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Some Convergence Theorems of the PUL-Stieltjes Integral

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Abstract. The PUL integral is an integration process which uses the notion of partition of unity [3]. The definition of this integral is similar to the Gauge integral, which was defined by Kurzweil and Henstock. Also, it is equivalent to the Lebesgue integral in Euclidean n-dimensional Spaces. Boonpogkrong [1] discussed the Kurzweil-Henstock integral on manifolds. The PUL-Stieltjes integral, established by Flores and Benitez [2], is a generalization of the PUL Integral. In this paper, we present some Convergence Theorems for the PUL-Stieltjes integral. Notions on the equi-integrability of this integral is also presented in the paper.

Keywords: Partition of unity, Convergence Theorems, Equi-integrability

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1. Introduction

Boonpogkrong [1] defined the Kurzweil-Henstock integral on manifolds using the definition of the PUL integral which was defined by J. Kurzwel and J. Jarnik. The PUL integral is a Henstock type of definition which utilizes the

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notion of the partition of unity. In this integration process, a division of an interval may contain some overlapping subintervals. In [2], Flores, et.al. extended this concept and defined the PUL-Stieltjes integral with values defined on a Banach space. In the said paper, they presented some simple properties and existence theorem for this integral.

In this paper, we exhibit some convergence theorems, namely, the Uniform Convergence Theorem, and the Equi-integrability Theorem.

2. PUL-Stieltjes Integral in Banach Space

Denote a compact interval in $\mathbb{R}^n$ by $[\boldsymbol{a},\boldsymbol{b}] = \prod_{k=1}^{n}[a_k,b_k]$ with $[a_k,b_k] \subseteq \mathbb{R}$ for each $k = 1,2,\ldots,n$ and $\mu([\boldsymbol{a},\boldsymbol{b}]) = \prod_{k=1}^{n}(b_k - a_k)$ be the volume of $[\boldsymbol{a},\boldsymbol{b}]$. Also, we denote $\overline{A}$ as the closure of the set $A \subseteq \mathbb{R}^n$. For a smooth function $\psi : [\boldsymbol{a},\boldsymbol{b}] \to \mathbb{R}$, the *support* of ψ, denoted by supp ψ, is given by

$$\text{supp } \psi = \overline{\{\boldsymbol{x} \in [\boldsymbol{a},\boldsymbol{b}] : \psi(\boldsymbol{x}) \neq 0\}}.$$

A *gauge* on $[\boldsymbol{a},\boldsymbol{b}]$ is a positive function defined on $[\boldsymbol{a},\boldsymbol{b}]$.

Definition 2.1. [1] A finite collection $\{\psi_k\}_{k=1}^m$ of smooth functions defined on $[\boldsymbol{a},\boldsymbol{b}]$ is said to be a *partial partition of unity* if the following holds:

1. $\psi_k(\boldsymbol{\xi}) \geq 0$ for all $\boldsymbol{\xi} \in [\boldsymbol{a},\boldsymbol{b}]$ and for all $k \in \{1,2,\cdots,m\}$ and
2. $\sum_{k=1}^{m}\psi_k(\boldsymbol{\xi}) \leq 1$ for all $\boldsymbol{\xi} \in [\boldsymbol{a},\boldsymbol{b}]$.

If $\sum_{k=1}^{m}\psi_k(\boldsymbol{\xi}) = 1$ for all $\boldsymbol{\xi} \in [\boldsymbol{a},\boldsymbol{b}]$, then $\{\psi_k\}_{k=1}^m$ is said to be a *partition of unity.*

Definition 2.2. [1] Let $\psi : [\boldsymbol{a},\boldsymbol{b}] \to \mathbb{R}$ be a smooth function and δ a gauge on $[\boldsymbol{a},\boldsymbol{b}]$. A triple $(\boldsymbol{\xi},\boldsymbol{I},\varphi)$, with $\boldsymbol{\xi} \in [\boldsymbol{a},\boldsymbol{b}]$ and $\boldsymbol{I} \subseteq [\boldsymbol{a},\boldsymbol{b}]$, is said to be *$\delta$-fine* if

$$\text{supp } \psi \subseteq \boldsymbol{I} \subseteq B(\boldsymbol{\xi},\delta(\boldsymbol{\xi})).$$

Note that $\boldsymbol{\xi}$ may not be in supp ψ and $\boldsymbol{I}$. If δ_1 and δ_2 are gauges on $[\boldsymbol{a},\boldsymbol{b}]$ such that $\delta_1(\boldsymbol{\xi}) \geq \delta_2(\boldsymbol{\xi})$ and $(\boldsymbol{\xi},\boldsymbol{I},\varphi)$ is δ_2-fine, then $(\boldsymbol{\xi},\boldsymbol{I},\varphi)$ is also δ_1-fine.

A division $D = \{(\boldsymbol{I}_k,\psi_k)\}_{k=1}^m$ of $[\boldsymbol{a},\boldsymbol{b}]$ is a collection of subintervals $\{\boldsymbol{I}_k\}_{k=1}^m$ of $[\boldsymbol{a},\boldsymbol{b}]$ and a partition of unity $\{\psi_k\}_{k=1}^m$ on $[\boldsymbol{a},\boldsymbol{b}]$ such that for each $k \in \{1,2,\cdots,m\}$, supp $\psi_k \subseteq \boldsymbol{I}_k$. In this case, $\boldsymbol{I}_k$'s may be overlapping.

