$K(t-s) = (t-s)^{-\alpha}, \quad 0 < \alpha < 1,$$

subject to the initial condition

$$u(x, 0) = g_0(x), \quad 0 \leq x \leq l, \quad (1.2)$$

and the boundary conditions

$$u(0, t) = f_0(t), \quad u(l, t) = f_1(t), \quad t \geq 0, \quad (1.3)$$

where,  $g_0(x)$ ,  $f_0(t)$ ,  $f_1(t)$  are known functions and  $f(x, t)$  is a given smooth function. If the memory term is zero, then (1.1) reduces to more general inhomogeneous convection-diffusion equation given by

$$u_t(x, t) + au_x(x, t) - bu_{xx}(x, t) = f(x, t), \quad 0 \leq x \leq l, \quad t > 0.$$

The source term  $f(x, t)$ , accounts for an insertion or extraction of mass of the system as it evolves with time. Specifically,  $f(x, t)$  represents the time rate of change of concentration due to external factors, such as a source or a sink.

The Legendre spectral tau method in time with Fourier approximation in space is proposed in [25]. A space-time spectral element method for second-order hyperbolic equations and nonlinear advection-diffusion problems is presented in [27] and [3]. In [24] the approximate solution of convection-diffusion integro-differential equation with a weakly singular kernel is proposed using cubic B-spline collection method.

The main goal of this paper is to present an efficient numerical algorithm for the solution of convection-diffusion weakly singular integro-differential equation. The Legendre spectral tau method, consist of reducing the problem to a set of algebraic equation by expanding the approximate solution  $u(x, t)$  as shifted Legendre polynomials with unknown coefficient. The operational matrices of integral and differential parts appearing in equation are given. These matrices are utilized to evaluate the unknown coefficients of shifted Legendre polynomials.

The rest of this article is organized as follows. In section 2, we describe the basic formulation of shifted Legendre polynomials. In section 3, we construct the operational matrices of Legendre polynomials. Section 4, by using Legendre spectral tau method we construct and develop an algorithm for the solution of the two-sided space and time convection-diffusion integro-differential equation with boundary conditions. We discuss the convergence analysis for the proposed function approximation and error estimation for tau method in section 5. In section 6, some illustrative numerical experiments are given. The paper ends with some conclusions in section 7.

## 2. PROPERTIES OF SHIFTED LEGENDRE POLYNOMIALS

It is well-known that the classical Legendre polynomials are defined on the interval  $[-1, 1]$  and can be determined with the aid of the following recurrence formulae

$$L_0(z) = 1, \quad L_1(z) = z,$$

$$L_{i+1}(z) = \frac{2i+1}{i+1}z L_i(z) - \frac{i}{i+1}L_{i-1}(z), \quad i = 1, 2, \dots$$

Assume  $z \in [z_a, z_b]$  and let  $z^\sim = \frac{2z - z_a - z_b}{z_b - z_a}$ . Then  $\{L_i(z^\sim)\}$  are called the shifted Legendre polynomials on  $[z_a, z_b]$ . In this paper, we mainly consider the shifted Legendre polynomials defined on  $[0, l]$ .

For  $x \in [0, l]$ , let  $L_{l,i}(x) = L_i\left(\frac{2x-l}{l}\right)$ ,  $i = 0, 1, 2, \dots$ . Then the shifted Legendre polynomials  $\{L_{l,i}(x)\}$  are defined by

$$\begin{aligned} L_{l,0}(x) &= 1, \\ L_{l,1}(x) &= \frac{2x-l}{l}, \\ L_{l,i+1}(x) &= \frac{(2i+1)(2x-l)}{(i+1)l}L_{l,i}(x) - \frac{i}{i+1}L_{l,i-1}(x), \quad i = 1, 2, \dots \end{aligned}$$

The set of  $L_{l,i}(x)$  is a complete  $L^2(0, l)$ -orthogonal system, namely

$$\int_0^l L_i^l(x)L_j^l(x)dx = \begin{cases} \frac{l}{2i+1}, & i = j, \\ 0, & i \neq j. \end{cases}$$

So we define  $\Pi_m = \text{span}\{L_{l,0}, L_{l,1}, \dots, L_{l,m}\}$ . Thus, for any  $y(x) \in L^2(0, l)$ , we write

$$y(x) = \sum_{j=0}^{\infty} c_j L_{l,j}(x),$$

where the coefficients  $c_j$  are given by

$$c_j = \frac{2j+1}{l} \int_0^l y(x)L_{l,j}(x)dx, \quad j = 0, 1, 2, \dots \quad (2.1)$$

In practice, only the first  $(m+1)$ - terms of shifted Legendre polynomials are considered.

Hence we can write

$$y_m(x) = \sum_{j=0}^m c_j L_{l,j}(x),$$

which alternatively may be written in the matrix form:

$$y_m(x) = C^T \Phi_{l,m}(x), \quad C^T = [c_0, c_1, \dots, c_m],$$

with

$$\Phi_{l,m}(x) = [L_{l,0}, L_{l,1}, \dots, L_{l,m}]^T = V X_x, \quad (2.2)$$

where  $V$  is the coefficient matrix of shifted Legendre polynomials,

$$X_x = [1, x, x^2, \dots, x^m]^T$$

,  $(\cdot)^T$  stands for the transpose.

Similarly a function of two independent variables  $u(x, t)$  which is infinitely differentiable for  $0 \leq x \leq l$  and  $0 \leq t \leq \tau$  may be expressed in terms of the double shifted Legendre polynomials as

$$u_{n,m}(x, t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{i,j} L_{l,i}(x) L_{\tau,j}(t). \quad (2.3)$$

If the infinite series in (2.3) is truncated, then it can be written as

$$u_{n,m}(x,t) = \sum_{i=0}^n \sum_{j=0}^m a_{i,j} L_{l,i}(x) L_{\tau,j}(t) = \Phi_{l,n}^T(x) A \Phi_{\tau,m}(t), \quad (2.4)$$

where the shifted Legendre vectors  $\Phi_{l,n}(x)$  and  $\Phi_{\tau,m}(t)$  are defined similarly to (2.2). Also the shifted Legendre coefficient matrix  $A$  is given by

$$A = \begin{bmatrix} a_{00} & a_{01} & \dots & a_{0m} \\ a_{10} & a_{11} & \dots & a_{1m} \\ \vdots & \vdots & \dots & \vdots \\ a_{n0} & a_{n1} & \dots & a_{nm} \end{bmatrix},$$

where

$$a_{i,j} = \left( \frac{2i+1}{\tau} \right) \left( \frac{2j+1}{l} \right) \int_0^{\tau} \int_0^l u(x,t) L_{\tau,i}(t) L_{l,j}(x) dx dt, \quad i = 0, 1, \dots, n. \quad (2.5)$$

### 3. OPERATIONAL MATRICES OF SHIFTED LEGENDRE POLYNOMIALS

Recently, operational matrices were adapted in several areas of numerical analysis. They are very important for solving different kinds of problems in various subjects such as integral equation [16, 26], fractional differential equation [2, 5, 22], integro differential equation [4, 9, 17]. In what follows, we make the operational matrix of convection-diffusion partial integral equation of the shifted Legendre vector.