Definition 2.3. [1] A finite collection $D = \{(\boldsymbol{\xi}_k,\boldsymbol{I}_k,\psi_k)\}_{k=1}^m$ is said to be a *δ-fine partial division* of $[\boldsymbol{a},\boldsymbol{b}]$ if the collection $\{\psi_k\}_{k=1}^m$ is a partial partition of unity and every $(\boldsymbol{\xi}_k, \boldsymbol{I}_k, \psi_k)$ is δ-fine. If $\{\psi_k\}_{k=1}^m$ is a partition of unity, then D is said to be a δ-*fine division* of $[\boldsymbol{a}, \boldsymbol{b}]$.

The existence of δ-fine divisions of $[\boldsymbol{a}, \boldsymbol{b}]$ is guaranteed by the open covering theorem and the existence of a partition of unity, see [1].

Let $D = \{(\boldsymbol{\xi}_k, \boldsymbol{I}_k, \psi_k)\}_{k=1}^m$ be a δ-fine division of $[\boldsymbol{a}, \boldsymbol{b}]$, and $g : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}$ be a function. Suppose that for each $k \in \{1, 2, \cdots, m\}$, the Riemann-Stieltjes integral $\int_{\boldsymbol{I}_k} \psi_k \, dg$ exists. Define the *PUL-Stieltjes sum of f with respect to g over D* by

$$S(f, g, D) = \sum_{k=1}^{m} f(\boldsymbol{\xi}_k) \int_{\boldsymbol{I}_k} \psi_k(\boldsymbol{x}) \; dg(\boldsymbol{x}) = \sum_{k=1}^{m} f(\boldsymbol{\xi}_k) \int_{\boldsymbol{I}_k} \psi_k \; dg.$$

For brevity, we write a δ-fine division of $[\boldsymbol{a}, \boldsymbol{b}]$ by $D = \{(\boldsymbol{\xi}, \boldsymbol{I}, \psi)\}$ and a PUL-Stieltjes sum of f with respect to g over D by

$$S(f, g, D) = (D) \sum f(\boldsymbol{\xi}) \int_{\boldsymbol{I}} \psi \; dg = \sum_{D} f(\boldsymbol{\xi}) \int_{\boldsymbol{I}} \psi \; dg.$$

Definition 2.4. [2] Let $(X, \|\cdot\|_X)$ be a Banach space. We say that a function $f : [\boldsymbol{a}, \boldsymbol{b}] \to X$ is said to be *PUL-Stieltjes integrable with respect to* $g : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}$ *on* $[\boldsymbol{a}, \boldsymbol{b}]$ if there exists $A \in X$ such that for every $\epsilon > 0$, there exists a gauge δ on $[\boldsymbol{a}, \boldsymbol{b}]$ such that for every δ-fine division $D = \{(\boldsymbol{\xi}_k, \boldsymbol{I}_k, \psi_k)\}_{k=1}^m$ of $[\boldsymbol{a}, \boldsymbol{b}]$, we have

$$\|S(f, g, D) - A\|_X < \epsilon.$$

If A is the PUL-Stieltjes integral of f with respect to g on $[\boldsymbol{a}, \boldsymbol{b}]$, then we write

$$A = (\mathcal{P}) \int_{[\boldsymbol{a}, \boldsymbol{b}]} f \; dg.$$

In this paper, we fix $X = \mathbb{R}$. A real-valued function f defined on a compact interval $[\boldsymbol{a}, \boldsymbol{b}] \subseteq \mathbb{R}^n$ is said to be *bounded* on $[\boldsymbol{a}, \boldsymbol{b}]$ if there exists $M > 0$ such that $|f(\boldsymbol{x})| \leq M$ for all $\boldsymbol{x} \in [\boldsymbol{a}, \boldsymbol{b}]$. Moreover, we define

$$\|f\|_\infty = \inf\{K : |f(\boldsymbol{x})| \leq K \ \forall \boldsymbol{x} \in [\boldsymbol{a}, \boldsymbol{b}]\}.$$

For a compact interval $\boldsymbol{I} = \prod_{k=1}^{n} [u_k, v_k]$, $u_k < v_k$, denote $\mathcal{V}(\boldsymbol{I})$ as the collection of vertices of $\boldsymbol{I}$. Define

$$\Delta_g(\boldsymbol{I}) = \Delta_g\left(\prod_{k=1}^{n} [u_k, v_k] \right) = \sum_{\boldsymbol{x} \in \mathcal{V}(\boldsymbol{I})} g(\boldsymbol{x}) \prod_{k=1}^{n} (-1)^{\chi_{\{u_k\}}(x_k)},$$

where $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ and $g : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}$ is a real-valued function. A real valued-function $g : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}$ is said to be a function of bounded variation on

$[\boldsymbol{a},\boldsymbol{b}]$, if $V(g;[\boldsymbol{a},\boldsymbol{b}]) < \infty$, where

$$V(g;[\boldsymbol{a},\boldsymbol{b}]) = \sup\left\{\sum_{\boldsymbol{I}\in D} |\Delta_g(\boldsymbol{I})| : D \text{ is a division of } [\boldsymbol{a},\boldsymbol{b}]\right\}.$$

We denote $BV([\boldsymbol{a},\boldsymbol{b}])$ as the collection of real-valued function f defined on $[\boldsymbol{a},\boldsymbol{b}]$.

Theorem 2.5. [2] *If $f : [\boldsymbol{a},\boldsymbol{b}] \to X$ is continuous on $[\boldsymbol{a},\boldsymbol{b}]$ and $g : [\boldsymbol{a},\boldsymbol{b}] \to \mathbb{R}$ is of bounded variation on $[\boldsymbol{a},\boldsymbol{b}]$, then f is PUL-stieltjes integrable on $[\boldsymbol{a},\boldsymbol{b}]$ with respect to g.*

3. Main Results

Lemma 3.1. *Let $f : [\boldsymbol{a},\boldsymbol{b}] \to X$ be bounded and $g \in BV([\boldsymbol{a},\boldsymbol{b}])$. Suppose that the PUL-Stieltjes integral of f with respect to g on $[\boldsymbol{a},\boldsymbol{b}]$ exists. Then*

$$\left|\int_{[\boldsymbol{a},\boldsymbol{b}]} f\ dg\right| \leq \|f\|_\infty \cdot V(g;[\boldsymbol{a},\boldsymbol{b}]).$$

Proof: Let $\epsilon > 0$. Then there exists a gauge δ on $[\boldsymbol{a},\boldsymbol{b}]$ such that for any δ-fine division D of $[\boldsymbol{a},\boldsymbol{b}]$, we have

$$\left|S(f,g,D) - \int_{[\boldsymbol{a},\boldsymbol{b}]} f\ dg\right| < \epsilon.$$

Since f is bounded on $[\boldsymbol{a},\boldsymbol{b}]$, $\|f\|_\infty$ exists. Let D be a fix (but arbitrary) δ-fine division D of $[\boldsymbol{a},\boldsymbol{b}]$. Then

$$|S(f,g,D)| \leq \sum_D \left|f(\boldsymbol{\xi}) \int_{\boldsymbol{I}} \varphi\ dg\right| \leq \|f\|_\infty \cdot V(g;[\boldsymbol{a},\boldsymbol{b}]).$$

Thus,

$$\left|\int_{[\boldsymbol{a},\boldsymbol{b}]} f\ dg\right| \leq \left|\int_{[\boldsymbol{a},\boldsymbol{b}]} f\ dg - S(f,g,D)\right| + |S(f,g,D)| < \epsilon + \|f\|_\infty \cdot V(g;[\boldsymbol{a},\boldsymbol{b}]).$$

Since $\epsilon > 0$ is arbitrary, the conclusion follows. □

Theorem 3.2 (**Uniform Convergence Theorem**). *Let $g \in BV([\boldsymbol{a},\boldsymbol{b}])$ and $\langle f_n\rangle_{n=1}^{\infty}$ is a sequence of bounded and PUL-Stieltjes integrable with respect to g on $[\boldsymbol{a},\boldsymbol{b}]$. If $f_n \to f$ uniformly on $[\boldsymbol{a},\boldsymbol{b}]$, then f is PUL-Stieltjes integrable with respect to g on $[\boldsymbol{a},\boldsymbol{b}]$ and*

$$\lim_{n\to\infty} \int_{[\boldsymbol{a},\boldsymbol{b}]} f_n\ dg = \int_{[\boldsymbol{a},\boldsymbol{b}]} f\ dg.$$

Proof: Let $\epsilon > 0$. Since $f_n \to f$ uniformly on $[\boldsymbol{a},\boldsymbol{b}]$, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$ and for all $\boldsymbol{x} \in [\boldsymbol{a},\boldsymbol{b}]$, we have

$$|f_n(\boldsymbol{x}) - f(\boldsymbol{x})| < \frac{\epsilon}{3 \cdot [V(g;[\boldsymbol{a},\boldsymbol{b}]) + 1]}. \tag{3.1}$$

If $m, n \geq N_1$ and $\boldsymbol{x} \in [\boldsymbol{a}, \boldsymbol{b}]$, then by Equation (3.1)

$$\begin{aligned} |f_n(\boldsymbol{x}) - f_m(\boldsymbol{x})| &\leq |f_n(\boldsymbol{x}) - f(\boldsymbol{x})| + |f(\boldsymbol{x}) - f_m(\boldsymbol{x})| \\ &< \frac{\epsilon}{3V(g;[\boldsymbol{a},\boldsymbol{b}])} + \frac{\epsilon}{3V(g;[\boldsymbol{a},\boldsymbol{b}])} \\ &= \frac{2\epsilon}{3V(g;[\boldsymbol{a},\boldsymbol{b}])}. \end{aligned}$$

Hence, for all $m, n \geq N_1$,

$$\|f_n - f_m\|_\infty < \frac{2\epsilon}{3V(g;[\boldsymbol{a},\boldsymbol{b}])}. \tag{3.2}$$