**3.1. Matrix representation of partial differential part.** The derivative of the vector  $\Phi_{l,m}(x)$  can be expressed by

$$\frac{d}{dx} \Phi_{l,m}(x) = D \Phi_{l,m}(x), \quad (3.1)$$

where  $D$  is the  $(m+1) \times (m+1)$  operational matrix of derivative given by

$$D = (d_{i,j}) = \begin{cases} \frac{2(2j+1)}{l}, & \text{for } j = i - k, \quad k = \begin{cases} 1, 3, \dots, m, & \text{if } m \text{ odd,} \\ 1, 3, \dots, m-1, & \text{if } m \text{ even.} \end{cases} \\ 0, & \text{otherwise.} \end{cases}$$

For example, for odd  $m$ , we have

$$D = \frac{2}{l} \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 5 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 3 & 0 & 7 & \dots & 2m-3 & 0 & 0 \\ 1 & 0 & 5 & 0 & \dots & 0 & 2m-1 & 0 \end{bmatrix}.$$

**Theorem 3.1.** Let  $\Phi_{l,m}(x)$  be the shifted Legendre vector and  $u_{n,m}(x,t) = \Phi_{l,n}^T(x)A\Phi_{\tau,m}(t)$  then

$$\frac{\partial^r}{\partial x^r} u_{n,m}(x,t) = \Phi_{l,n}^T(x)(D^T)^r A\Phi_{\tau,m}(t). \quad (3.2)$$

*Proof.* From equations (2.4) and (3.1) we have

$$\begin{aligned} \frac{\partial^r}{\partial x^r} u_{n,m}(x,t) &= \frac{\partial^{r-1}}{\partial x^{r-1}} \left( \Phi_{l,n}^T(x) D^T A \Phi_{\tau,m}(t) \right) \\ &= \frac{\partial^{r-2}}{\partial x^{r-2}} \left( \frac{\partial}{\partial x} \Phi_{l,n}^T(x) D^T A \Phi_{\tau,m}(t) \right) \\ &= \frac{\partial^{r-2}}{\partial x^{r-2}} \left( \Phi_{l,n}^T(x) (D^T)^2 A \Phi_{\tau,m}(t) \right) \\ &\vdots \\ &= \frac{\partial}{\partial x} \left( \Phi_{l,n}^T(x) (D^T)^{r-1} A \Phi_{\tau,m}(t) \right) \\ &= \Phi_{l,n}^T(x) (D^T)^r A \Phi_{\tau,m}(t). \end{aligned}$$

□

**Corollary 3.2.** Let  $\Phi_{l,m}(x)$  be the shifted Legendre vector and  $u_{n,m}(x,t) = \Phi_{l,n}^T(x)A\Phi_{\tau,m}(t)$  then

$$\frac{\partial^r}{\partial t^r} u_{n,m}(x,t) = \Phi_{l,n}^T(x)AD^r\Phi_{\tau,m}(t). \quad (3.3)$$

### 3.2. Matrix representation of integral part.

**Lemma 3.3.** If  $\Gamma$  is the Gamma function, then we have

$$\int_0^t \frac{s^m}{(t-s)^\alpha} ds = \frac{\Gamma(1-\alpha)\Gamma(m+1)}{\Gamma(m-\alpha+2)} t^{m-\alpha+1}, \quad m = 0, 1, 2, \dots$$

*Proof.* With integration by parts and using  $\Gamma(\alpha) = (\alpha-1)!$  it can easily be obtained. □

**Theorem 3.4.** Let  $\Phi_{l,m}(x) = VX_x$  be the shifted Legendre vector then

$$\int_0^t \frac{u(x,s)}{(t-s)^\alpha} ds = \Phi_{l,n}^T(x)AV\Gamma K\Phi_{\tau,m}(t), \quad (3.4)$$

where  $\Gamma$  is a diagonal matrix with elements

$$\Gamma_{m,m} = \frac{\Gamma(1-\alpha)\Gamma(i+1)}{\Gamma(i-\alpha+2)}, \quad i = 0, 1, 2, \dots, m,$$

and

$$K = [B_0, B_1, \dots, B_m]^T, \quad B_j = [t_{j,0}, t_{j,1}, \dots, t_{j,m}],$$

which  $t_{j,i}$ ,  $i = 0, 1, \dots, m$  is the coefficients of  $L_{\tau,i}$ ,  $i = 0, 1, \dots, m$  in expansion of  $t^{j-\alpha+1}$ .

*Proof.*

$$\begin{aligned} \int_0^t \frac{u(x,s)}{(t-s)^\alpha} ds &= \int_0^t \frac{\Phi_{l,n}^T(x) A \Phi_{\tau,m}(s)}{(t-s)^\alpha} ds \\ &= \Phi_{l,n}^T(x) A \int_0^t \frac{[L_{\tau,0}(s), L_{\tau,1}(s), \dots, L_{\tau,m}(s)]^T}{(t-s)^\alpha} ds \\ &= \Phi_{l,n}^T(x) A \int_0^t \frac{V[1, s, \dots, s^m]^T}{(t-s)^\alpha} ds \\ &= \Phi_{l,n}^T(x) AV \left[ \int_0^t \frac{1}{(t-s)^\alpha} ds, \int_0^t \frac{s}{(t-s)^\alpha} ds, \dots, \int_0^t \frac{s^m}{(t-s)^\alpha} ds, \dots \right]^T, \end{aligned}$$

by using lemma (3.3) we can write

$$\begin{aligned} \int_0^t \frac{u(x,s)}{(t-s)^\alpha} ds &= \Phi_{l,n}^T(x) AV \left[ \frac{\Gamma(1-\alpha)\Gamma(1)}{\Gamma(-\alpha+2)} t^{-\alpha+1}, \frac{\Gamma(1-\alpha)\Gamma(2)}{\Gamma(-\alpha+3)} t^{-\alpha+2}, \right. \\ &\quad \left. \dots, \frac{\Gamma(1-\alpha)\Gamma(m+1)}{\Gamma(m-\alpha+2)} t^{m-\alpha+1}, \dots \right]^T \\ &= \Phi_{l,n}^T(x) AV \Gamma \Pi, \end{aligned} \quad (3.5)$$

where  $\Pi = [t^{-\alpha+1}, t^{-\alpha+2}, \dots, t^{m-\alpha+1}, \dots]^T$ . By approximating  $t^{j-\alpha+1}$ ,  $j = 0, 1, \dots, m$ , we get

$$\begin{aligned} t^{j-\alpha+1} &= \sum_{i=0}^m t_{j,i} L_{\tau,i}(t) = B_j \Phi_{\tau,m}(t), \\ B_j &= [t_{j,0}, t_{j,1}, \dots, t_{j,m}], \end{aligned}$$

we obtain

$$\begin{aligned} \Pi &= [B_0 \Phi_{\tau,m}(t), B_1 \Phi_{\tau,m}(t), \dots, B_m \Phi_{\tau,m}(t)]^T = K \Phi_{\tau,m}(t), \quad (3.6) \\ K &= [B_0, B_1, \dots, B_m, \dots]^T \end{aligned}$$

By substituting (3.6) into (3.5) we obtain

$$\int_0^t \frac{u(x,s)}{(t-s)^\alpha} ds = \Phi_{l,n}^T(x) AV \Gamma K \Phi_{\tau,m}(t). \quad (3.7)$$

□

## 4. THE LEGENDRE SPECTRAL TAU METHOD

As we know, the convection-diffusion integro-differential equation with a weakly singular kernel occur in many applications such as in the transport of air and ground water pollutants, oil reservoir flow. The spectral method has been an efficient tool for computing approximate solution of this equations because of its high-order accuracy. For this method much fewer time and space levels are needed to compute a smooth solution.