By (3.2), by Lemma 3.1 and by linearity,

$$\begin{aligned} \left| \int_{[\boldsymbol{a},\boldsymbol{b}]} f_m \; dg - \int_{[\boldsymbol{a},\boldsymbol{b}]} f_n \; dg \right| = \left| \int_{[\boldsymbol{a},\boldsymbol{b}]} (f_m - f_n) \; dg \right| &\leq \|f_m - f_n\|_\infty \cdot V(g;[\boldsymbol{a},\boldsymbol{b}]) \\ &< \frac{2\epsilon}{3V(g;[\boldsymbol{a},\boldsymbol{b}])} \cdot V(g;[\boldsymbol{a},\boldsymbol{b}]) \\ &= \frac{2\epsilon}{3} < \epsilon \end{aligned}$$

for all $m, n \geq N_1$. Thus, $\left\langle \int_{[\boldsymbol{a},\boldsymbol{b}]} f_n \; dg \right\rangle$ is Cauchy, and so there exists $A \in \mathbb{R}$ such that $\lim_{n\to\infty} \int_{[\boldsymbol{a},\boldsymbol{b}]} f_n \; dg = A$. Hence, there exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$,

$$\left| \int_{[\boldsymbol{a},\boldsymbol{b}]} f_n \; dg - A \right| < \frac{\epsilon}{3}.$$

Take $N = \max\{N_1, N_2\}$. Since f_N is PUL-Stieltjes integrable with respect to g on $[\boldsymbol{a}, \boldsymbol{b}]$, there is a gauge δ on $[\boldsymbol{a}, \boldsymbol{b}]$ such that for any δ-fine division D of $[\boldsymbol{a}, \boldsymbol{b}]$, we have

$$\left| S(f_N, g, D) - \int_{[\boldsymbol{a},\boldsymbol{b}]} f_N \; dg \right| < \frac{\epsilon}{3}.$$

By (3.1), we have

$$\begin{aligned} |S(f,g,D) - S(f_N,g,D)| &= \left| \sum_D [f(\boldsymbol{\xi}) - f_N(\boldsymbol{\xi})] \int_{\boldsymbol{I}} \varphi \; dg \right| \leq \sum_D |f(\boldsymbol{\xi}) - f_N(\boldsymbol{\xi})| \cdot \left| \int_{\boldsymbol{I}} \varphi \; dg \right| \\ &< \frac{\epsilon}{3 \cdot [V(g;[\boldsymbol{a},\boldsymbol{b}]) + 1]} \cdot V(g;[\boldsymbol{a},\boldsymbol{b}]) = \frac{\epsilon}{3} \cdot \frac{V(g;[\boldsymbol{a},\boldsymbol{b}])}{V(g;[\boldsymbol{a},\boldsymbol{b}]) + 1} < \frac{\epsilon}{3}. \end{aligned}$$

Therefore,

$$\begin{aligned} |S(f,g,D) - A| &\leq |S(f,g,D) - S(f_N,g,D)| + \left| S(f_N,g,D) - \int_{[\boldsymbol{a},\boldsymbol{b}]} f_N \; dg \right| + \left| \int_{[\boldsymbol{a},\boldsymbol{b}]} f_N \; dg - A \right| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

Therefore,

$$\int_{[\boldsymbol{a},\boldsymbol{b}]} f = A = \lim_{n\to\infty} \int_{[\boldsymbol{a},\boldsymbol{b}]} f_n. \qquad \square$$

EXAMPLE 3.3. For each $n \in \mathbb{N}$, let $f_n = \frac{x^2}{n}$ for all $x \in [0,1] \subseteq \mathbb{R}$. Fix $\epsilon > 0$. Then choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$ so that for each $x \in [0,1]$ and for each $n \in \mathbb{N}$, we have

$$|f_n(x) - 0| = |f_n(x)| = \left|\frac{x^2}{n}\right| < \frac{1}{n} \leq \frac{1}{N} < \epsilon.$$

This means that $\langle f_n \rangle_{n=1}^{\infty}$ converges uniformly to 0 on $[0,1]$. Moreover,

$$\int_0^1 f_n(x)\ dx = \int_0^1 \frac{x^2}{n}\ dx = \frac{1}{n}$$

implies

$$\lim_{n\to\infty} \int_0^1 f_n(x)\ dx = \lim_{n\to\infty} \frac{1}{n} = 0 = \int_0^1 0\ dx.$$

Definition 3.4. Let $g : [\boldsymbol{a},\boldsymbol{b}] \to \mathbb{R}$. A family $\mathcal{F}$ of real-valued PUL-Stieltjes integrable functions with respect to g on $[\boldsymbol{a},\boldsymbol{b}]$ is said to be *equi-integrable* with respect to g on $[\boldsymbol{a},\boldsymbol{b}]$ if for every $\epsilon > 0$, there is a gauge δ on $[\boldsymbol{a},\boldsymbol{b}]$ such that for each $f \in \mathcal{F}$ and for every δ-fine division D of $[\boldsymbol{a},\boldsymbol{b}]$, we have

$$\left| S(f,g,D) - \int_{[\boldsymbol{a},\boldsymbol{b}]} f\ dg \right| < \epsilon.$$

A sequence $\langle f_n \rangle_{n=1}^{\infty}$ of real-valued PUL-Stieltjes integrable functions with respect to g on $[\boldsymbol{a},\boldsymbol{b}]$ is said to be *equi-integrable* with respect to g on $[\boldsymbol{a},\boldsymbol{b}]$ if the family $\{f_n : [\boldsymbol{a},\boldsymbol{b}] \to \mathbb{R} \mid n \in \mathbb{N}\}$ is equi-integrable with respect to g on $[\boldsymbol{a},\boldsymbol{b}]$.