In this section, a new algorithm for solving this set of equations is proposed based on Legendre spectral tau method in conjunction with the operational matrices of partial differentiation part and operational matrix of weakly integral part of the Legendre polynomials.

Let us start our algorithm to solve (1.1)-(1.3).

We assume that the functions  $f(x, t)$ ,  $g_0(x)$ ,  $f_0(t)$ ,  $f_1(t)$ , generally are polynomial. Otherwise, we can approximate these functions by polynomials to any degree of accuracy (by interpolation or Taylor series or other suitable method.)

Now, we approximate the functions  $f(x, t)$ ,  $g_0(x)$ ,  $f_0(t)$ ,  $f_1(t)$  by the shifted Legendre polynomials as

$$\begin{aligned} f(x, t) &= \sum_{i=0}^n \sum_{j=0}^m f_{i,j} L_{l,i}(x) L_{\tau,j}(t) = \Phi_{l,n}^T(x) F \Phi_{\tau,m}(t), \\ g_0(x) &= \sum_{i=0}^n g_i L_{l,i}(x) = \Phi_{l,n}^T(x) G, \\ f_0(t) &= \sum_{i=0}^m p_i L_{l,i}(t) = P \Phi_{\tau,m}(t), \\ f_1(t) &= \sum_{i=0}^m r_i L_{l,i}(t) = R \Phi_{\tau,m}(t), \end{aligned} \quad (4.1)$$

where  $F$ ,  $G$ ,  $P$  and  $R$  are known matrices which can be written as

$$P = [p_0, p_1, \dots, p_m], \quad R = [r_0, r_1, \dots, r_m], \quad G = [g_0, g_1, \dots, g_n]^T,$$

$$F = \begin{bmatrix} f_{00} & f_{01} & \dots & f_{0m} \\ f_{10} & f_{11} & \dots & f_{1m} \\ \vdots & \vdots & \dots & \vdots \\ f_{n0} & f_{n1} & \dots & f_{nm} \end{bmatrix},$$

where  $p_j$ ,  $r_j$ ,  $g_j$  are given as in (2.1) but  $f_{ij}$  are given as in (2.5).

Using (3.2), (3.3), (3.4), (4.1) and substituting in equation (1.1), it is easy to obtain that

$$\begin{aligned} &\Phi_{l,n}^T(x) A D \Phi_{\tau,m}(t) + a \Phi_{l,n}^T(x) D^T A \Phi_{\tau,m}(t) - b \Phi_{l,n}^T(x) (D^T)^2 A \Phi_{\tau,m}(t) \\ &= \Phi_{l,n}^T(x) A V \Gamma K \Phi_{\tau,m}(t) + \Phi_{l,n}^T(x) F \Phi_{\tau,m}(t). \end{aligned}$$



Hence the residual  $R_{n,m}(x, t)$  for (1.1) can be written as

$$\begin{aligned} R_{n,m}(x, t) &= \Phi_{l,n}^T(x) [AD + aD^T A - b(D^T)^2 A - AV\Gamma K - F] \Phi_{\tau,m}(t) \\ &= \Phi_{l,n}^T(x) H \Phi_{\tau,m}(t), \end{aligned}$$

where

$$H = A(D - V\Gamma K) + (aD^T - b(D^T)^2) A - F.$$

For finding a typical matrix formulation, similar to the typical tau method, we eliminate one last column and two last rows of the matrix  $H$ , then we generate  $(n - 1) \times m$  algebraic equations by using the following algebraic equations

$$H_{ij} = 0, \quad i = 0, 1, \dots, n - 2, \quad j = 0, 1, \dots, m - 1,$$

namely

$$\int_0^l \int_0^\tau R_{n,m}(x, t) L_{\tau,i}(t) L_{l,j}(x) dt dx, \quad i = 0, 1, \dots, n - 2, \quad j = 0, 1, \dots, m - 1. \quad (4.2)$$

Also, by substituting equations (2.4), (4.1) in equations (1.2), (1.3) we have

$$\Phi_{l,n}^T(x) A \Phi_{\tau,m}(0) = \Phi_{l,n}^T(x) G,$$

$$\Phi_{l,n}^T(0) A \Phi_{\tau,m}(t) = P \Phi_{\tau,m}(t),$$

$$\Phi_{l,n}^T(1) A \Phi_{\tau,m}(t) = R \Phi_{\tau,m}(t),$$

which implies

$$A \Phi_{\tau,m}(0) = G, \quad (4.3)$$

$$\Phi_{l,n}^T(0) A = P, \quad (4.4)$$

$$\Phi_{l,n}^T(1) A = R. \quad (4.5)$$

We can find  $n + 1$  linear algebraic equations from (4.3),  $m$  linear algebraic equations by choosing  $m$  equations from (4.4), similarly  $m$  equations from (4.5) and finally  $(n - 1) \times m$  equations from (4.2). Since the number of the unknown coefficients  $a_{ij}$  is equal to  $(n + 1) \times (m + 1)$  we generate a system of  $(n + 1) \times (m + 1)$  equations. Consequently  $u_{n,m}(x, t)$  given in (2.4) can be calculated. In our implementation, we have solved this system using the Mathematica Solve function.

In all the considered examples in section 6, this function has succeeded to obtain an accurate approximate solution of the system. We summarize the algorithm of the method as follows.

#### Algorithm of the method.

**Step 1.** Choose the set of shifted Legendre polynomials  $\{L_{l,i}(x)\}_{i=0}^n$ ,  $\{L_{\tau,j}(t)\}_{j=0}^m$

and let the approximate solution be  $u_{n,m}(x,t) = \sum_{i=0}^n \sum_{j=0}^m a_{i,j} L_{l,i}(x) L_{\tau,j}(t)$ .

**Step 2.** Find the coefficient matrix  $V$  with respect to  $X_x = [1, x, x^2, \dots, x^m]^T$ , such that  $\Phi_{l,m}(x) = VX_x$ .

**Step 3.** Use Equations (3.2), (3.3), (3.4), (4.1) convert problem (1.1) and boundary conditions (1.2), (1.3) to an algebraic system.

**Step 4.** Linearize the supplementary conditions in the same way as mentioned in step 3.

**Step 5.** We can find  $n+1$  linear algebraic equations from (4.3),  $2m$  equations from (4.4), (4.5) and  $m(n-1)$  equations from (4.2) in the obtained system.

**Step 6.** Solve the system obtained from step 4 and 5 to find the unknown coefficient  $a_{i,j}$ ,  $i = 0, 1, \dots, n$ ,  $j = 0, 1, \dots, m$ .

## 5. CONVERGENCE ANALYSIS

In this section we present the convergence analysis for the proposed function approximation and error estimation for Legendre spectral tau method.