Lemma 3.5. *Let $g \in BV([\boldsymbol{a},\boldsymbol{b}])$ and let $\langle f_n \rangle_{n=1}^{\infty}$ be a sequence of real-valued PUL-Stieltjes integrable functions with respect to g on $[\boldsymbol{a},\boldsymbol{b}]$. If $\langle f_n \rangle_{n=1}^{\infty}$ is a equi-integrable with respect to g on $[\boldsymbol{a},\boldsymbol{b}]$ and converges pointwisely to $f : [\boldsymbol{a},\boldsymbol{b}] \to \mathbb{R}$ on $[\boldsymbol{a},\boldsymbol{b}]$, then $\left\langle \int_{[\boldsymbol{a},\boldsymbol{b}]} f_n \right\rangle$ is Cauchy.*

Proof: Let $\epsilon > 0$. Since $\langle f_n \rangle_{n=1}^{\infty}$ is equi-integrable with respect to g on $[\boldsymbol{a},\boldsymbol{b}]$, there is a gauge δ on $[\boldsymbol{a},\boldsymbol{b}]$ such that if D is a δ-fine division of $[\boldsymbol{a},\boldsymbol{b}]$,

$$\left| S(f_n,g,D) - \int_{[\boldsymbol{a},\boldsymbol{b}]} f_n\ dg \right| < \frac{\epsilon}{3}\ \text{, for all } n \in \mathbb{N}.$$

Since $g \in BV([\boldsymbol{a},\boldsymbol{b}])$, $V(g;[\boldsymbol{a},\boldsymbol{b}]) = M \in \mathbb{R}$. Now, fix a δ-fine division $D = \{(\boldsymbol{\xi}, \boldsymbol{I}, \varphi)\}$ of $[\boldsymbol{a},\boldsymbol{b}]$. Since $\langle f_n \rangle_{n=1}^{\infty}$ converges pointwisely to f on $[\boldsymbol{a},\boldsymbol{b}]$, for each tag $\boldsymbol{\xi}$ in D there is an $N_{\boldsymbol{\xi}} \in \mathbb{N}$ such that for all $n \geq N_{\boldsymbol{\xi}}$, we have

$$|f_n(\boldsymbol{\xi}) - f(\boldsymbol{\xi})| < \frac{\epsilon}{6(M+1)}.$$

Put $N = \max\{N_{\boldsymbol{\xi}} : \boldsymbol{\xi}$ is a tag in $D\}$. For each $m, n \geq N$, we have

$$\begin{aligned}|f_m(\boldsymbol{\xi}) - f_n(\boldsymbol{\xi})| &\leq |f_m(\boldsymbol{\xi}) - f((\boldsymbol{\xi})| + |f_n(\boldsymbol{\xi}) - f(\boldsymbol{\xi})| \\ &< \frac{\epsilon}{6(M+1)} + \frac{\epsilon}{6(M+1)} = \frac{\epsilon}{3(M+1)},\end{aligned}$$

for each tag $\boldsymbol{\xi}$ in D. Hence, for all $m, n \geq N$

$$|S(f_m, g, D) - S(f_n, g, D)| = |S(f_m - f_n, g, D)| = \left| \sum_D [f_m(\boldsymbol{\xi}) - f_n(\boldsymbol{\xi})] \int_{\boldsymbol{I}} \varphi \; dg \right|$$

$$\leq \sum_D \left| f_m(\boldsymbol{\xi}) - f_n(\boldsymbol{\xi}) \right| \left| \int_{\boldsymbol{I}} \varphi \; dg \right| \leq \frac{\epsilon}{3(M+1)}(M+1) = \frac{\epsilon}{3}.$$

Thus, for all $m, n \geq N$

$$\begin{aligned}\left| \int_{[\boldsymbol{a},\boldsymbol{b}]} f_m \; dg - \int_{[\boldsymbol{a},\boldsymbol{b}]} f_n \; dg \right| &\leq \left| \int_{[\boldsymbol{a},\boldsymbol{b}]} f_m \; dg - S(f_m, g, D) \right| \\ &+ |S(f_m, g, D) - S(f_n, g, D)| + \left| S(f_n, D, g) - \int_{[\boldsymbol{a},\boldsymbol{b}]} f_n \; dg \right| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.\end{aligned}$$

Therefore, the sequence $\left\langle \int_{[\boldsymbol{a},\boldsymbol{b}]} f_n \; dg \right\rangle$ is Cauchy in X. □

Theorem 3.6 (**Equi-integrability Theorem**). *Let $g \in BV([\boldsymbol{a},\boldsymbol{b}])$ and let $\langle f_n \rangle_{n=1}^{\infty}$ be a sequence of real-valued PUL-Stieltjes integrable with respect to g on $[\boldsymbol{a},\boldsymbol{b}]$. If $\langle f_n \rangle_{n=1}^{\infty}$ is equi-integrable with respect to g on $[\boldsymbol{a},\boldsymbol{b}]$ and converges pointwisely to $f : [\boldsymbol{a},\boldsymbol{b}] \to \mathbb{R}$, then f is PUL-Stieltjes integrable with respect to g on $[\boldsymbol{a},\boldsymbol{b}]$ and*

$$\int_{[\boldsymbol{a},\boldsymbol{b}]} f \; dg = \lim_{n \to \infty} \int_{[\boldsymbol{a},\boldsymbol{b}]} f_n \; dg.$$