**Theorem 5.1.** (convergence theorem) *If a continuous function  $u(x,t)$ , defined on  $[0, l] \times [0, \tau]$ , has bounded mixed fourth partial derivative  $\frac{\partial^4 u(x,t)}{\partial x^2 \partial t^2}$ , then the shifted Legendre expansion of the function as  $\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{i,j} L_{l,i}(x) L_{\tau,j}(t)$  converges uniformly to the  $u(x,t)$ .*

*Proof.* Let  $u(x,t)$  be a function defined on  $[0, l] \times [0, \tau]$  such that  $\left| \frac{\partial^4 u(x,t)}{\partial x^2 \partial t^2} \right| \leq \alpha$ , where  $\alpha$  is a positive constant and

$$a_{i,j} = \left( \frac{2i+1}{\tau} \right) \left( \frac{2j+1}{l} \right) \int_0^{\tau} \int_0^l u(x,t) L_{\tau,i}(t) L_{l,j}(x) dx dt,$$

$$i = 0, 1, \dots, n, \quad j = 0, 1, \dots, m.$$

By partial integration and using following equation

$$L'_{l,i+1} - L'_{l,i-1} = \frac{2}{l} (2i+1) L_{l,i}(x)$$

we have

$$\begin{aligned}
a_{i,j} &= \frac{2j+1}{2\tau} \int_0^\tau u(x,t) \left( L_{l,i}(x) - L_{l,i-1}(x) \right) \Big|_0^l L_{\tau,j}(t) dt \\
&\quad - \frac{2j+1}{2\tau} \int_0^\tau \int_0^l \frac{\partial u(x,t)}{\partial x} \left( L_{l,i}(x) - L_{l,i-1}(x) \right) L_{\tau,j}(t) dx dt \\
&= -\frac{2j+1}{2\tau} \int_0^\tau \int_0^l \frac{\partial u(x,t)}{\partial x} \left( L_{l,i}(x) - L_{l,i-1}(x) \right) L_{\tau,j}(t) dx dt \\
&= -\frac{(2j+1)l}{4\tau} \int_0^\tau \frac{\partial u(x,t)}{\partial x} \left( \frac{L_{l,i+2}(x) - L_{l,i}(x)}{2i+3} - \frac{L_{l,i}(x) - L_{l,i-2}(x)}{2i-1} \right) \Big|_0^l L_{\tau,j}(t) dt \\
&\quad + \frac{(2j+1)l}{4\tau} \int_0^\tau \int_0^l \frac{\partial^2 u(x,t)}{\partial x^2} \left( \frac{L_{l,i+2}(x) - L_{l,i}(x)}{2i+3} - \frac{L_{l,i}(x) - L_{l,i-2}(x)}{2i-1} \right) L_{\tau,j}(t) dx dt \\
&= \frac{(2j+1)l}{4\tau} \int_0^\tau \int_0^l \frac{\partial^2 u(x,t)}{\partial x^2} \left( \frac{L_{l,i+2}(x) - L_{l,i}(x)}{2i+3} - \frac{L_{l,i}(x) - L_{l,i-2}(x)}{2i-1} \right) L_{\tau,j}(t) dx dt.
\end{aligned}$$

Now, let  $Q_{l,i}(x) = (2i-1)L_{l,i+2} - 2(2i+1)L_{l,i}(x) + (2i+3)L_{l,i-2}(x)$  then we have

$$\begin{aligned}
a_{i,j} &= \frac{(2j+1)l}{4\tau(2i+3)(2i-1)} \int_0^\tau \int_0^l \frac{\partial^2 u(x,t)}{\partial x^2} Q_{l,i}(x) L_{\tau,j}(t) dx dt \\
&= \frac{l\tau}{16(2i+3)(2i-1)(2j+3)(2j-1)} \int_0^l \int_0^\tau \frac{\partial^4 u(x,t)}{\partial t^2 \partial x^2} Q_{l,i}(x) Q_{\tau,j}(t) dt dx,
\end{aligned}$$

thus

$$\begin{aligned}
|a_{i,j}| &\leq \frac{l\tau}{16(2i+3)(2i-1)(2j+3)(2j-1)} \int_0^l \int_0^\tau \left| \frac{\partial^4 u(x,t)}{\partial t^2 \partial x^2} \right| |Q_{l,i}(x)| |Q_{\tau,j}(t)| dt dx \\
&\leq \frac{l\tau\alpha}{16(2i+3)(2i-1)(2j+3)(2j-1)} \int_0^l |Q_{l,i}(x)| dx \int_0^\tau |Q_{\tau,j}(t)| dt.
\end{aligned}$$

Also we have

$$\begin{aligned}
\left( \int_0^l |Q_{l,i}(x)| dx \right)^2 &= \left( \int_0^l |(2i-1)L_{l,i+2}(x) - 2(2i+1)L_{l,i}(x) + (2i+3)L_{l,i-2}(x)| dx \right)^2 \\
&\leq \left( \int_0^l (1)^2 dx \right) \left( \int_0^l (2i-1)^2 L_{l,i+2}(x)^2 + (4i+2)^2 L_{l,i}(x)^2 + (2i+3)^2 L_{l,i-2}(x)^2 \right) dx \\
&\leq l \left( \frac{(2i-1)^2 l}{2i+5} + \frac{(4i+2)^2 l}{2i+1} + \frac{(2i+3)^2 l}{2i-3} \right) \\
&\leq \frac{6l^2(2i+3)^2}{2i-3}.
\end{aligned}$$

Then we get

$$\int_0^l |Q_{l,i}(x)| dx \leq \frac{\sqrt{6l}(2i+3)}{\sqrt{2i-3}}.$$

Thus we obtain

$$\begin{aligned} |a_{i,j}| &\leq \frac{l\tau\alpha}{16(2i+3)(2i-1)(2j+3)(2j-1)} \times \frac{\sqrt{6l}(2i+3)}{\sqrt{2i-3}} \times \frac{\sqrt{6\tau}(2j+3)}{\sqrt{2j-3}} \\ &\leq \frac{3l^2\tau^2\alpha}{8\sqrt{(2i-3)^3}\sqrt{(2j-3)^3}}. \end{aligned}$$

Consequently,  $\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{i,j}$  is absolute convergent and thus the expansion of the function converges uniformly.  $\square$

**Theorem 5.2.** *Let  $u(x, t)$  be a continuous function defined on  $[0, l] \times [0, \tau]$  with bounded mixed fourth partial derivative, say  $\left| \frac{\partial^4 u(x, t)}{\partial x^2 \partial t^2} \right| \leq \alpha$ , then we have the following accuracy estimation*

$$\begin{aligned} \varepsilon_n &= \left( \int_0^\tau \int_0^l (u(x, t) - \sum_{i=0}^n \sum_{j=0}^m a_{i,j} L_{l,i}(x) L_{\tau,j}(t))^2 dx dt \right)^{1/2} \\ &\leq \frac{3\alpha l^2 \tau^2 \sqrt{l\tau}}{8} \sqrt{\sum_{i=n+1}^{\infty} \frac{1}{(2i-3)^4} \sum_{j=m+1}^{\infty} \frac{1}{(2j-3)^4}}. \end{aligned} \quad (5.1)$$

Also in the case of  $n = m$  the error bound is  $\varepsilon_n \leq \frac{3\alpha l^2 \tau^2 \sqrt{l\tau}}{8} \sum_{i=n+1}^{\infty} \frac{1}{(2i-3)^4}$ .