Proof: Let $\epsilon > 0$. By Lemma 3.5, $\left\langle \int_{[\boldsymbol{a},\boldsymbol{b}]} f_n \; dg \right\rangle$ is Cauchy in X. Since $\mathbb{R}$ is a complete, the sequence $\left\langle \int_{[\boldsymbol{a},\boldsymbol{b}]} f_n \; dg \right\rangle$ converges to, say A, that is,

$$\lim_{n \to \infty} \int_{[\boldsymbol{a},\boldsymbol{b}]} f_n \; dg = A.$$

Thus, there is $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, we have

$$\left| \int_{[\boldsymbol{a},\boldsymbol{b}]} f_n \; dg - A \right| < \frac{\epsilon}{3}.$$

By equi-integrability of $\langle f_n \rangle_{n=1}^{\infty}$, choose a gauge δ on $[\boldsymbol{a},\boldsymbol{b}]$ and fix (but arbitrary) a δ-fine division $D = \{(\boldsymbol{\xi}, \boldsymbol{I}, \varphi)\}$ of $[\boldsymbol{a},\boldsymbol{b}]$ such that for each $n \in \mathbb{N}$,

$$\left| S(f_n, g, D) - \int_{[\boldsymbol{a},\boldsymbol{b}]} f_n \; dg \right| < \frac{\epsilon}{3}.$$

By pointwise convergence of $\langle f_n\rangle_{n=1}^{\infty}$, for each tag $\boldsymbol{\xi}$ in D there is $N_{\boldsymbol{\xi}} \in \mathbb{N}$ such that for all $n \geq N_{\boldsymbol{\xi}}$,

$$|f_n(\boldsymbol{\xi}) - f(\boldsymbol{\xi})| < \frac{\epsilon}{3(M+1)}.$$

Put $N_2 = \max\{N_{\boldsymbol{\xi}} : \boldsymbol{\xi} \text{ is a tag in } D\}$. Hence, for each $n \geq N_2$, we have

$$\begin{aligned}\Big|S(f,g,D) - S(f_n,g,D)\Big| &= \Big|S(f-f_n,g,D)\Big| = \left|\sum_D [f(\boldsymbol{\xi}) - f_n(\boldsymbol{\xi})] \int_{\boldsymbol{I}} \varphi \; dg\right| \\ &\leq \sum_D |f(\boldsymbol{\xi}) - f_n(\boldsymbol{\xi})| \left|\int_{\boldsymbol{I}} \varphi \; dg\right| \leq \frac{\epsilon}{3(M+1)}(M+1) = \frac{\epsilon}{3}.\end{aligned}$$

Take $N = \max\{N_1, N_2\}$. Then

$$\begin{aligned}\Big|S(f,g,D) - A\Big| &\leq \Big|S(f,g,D) - S(f_N,g,D)\Big| + \left|S(f_N,g,D) - \int_{[\boldsymbol{a},\boldsymbol{b}]} f_N \; dg\right| \\ &\quad + \left|\int_{[\boldsymbol{a},\boldsymbol{b}]} f_N dg - A\right| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.\end{aligned}$$

Therefore, f is equi-integrable with respect to g on $[\boldsymbol{a},\boldsymbol{b}]$ and

$$\int_{[\boldsymbol{a},\boldsymbol{b}]} f \; dg = A = \lim_{n\to\infty} \int_{[\boldsymbol{a},\boldsymbol{b}]} f_n \; dg. \qquad \square$$

Denote $C([\boldsymbol{a},\boldsymbol{b}])$ as the collection of all continuous functions $f : [\boldsymbol{a},\boldsymbol{b}] \to \mathbb{R}$.

Lemma 3.7. *Let $f \in C([\boldsymbol{a},\boldsymbol{b}])$ and $\langle g_n\rangle_{n=1}^{\infty}$ be sequence in $BV([\boldsymbol{a},\boldsymbol{b}])$ such that* $\sup\big\{V(g_n;[\boldsymbol{a},\boldsymbol{b}]) : n \in \mathbb{N}\big\} < \infty$. *Suppose that $g_n(\boldsymbol{x}) \to g(\boldsymbol{x})$ uniformly on $[\boldsymbol{a},\boldsymbol{b}]$. Then the sequence*

$$\left\langle \int_{[a,b]} f \; dg_n \right\rangle_{n=1}^{\infty}$$

is Cauchy.

Proof: Let $\epsilon > 0$. Put $K = \sup\big\{V(g_n;[\boldsymbol{a},\boldsymbol{b}]) : n \in \mathbb{N}\big\}$. Since $f \in C([\boldsymbol{a},\boldsymbol{b}])$, for each $\boldsymbol{x} \in [\boldsymbol{a},\boldsymbol{b}]$, there exists $\delta(\boldsymbol{\xi})$ such that for any $\boldsymbol{y} \in [\boldsymbol{a},\boldsymbol{b}]$ with $\boldsymbol{y} \in B(\boldsymbol{x}, \delta(\boldsymbol{x}))$, we have

$$|f(\boldsymbol{x}) - f(\boldsymbol{y})| < \frac{\epsilon}{4(1+K)}. \tag{3.3}$$

Let $D = \{(\boldsymbol{\xi}, \boldsymbol{I}, \varphi)\}$ be a δ-fine division of $[\boldsymbol{a}, \boldsymbol{b}]$. Then, for all $n \in \mathbb{N}$