*Proof.*

$$\begin{aligned}
\varepsilon_n^2 &= \int_0^\tau \int_0^l \left( u(x, t) - \sum_{i=0}^n \sum_{j=0}^m a_{i,j} L_{l,i}(x) L_{\tau,j}(t) \right)^2 dx dt \\
&= \int_0^\tau \int_0^l \left( \sum_{i=0}^\infty \sum_{j=0}^\infty a_{i,j} L_{l,i}(x) L_{\tau,j}(t) - \sum_{i=0}^n \sum_{j=0}^m a_{i,j} L_{l,i}(x) L_{\tau,j}(t) \right)^2 dx dt \\
&= \int_0^\tau \int_0^l \left( \sum_{i=n+1}^\infty \sum_{j=m+1}^\infty a_{i,j} L_{l,i}(x) L_{\tau,j}(t) \right)^2 dx dt \\
&= \int_0^\tau \int_0^l \sum_{i=n+1}^\infty \sum_{j=m+1}^\infty a_{i,j}^2 L_{l,i}^2(x) L_{\tau,j}^2(t) dx dt \\
&= \sum_{i=n+1}^\infty \sum_{j=m+1}^\infty a_{i,j}^2 \int_0^l L_{l,i}^2(x) dx \int_0^\tau L_{\tau,j}^2(t) dt \\
&= \sum_{i=n+1}^\infty \sum_{j=m+1}^\infty a_{i,j}^2 \frac{l\tau}{(2i+1)(2j+1)} \\
&\leq \sum_{i=n+1}^\infty \sum_{j=m+1}^\infty \frac{9\alpha^2 l^5 \tau^5}{64(2i-3)^3(2j-3)^3(2i+1)(2j+1)} \\
&\leq \frac{9\alpha^2 l^5 \tau^5}{64} \sum_{i=n+1}^\infty \sum_{j=m+1}^\infty \frac{1}{(2i-3)^4(2j-3)^4} \\
&= \frac{9\alpha^2 l^5 \tau^5}{64} \sum_{i=n+1}^\infty \frac{1}{(2i-3)^4} \sum_{j=m+1}^\infty \frac{1}{(2j-3)^4}.
\end{aligned}$$

Then we have

$$\varepsilon_n \leq \frac{3\alpha l^2 \tau^2 \sqrt{l\tau}}{8} \sqrt{\sum_{i=n+1}^\infty \frac{1}{(2i-3)^4} \sum_{j=m+1}^\infty \frac{1}{(2j-3)^4}}, \quad (5.2)$$

which in the case of  $n = m$ ,  $\varepsilon_n \leq \frac{3\alpha l^2 \tau^2 \sqrt{l\tau}}{8} \sum_{i=n+1}^\infty \frac{1}{(2i-3)^4}$   $\square$

## 6. ERROR ESTIMATION

In this section, we state the estimate error for Legendre spectral tau method. Firstly, we define

$$e_{n,m}(x, t) = u(x, t) - u_{n,m}(x, t), \quad (6.1)$$

the error function of the Legendre tau approximation  $u_{n,m}(x, t)$  to  $u(x, t)$ , where  $u(x, t)$  is the exact solution of (1.1). Therefore by using equations (6.1)

and (1.1) we have,

$$(e_{n,m})_t(x,t) + a(e_{n,m})_x(x,t) - b(e_{n,m})_{xx}(x,t) = \int_0^t k(t-s)e_{n,m}(x,s)ds + H_{n,m}(x,t),$$

$H_{n,m}(x,t)$  is a perturbation term associated with  $u_{n,m}(x,t)$  and can be obtained with following formulae

$$H_{n,m}(x,t) = \int_0^t k(t-s)u_{n,m}(x,s)ds + f(x,t) - (u_{n,m})_t(x,t) - a(u_{n,m})_x(x,t) + b(u_{n,m})_{xx}(x,t),$$

and the boundary conditions

$$\begin{aligned} e_{n,m}(x,0) &= u(x,0) - u_{n,m}(x,0) \\ &= g_0(x) - u_{n,m}(x,0), \end{aligned}$$

$$\begin{aligned} e_{n,m}(0,t) &= u(0,t) - u_{n,m}(0,t) \\ &= f_0(t) - u_{n,m}(0,t), \end{aligned}$$

$$\begin{aligned} e_{n,m}(l,t) &= u(l,t) - u_{n,m}(l,t) \\ &= f_1(t) - u_{n,m}(l,t). \end{aligned}$$

We proceed to find an approximation  $(e_{n,m})_{n_1,m_1}(x,t)$  to the  $e_{n,m}(x,t)$  in the same as we did for the solutions of equations (1.1), (1.2) and (1.3).  $((n_1, m_1)$  denotes the Tau degree of  $e_{n,m}(x,t)$ ).

## 7. NUMERICAL RESULTS AND COMPARISONS

In this section, we present four numerical examples to demonstrate the accuracy of the proposed method. The results show that this method, by selecting a few number of shifted Legendre polynomials is accurate. Let  $t_n = nk$ ,  $n = 0, 1, 2, \dots, M$ ,  $k = \frac{T}{M}$ ,  $x_i = ih$ ,  $i = 0, 1, 2, \dots, N$ ,  $h = \frac{L}{N}$  where  $M$ ,  $N$  respectively denotes the final time level  $t_M$  and the final space level  $x_N$ ,  $N+1$  is the number of nodes. In order to check the accuracy of the proposed method, the maximum absolute errors and  $L_2$  norm errors between the exact solution  $u(x,t)$  and the approximate solution  $u_{n,m}(x,t)$  are given by the following definitions.

**Maximum norm error:**  $\| e_M \|_\infty = \max | u(x_i, t_M) - u_{n,m}(x_i, t_M) |$ .

**$L_2$  norm error:**  $\frac{1}{N} \left( \sum_{i=0}^N | u(x_i, t_M) - u_{n,m}(x_i, t_M) |^2 \right)^{1/2}$ .

Some important non-dimensional parameters in numerical analysis are defined as follows

Courant number :  $C_r = a \frac{k}{h}$ ,

Diffusion number:  $S = b \frac{k}{h^2},$

Grid Peclet number:  $P_e = \frac{C_r}{S} = \frac{a}{b}h.$

When the Peclet number is high, the convection term dominates and when the Peclet number is low, the diffusion term dominates.

EXAMPLE 7.1. As a first application, we offer the following convection-diffusion integro-differential equation

$$u_t(x, t) + au_x(x, t) - bu_{xx}(x, t) = \int_0^t \frac{u(x, s)}{\sqrt[3]{t-s}} ds + f(x, t), \quad x \in [0, 1], \quad t > 0,$$

with  $a = 0.005$ ,  $b = 0.5$  and the following initial condition

$$u(x, 0) = 1 - \cos 2\pi x + 2\pi^2 x(1 - x), \quad 0 \leq x \leq 1,$$

and boundary conditions

$$u(0, t) = u(1, t) = 0, \quad 0 \leq t \leq 1,$$

$$\begin{aligned} f(x, t) = & 2(1+t)(1+2\pi^2(1-x)x - \cos 2\pi x) + \\ & \frac{3}{40}t^{2/3}(20+3t(8+3t))(-1+2\pi^2(-1+x)x + \cos 2\pi x) \\ & - 0.5(1+t)^2(-4\pi^2+4\pi^2\cos 2\pi x) + \\ & 0.005(1+t)^2(2\pi^2(1-x) - 2\pi^2x + 2\pi \sin 2\pi x). \end{aligned}$$

The exact solution of the problem is

$$u(x, t) = (t+1)^2(1 - \cos 2\pi x + 2\pi^2 x(1 - x)).$$

The maximum absolute errors and  $L_2$  norm errors between the exact solution  $u(x, t)$  and the approximate solution  $u_{n,m}(x, t)$  with various choices of  $(n = m)$  and two different grid sizes  $N = 100$ ,  $M = 10$  and  $N = 50$ ,  $M = 100$  are presented in table 7.1. In this problem for  $h = 0.01$ ,  $p_e = 0.0001$  and for  $h = 0.02$ ,  $p_e = 0.0002$  which shows that the diffusion term dominates. We see in this table that the results are accurate for even small choices of  $n, m$ . Fig 1 shows the error functions  $u(0.9, t) - u_{11,11}(0.9, t)$  and  $u(x, 0.7) - u_{11,11}(x, 0.7)$ . The maximal errors for different  $n, m$  and the grid size  $N = 100$ ,  $M = 50$  are shown in Fig 2, which can be used to show the convergence behavior of the method. In [24] the authors applied the cubic B-spline collection method to introduce an approximation solution of this problem, the best result is achieved at  $N = 100$ ,  $M = 10$  and  $k = 0.00001$  which the  $L_2$  norm error is  $4.54 \times 10^{-8}$ . Regarding table 7.1, we observe that the approximation solution by legendre spectral method is more better than those obtained in [24]. The  $L_2$  norm error in  $N = 100$ ,  $M = 10$  and  $k = 0.00001$  by our method is  $4.09 \times 10^{-10}$ .