$$\begin{aligned}
\left|S(f, g_n, D) - \int_{[\boldsymbol{a},\boldsymbol{b}]} f \; dg_n\right| &= \left|\sum_D \left[f(\boldsymbol{\xi}) \int_{\boldsymbol{I}} \varphi \; dg_n - \int_{\boldsymbol{I}} f \cdot \varphi \; dg_n\right]\right| \\
&\leq \sum_D \left|f(\boldsymbol{\xi}) \int_{\boldsymbol{I}} \varphi \; dg_n - \int_{\boldsymbol{I}} f \cdot \varphi \; dg_n\right| \\
&\leq \sum_D \left|\int_{\boldsymbol{I}} [f(\boldsymbol{\xi}) \cdot \varphi - f \cdot \varphi] \; dg_n\right| \leq \sum_D \left|\int_{\boldsymbol{I}} [f(\boldsymbol{\xi}) - f] \cdot \varphi \; dg_n\right| \\
&\leq \sum_D \left|\int_{\boldsymbol{I}} \frac{\epsilon}{4(1+K)} \varphi \; dg_n\right| = \frac{\epsilon}{4(1+K)} \sum_D \left|\int_{\boldsymbol{I}} \varphi \; dg_n\right| \\
&\leq \frac{\epsilon}{4(1+K)} \cdot K < \frac{\epsilon}{4}.
\end{aligned}$$

Since f is continuous on $[\boldsymbol{a}, \boldsymbol{b}]$, f is bounded in $[\boldsymbol{a}, \boldsymbol{b}]$. Hence, there is $M > 0$ such that $|f(\boldsymbol{x})| \leq M$ for all $\boldsymbol{x} \in [\boldsymbol{a}, \boldsymbol{b}]$.

Now, choose $L = \sup\{|D| : D \text{ is a division of } [\boldsymbol{a}, \boldsymbol{b}]\}$. Since $\lim_{n\to\infty} g(\boldsymbol{x}) = g(\boldsymbol{x})$ on $[\boldsymbol{a}, \boldsymbol{b}]$, there is an $N \in \mathbb{N}$ such that for all $n \geq N$ and for all $\boldsymbol{x} \in [\boldsymbol{a}, \boldsymbol{b}]$, we have

$$|g_n(\boldsymbol{x}) - g(\boldsymbol{x})| < \frac{\epsilon}{8(M+1) \cdot 2^n \cdot (L+1)}.$$

We now find a bound for $V(g - g_n; [\boldsymbol{a}, \boldsymbol{b}])$. Let P be a division of $[\boldsymbol{a}, \boldsymbol{b}]$. Then

$$\begin{aligned}
\sum_{\boldsymbol{J} \in P} |\Delta_{(g-g_n)}(\boldsymbol{J})| &\leq \sum_{\boldsymbol{J} \in P} \sum_{\boldsymbol{t} \in \mathcal{V}(\boldsymbol{J})} |g(\boldsymbol{t}) - g_n(\boldsymbol{t})| \\
&< \sum_{\boldsymbol{J} \in P} \sum_{\boldsymbol{t} \in \mathcal{V}(\boldsymbol{J})} \frac{\epsilon}{8(M+1) \cdot 2^n \cdot (L+1)} \\
&< \frac{\epsilon}{8(M+1) \cdot 2^n \cdot (L+1)} \sum_{\boldsymbol{J} \in P} \sum_{\boldsymbol{t} \in \mathcal{V}(\boldsymbol{J})} 1 \\
&= \frac{\epsilon}{8(M+1) \cdot 2^n \cdot (L+1)} \cdot |P| \cdot 2^n \\
&\leq \frac{\epsilon}{8(M+1) \cdot 2^n \cdot (L+1)} \cdot L \cdot 2^n \\
&= \frac{\epsilon}{8(M+1)}.
\end{aligned}$$

So, $V(g - g_n; [\boldsymbol{a}, \boldsymbol{b}]) < \dfrac{\epsilon}{8(M+1)}$. Thus

$$\begin{aligned}|S(f,g_n,D)-S(f,g,D)| = |S(f,g_n-g,D)| &= \left|\sum_D f(\boldsymbol{\xi})\int_I \varphi\ d(g-g_n)\right| \\ &\le M\cdot\sum_D\left|\int_{\boldsymbol{I}}\varphi\ d(g-g_n)\right| \\ &\le M\cdot\sum_D \|\varphi\|_C\cdot V(g-g_n;\boldsymbol{I}) \\ &\le M\cdot V(g-g_n;[\boldsymbol{a},\boldsymbol{b}]) = \frac{\epsilon}{8}.\end{aligned}$$

So, if $m,n\ge N$, then

$$\begin{aligned}\big|S(f,g_n,D)-S(f,g_m,D)\big| &\le \big|S(f,g_n,D)-S(f,g,D)\big| \\ &\qquad + \big|S(f,g,D)-S(f,g_m,D)\big| \\ &\le \big|S(f,g_n-g,D)\big| + \big|S(f,g-g_m,D)\big| \\ &< \frac{\epsilon}{8}+\frac{\epsilon}{8}=\frac{\epsilon}{4}.\end{aligned}$$