TABLE 1.  $\|e_M\|_\infty$  is the Maximum norm error and  $\|e_M\|_2$  is  $L_2$  norm error .

$n = m$	$M = 10$	$N = 100$	$M = 100$	$N = 50$
	$\ e_M\ _\infty$	$\ e_M\ _2$	$\ e_M\ _\infty$	$\ e_M\ _2$
7	$3.29 \times 10^{-3}$	$9.57 \times 10^{-5}$	$2.33 \times 10^{-2}$	$1.85 \times 10^{-3}$
9	$1.02 \times 10^{-4}$	$2.85 \times 10^{-6}$	$6.34 \times 10^{-4}$	$5.04 \times 10^{-5}$
11	$2.08 \times 10^{-6}$	$5.72 \times 10^{-8}$	$1.19 \times 10^{-5}$	$9.41 \times 10^{-7}$
13	$3.01 \times 10^{-8}$	$1.02 \times 10^{-9}$	$3.18 \times 10^{-7}$	$1.87 \times 10^{-8}$
15	$1.00 \times 10^{-8}$	$4.09 \times 10^{-10}$	$2.07 \times 10^{-7}$	$1.79 \times 10^{-8}$

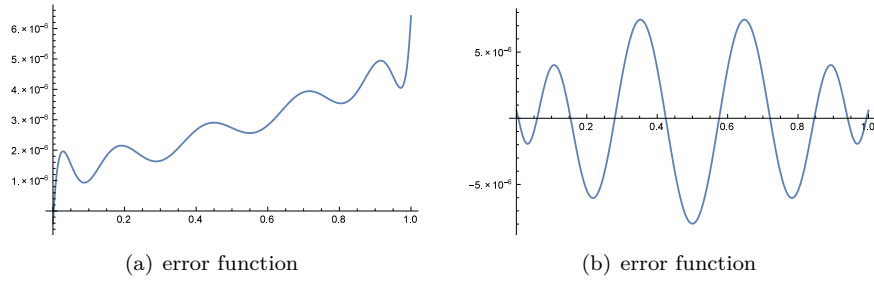


FIGURE 1. Errors at  $x = 0.9$  (left) and  $t = 0.7$  (right) with  $m=n=11$  for Example 7.1.

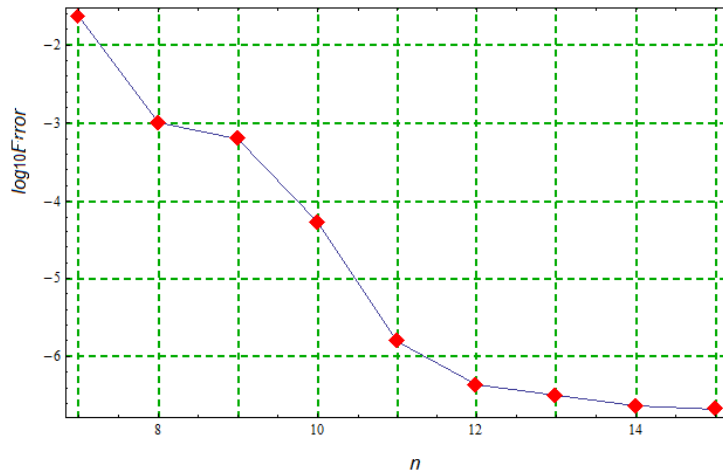


FIGURE 2. An illustration of the rate of convergence for the Legendre spectral tau method with various  $n, m$  of Example 7.1.



EXAMPLE 7.2. Consider the following convection-diffusion integro-differential equation

$$u_t(x, t) + au_x(x, t) - bu_{xx}(x, t) = \int_0^t \frac{u(x, s)}{\sqrt[4]{t-s}} ds + f(x, t), \quad x \in [0, 1], \quad t > 0,$$

with  $a = 0.5$ ,  $b = 0.001$  and the following initial condition

$$u(x, 0) = 2\sin^2 \pi x, \quad 0 \leq x \leq 1,$$

and boundary conditions

$$u(0, t) = u(1, t) = 0, \quad 0 \leq t \leq 1,$$

$$\begin{aligned} f(x, t) = & 6.28319(1 + t + t^2)\cos \pi x \sin \pi x + 2(1 + 2t)\sin^2 \pi x - \\ & \frac{8}{231}t^{3/4}(77 + 4t(11 + 8t))\sin \pi x^2 \\ & - 0.002(1 + t + t^2)(2\pi)^2\cos \pi x^2 - 2(\pi)^2\sin^2 \pi x. \end{aligned}$$

The exact solution of this problem is

$$u(x, t) = 2(t^2 + t + 1)\sin^2 \pi x.$$

In this example, we implement the Legendre spectral tau method to solve the convection-diffusion integro-differential equation and the results of this example which show the maximum absolute error for different choices of  $n$ ,  $m$  are shown in Table 7.2. Also The graph of the maximum absolute error function is shown in Fig 3.

TABLE 2. Maximal absolute error ( $|u(x, 0) - u_{n,m}(x, 0)|$ ) for different choices of  $n, m$ .

$x$	$m=n=9$	$m=n=11$	$m=n=13$
0	$1.12 \times 10^{-4}$	$2.39 \times 10^{-6}$	$1.87 \times 10^{-8}$
0.1	$2.11 \times 10^{-5}$	$6.90 \times 10^{-7}$	$4.66 \times 10^{-10}$
0.2	$5.20 \times 10^{-6}$	$5.68 \times 10^{-7}$	$1.31 \times 10^{-9}$
0.3	$2.12 \times 10^{-5}$	$5.88 \times 10^{-7}$	$2.96 \times 10^{-9}$
0.4	$2.77 \times 10^{-5}$	$4.17 \times 10^{-7}$	$4.14 \times 10^{-9}$
0.5	$1.18 \times 10^{-14}$	$4.25 \times 10^{-14}$	$9.43 \times 10^{-13}$
0.6	$2.77 \times 10^{-5}$	$4.17 \times 10^{-7}$	$1.07 \times 10^{-9}$
0.7	$2.12 \times 10^{-5}$	$5.88 \times 10^{-7}$	$7.89 \times 10^{-9}$
0.8	$5.20 \times 10^{-6}$	$5.68 \times 10^{-7}$	$1.79 \times 10^{-9}$
0.9	$2.11 \times 10^{-5}$	$6.90 \times 10^{-7}$	$1.5 \times 10^{-10}$
1	$1.12 \times 10^{-4}$	$2.39 \times 10^{-6}$	$5.45 \times 10^{-8}$

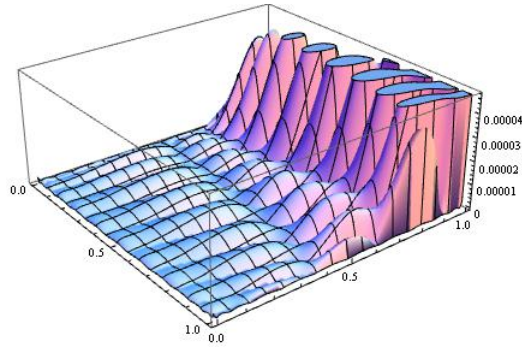


FIGURE 3. Error function ( $|u(x, t) - u_{n,m}(x, t)|$ ) for the example 7.2 , when  $m = n = 11$ .