For each $m,n\ge N$,

$$\begin{aligned}\left|(\mathcal{P})\int_{[\boldsymbol{a},\boldsymbol{b}]} f\ dg_n-(\mathcal{P})\int_{[\boldsymbol{a},\boldsymbol{b}]} f\ dg_m\right| &\le \left|(\mathcal{P})\int_{[\boldsymbol{a},\boldsymbol{b}]} f\ dg_n - S(f,g_n,D)\right| \\ &\quad + \left|S(f,g_n,D)-S(f,g,D)\right| \\ &\quad + \left|S(f,g,D)-S(f,g_m,D)\right| \\ &\quad + \left|S(f,g_m,D)-(\mathcal{P})\int_{[\boldsymbol{a},\boldsymbol{b}]} f\ dg_m\right| \\ &< \frac{\epsilon}{4}+\frac{\epsilon}{4}+\frac{\epsilon}{4}+\frac{\epsilon}{4}=\epsilon;\end{aligned}$$

which means that the sequence $\left\langle(\mathcal{P})\int_{[\boldsymbol{a},\boldsymbol{b}]} f\ dg_n\right\rangle_{n=1}^{\infty}$. □

Denote $C([\boldsymbol{a},\boldsymbol{b}])$ to be the set of continuous real-valued functions on $[\boldsymbol{a},\boldsymbol{b}]$.

Theorem 3.8 (Uniform Convergence Theorem II). *Let $f\in C([\boldsymbol{a},\boldsymbol{b}])$ and $\langle g_n\rangle_{n=1}^{\infty}$ be sequence in $BV([\boldsymbol{a},\boldsymbol{b}])$ such that*

$$\sup\left\{V(g_n,[\boldsymbol{a},\boldsymbol{b}]) : n\in\mathbb{N}\right\}<\infty.$$

Suppose that $g_n(\boldsymbol{x})\to g(\boldsymbol{x})$ uniformly on $[\boldsymbol{a},\boldsymbol{b}]$. Then f is PUL-Stieltjes integrable with respect to g on $[\boldsymbol{a},\boldsymbol{b}]$ and

$$\int_{[\boldsymbol{a},\boldsymbol{b}]} f\ dg = \lim_{n\to\infty}\int_{[\boldsymbol{a},\boldsymbol{b}]} f\ dg_n.$$

Proof: By Lemma 3.7, the sequence $\left\langle (\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f \ dg_n \right\rangle_{n=1}^{\infty}$ is Cauchy and so

$$\lim_{n\to\infty} (\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f \ dg_n = A. \tag{3.4}$$

It remains to show that $A = (\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f \ dg$. By (3.4), there is $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$\left| (\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f \ dg_n - A \right| < \frac{\epsilon}{3}.$$

In particular,

$$\left| (\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f \ dg_N - A \right| < \frac{\epsilon}{3}. \tag{3.5}$$

Note that f is PUL-Stieltjes integrable with respect to g_N on $[\boldsymbol{a},\boldsymbol{b}]$. Hence, there is a gauge δ on $[\boldsymbol{a},\boldsymbol{b}]$ such that for any δ-fine division D of $[\boldsymbol{a},\boldsymbol{b}]$, we have

$$\left| S(f, g_N, D) - (\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f \ dg_N \right| < \frac{\epsilon}{3}. \tag{3.6}$$

Also, as in the proof of Lemma 3.7,

$$|S(f, g, D) - S(f, g_N, D)| < \frac{\epsilon}{3}. \tag{3.7}$$

Therefore, by (3.5), (3.6), and (3.7)

$$\begin{aligned} |S(f,g,D) - A| \leq & \left| S(f,g,D) - S(f,g_N,D) \right| \\ & + \left| S(f,g_N,D) - (\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f \ dg_N \right| \\ & + \left| (\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f \ dg_N - A \right| \\ < & \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ = & \epsilon; \end{aligned}$$

Hence,

$$\lim_{n\to\infty} (\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f \ dg_n = A = (\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f \ dg. \qquad \square$$

EXAMPLE 3.9. Let $\{f_k\}_{k=1}^{n}$ be a finite collections of integrable real-valued functions on the compact interval $[\mathbf{0},\mathbf{1}] \subseteq \mathbb{R}^n$. Then $\{f_k\}_{k=1}^{n}$ is equi-integrable.

EXAMPLE 3.10. Let $\langle f_n \rangle_{n=1}^{\infty}$ be a sequence of nonnegative integrable functions that converges pointwise to 0 on $[\mathbf{0},\mathbf{1}] \subseteq$. If $\langle f_n \rangle_{n=1}^{\infty}$ is equi-integrable on $[\mathbf{0},\mathbf{1}] \subseteq \mathbb{R}^n$, then

$$\lim_{n\to\infty} \int_0^1 f_n = 0.$$

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References

1. Boonpogkrong, V., Kursweil-Henstock Integration on Manifolds, *Taiwanese Journal of Mathematics*, **15**(2), (2011), 559-571.
2. G. C. Flores, J. V. Benitez, *Simple Properties of PUL-Stieltjes Integral in Banach Space*, Journal of Ultra Scientist of Physical Sciences, **29**(4), (2017), 126-134.
3. J. Jarnik, J. Kurzweil, A nonabsolutely convergent integral which admits transformation and can be used for integration on manifolds, **Czechoslovak Math. J.**, **35**(1), (1985), 116-139.