In table 7.3 we make a comparison of the presented method with the cubic B-spline method proposed in [24]. Obviously, our method is more accurate than cubic B-spline method in [24].

TABLE 3. The comparison between proposed method in [24] and the Legendre spectral method ( $m = n = 15$ ) .

$N = 100$	B-spline method [24]	Legendre spectral tau method
$M$	$\  e_M \ _2$	$\  e_M \ _2$
10	$3.94 \times 10^{-10}$	$2.51 \times 10^{-10}$
50	$1.97 \times 10^{-9}$	$3.52 \times 10^{-10}$
100	$3.94 \times 10^{-9}$	$4.90 \times 10^{-10}$
500	$1.97 \times 10^{-8}$	$1.21 \times 10^{-9}$

EXAMPLE 7.3. Consider the following convection-diffusion integro-differential equation [24]

$$u_t(x, t) + 0.05 u_x(x, t) - 0.4 u_{xx}(x, t) = \int_0^t \frac{u(x, s)}{\sqrt{t-s}} ds + f(x, t), \quad x \in [0, 1], \quad t > 0,$$

with initial condition

$$u(x, 0) = \sin \pi x, \quad 0 \leq x \leq 1,$$

and boundary conditions

$$u(0, t) = u(1, t) = 0, \quad 0 \leq t \leq 1,$$

$$f(x, t) = 0.15708(1+t)^2 \cos \pi x + 2(1+t) \sin \pi x + 3.94784(t+1)^2 \sin \pi x - \frac{2}{15} \sqrt{t}(15+4t(5+2t)) \sin \pi x.$$

The exact solution of this initial value problem is

$$u(x, t) = (t + 1)^2 \sin \pi x.$$

Table 7.4 displays the  $L_2$  norm errors using cubic B-spline method [24] and Legendre spectral tau method for  $N = 100$ ,  $k = 0.0001$  at different time levels  $M$ . The results show that the approximate solution by presented method is more better than those obtained in [24].

TABLE 4. The comparison between proposed method in [24] and the Legendre spectral method ( $m = n = 13$ ).

$N = 100$	B-spline method [24]	Legendre spectral tau method
$M$	$\ e_M\ _2$	$\ e_M\ _2$
10	$1.32 \times 10^{-7}$	$4.20 \times 10^{-8}$
50	$3.31 \times 10^{-7}$	$1.40 \times 10^{-8}$
100	$5.73 \times 10^{-7}$	$7.38 \times 10^{-9}$
500	$2.43 \times 10^{-6}$	$9.08 \times 10^{-9}$

EXAMPLE 7.4. Consider the following convection-diffusion integro-differential equation [24]

$$u_t(x, t) + 0.5 u_x(x, t) - 0.005 u_{xx}(x, t) = \int_0^t \frac{u(x, s)}{\sqrt[3]{t-s}} ds + f(x, t), \quad x \in [0, 1], \quad t > 0,$$

with initial condition

$$u(x, 0) = \cos \pi x, \quad 0 \leq x \leq 1,$$

and boundary conditions

$$u(0, t) = (t + 1), \quad u(1, t) = -(t + 1), \quad 0 \leq t \leq 1,$$

$$f(x, t) = \cos \pi x + 0.04934(1 + t) \cos \pi x - \frac{3}{10} t^{\frac{2}{3}} (5 + 3t) \cos \pi x - 1.57079(t + 1) \sin \pi x.$$

The exact solution of this initial value problem is

$$u(x, t) = (t + 1) \cos \pi x.$$

In [24] the domain  $[0, l] \times [0, \tau]$  is divided into  $N \times M$  mesh with the spatial step size  $h = \frac{l}{N}$ ,  $k = \frac{\tau}{M}$ , respectively. For the purpose of comparison in table 7.5, we compare the  $L_2$  norm error of our method at  $m = n = 11$ , with the method in [24].

EXAMPLE 7.5. Consider the following convection-diffusion integro-differential equation

$$u_t(x, t) + u_x(x, t) - u_{xx}(x, t) = \int_0^t \frac{u(x, s)}{\sqrt{t-s}} ds + f(x, t), \quad x \in [0, 1], \quad t > 0,$$

TABLE 5. The comparison between proposed method in [24] and the Legendre spectral method ( $m = n = 11$ ).

$N = 100$	B-spline method [24]	Legendre spectral tau method
$M$	$\ e_M\ _2$	$\ e_M\ _2$
10	$1.17 \times 10^{-6}$	$2.37 \times 10^{-10}$
50	$2.01 \times 10^{-5}$	$2.46 \times 10^{-10}$
100	$7.26 \times 10^{-5}$	$2.93 \times 10^{-10}$
500	$1.50 \times 10^{-3}$	$5.12 \times 10^{-10}$

with initial condition

$$u(x, 0) = \sin \pi x, \quad 0 \leq x \leq 1,$$

and boundary conditions

$$u(0, t) = u(1, t) = 0, \quad 0 \leq t \leq 1,$$

$$f(x, t) = e^t \pi \cos \pi x + e^t \sin \pi x + e^t \pi^2 \sin \pi x - e^t \sqrt{\pi} \operatorname{erf}(\sqrt{t}) \sin \pi x.$$

The exact solution of this initial value problem is

$$u(x, t) = \sin \pi x e^t.$$

We have solved this problem for  $m = n = 9$  and compute  $u(x, t) - u_{9,9}(x, t)$  for different  $t, x$ . See table 7.6.

The graphs of the absolute error are shown in Fig 7.4.

TABLE 6. Error function  $u(x, t) - u_{9,9}(x, t)$  for different  $t$  and  $x$  of example 7.5.

$m = n = 9$					
$x$	$t = 0.1$	$t = 0.2$	$t = 0.3$	$t = 0.4$	$t = 0.5$
0	$3.96 \times 10^{-8}$	$-3.22 \times 10^{-8}$	$1.27 \times 10^{-8}$	$1.72 \times 10^{-8}$	$-3.27 \times 10^{-8}$
0.25	$-5.18 \times 10^{-8}$	$-1.34 \times 10^{-7}$	$1.68 \times 10^{-7}$	$-2.04 \times 10^{-8}$	$-2.96 \times 10^{-7}$
0.5	$-3.68 \times 10^{-8}$	$-1.30 \times 10^{-7}$	$2.16 \times 10^{-7}$	$1.78 \times 10^{-8}$	$-2.91 \times 10^{-7}$
0.75	$-5.73 \times 10^{-9}$	$-9.54 \times 10^{-8}$	$3.32 \times 10^{-7}$	$1.19 \times 10^{-7}$	$-2.41 \times 10^{-7}$
1	$-8.89 \times 10^{-8}$	$-2.47 \times 10^{-7}$	$4.79 \times 10^{-7}$	$6.64 \times 10^{-8}$	$-5.48 \times 10^{-7}$

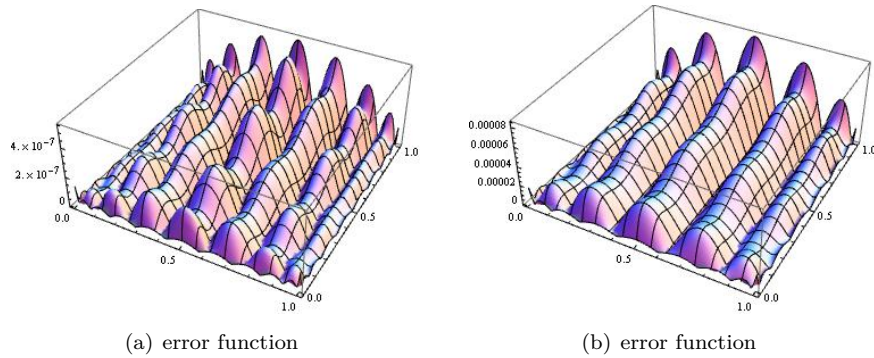


FIGURE 4. Error function for  $m = n = 9$  (left) and  $m = n = 7$  (right) of Example 7.5

EXAMPLE 7.6. Consider the following convection-diffusion integro-differential equation

$$u_t(x, t) + u_x(x, t) - u_{xx}(x, t) = \int_0^t \frac{u(x, s)}{\sqrt{t-s}} ds + f(x, t), \quad x \in [0, 1], \quad t > 0,$$

with initial condition

$$u(x, 0) = e^x, \quad 0 \leq x \leq 1,$$

and boundary conditions

$$u(0, t) = e^t, \quad u(1, t) = e^{t+1}, \quad 0 \leq t \leq 1,$$

$$f(x, t) = e^{t+x} - e^{t+x} \sqrt{\pi} \operatorname{erf}(\sqrt{t}).$$

The exact solution of this initial value problem is

$$u(x, t) = e^{t+x}.$$

The maximum norm error and  $L_2$  norm error of this example for two grid sizes  $M = 50$ ,  $N = 100$  and  $M = 100$ ,  $N = 50$  are presented in Table 7.  $p_e = 0.01$ ,  $p_e = 0.02$  for  $h = 0.01$  and  $h = 0.02$ , respectively. Here, Peclet number  $p_e$  is low, which indicates that the diffusion term dominates. Also the comparison between the Legendre spectral tau error ( $u(x, t) - u_{n,m}(x, t)$ ) and the estimate error that stated in section 5 is shown in Table 7.8. We see in this tables that the results are accurate for even small choices of  $n$ ,  $m$ .

In addition, to demonstrate the convergence of the proposed method, in Fig 5, we plot the logarithmic graph of maximum absolute error ( $\log_{10} \text{Error}$ ) with various values of  $n$  ( $n = m$ ).

TABLE 7.  $\| e_M \|_\infty$  is the Maximum norm error and  $\| e_M \|_2$  is  $L_2$  norm error.

	$M = 50$	$N = 100$	$M = 100$	$N = 50$
$n = m$	$\  e_M \ _\infty$	$\  e_M \ _2$	$\  e_M \ _\infty$	$\  e_M \ _2$
5	$5.35 \times 10^{-2}$	$2.11 \times 10^{-3}$	$4.51 \times 10^{-2}$	$3.67 \times 10^{-3}$
7	$2.57 \times 10^{-3}$	$1.13 \times 10^{-5}$	$2.77 \times 10^{-3}$	$2.17 \times 10^{-4}$
9	$6.84 \times 10^{-5}$	$3.79 \times 10^{-6}$	$1 \times 10^{-4}$	$7.81 \times 10^{-6}$
11	$1.58 \times 10^{-6}$	$8.65 \times 10^{-8}$	$2.32 \times 10^{-6}$	$1.83 \times 10^{-7}$
13	$2.60 \times 10^{-8}$	$1.41 \times 10^{-9}$	$3.71 \times 10^{-8}$	$2.96 \times 10^{-9}$

TABLE 8. "Error 1" is the Legendre spectral Tau error and "Error 2" is the estimate error that stated in section 5.

$x$	$m = n = 5, t = 0$		$m = n = 7, t = 0$	
	Error 1	Error 2	Error 1	Error 2
0	$2.40 \times 10^{-6}$	$2.40 \times 10^{-6}$	$2.07 \times 10^{-8}$	$3.55 \times 10^{-8}$
0.1	$9.56 \times 10^{-7}$	$9.56 \times 10^{-7}$	$4.57 \times 10^{-9}$	$7.93 \times 10^{-9}$
0.2	$3.99 \times 10^{-7}$	$3.99 \times 10^{-7}$	$6.07 \times 10^{-9}$	$1.10 \times 10^{-8}$
0.3	$7.31 \times 10^{-7}$	$7.31 \times 10^{-7}$	$5.22 \times 10^{-10}$	$1.13 \times 10^{-9}$
0.4	$1.73 \times 10^{-7}$	$1.73 \times 10^{-7}$	$4.91 \times 10^{-9}$	$9.52 \times 10^{-9}$
0.5	$7.80 \times 10^{-7}$	$7.80 \times 10^{-7}$	$2.94 \times 10^{-10}$	$7.05 \times 10^{-10}$
0.6	$2.29 \times 10^{-7}$	$2.29 \times 10^{-7}$	$4.36 \times 10^{-9}$	$9.18 \times 10^{-9}$
0.7	$7.29 \times 10^{-7}$	$7.29 \times 10^{-7}$	$9.93 \times 10^{-11}$	$2.61 \times 10^{-10}$
0.8	$4.61 \times 10^{-7}$	$4.61 \times 10^{-7}$	$4.08 \times 10^{-9}$	$9.50 \times 10^{-9}$
0.9	$1.00 \times 10^{-6}$	$1.00 \times 10^{-6}$	$3.00 \times 10^{-9}$	$2.40 \times 10^{-9}$
1	$2.59 \times 10^{-6}$	$2.59 \times 10^{-6}$	$1.05 \times 10^{-8}$	$2.80 \times 10^{-8}$

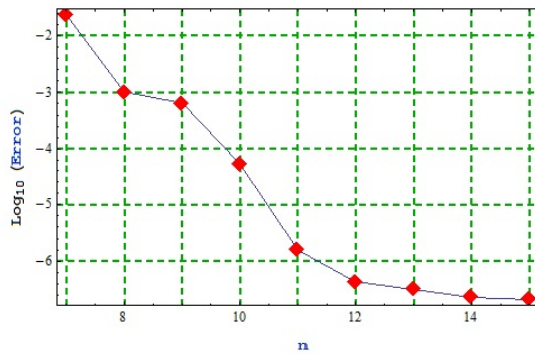


FIGURE 5. An illustration of the rate of convergence for the Legendre spectral tau method with various  $n, m$  of Example 7. 6.

## 8. CONCLUSION

In this research, the convection-diffusion integro-differential equation with a weakly singular kernel was solved based on the Legendre spectral tau method in conjunction with the operational matrices of partial derivatives and integral parts. The most important section of our method is converting the problem to a linear system of algebraic equations. The performance of the proposed method for the considered problems was measured by calculating the maximum norm error and  $L_2$  norm error. Also, we are able to demonstrate the errors of the tau approximations decay exponentially, which is a desired feature for a spectral tau method. The proposed method is also valid and efficient for different values of  $\alpha$ ,  $0 < \alpha < 1$ . This results confirmed by some numerical experiments.

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