Iranian Journal of Mathematical Sciences and Informatics
Vol. 16, No. 2 (2021), pp 31-48
DOI: 10.29252/ijmsi.16.2.31

Topological Rings and Modules Via Operations

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Abstract. The structure of an $\alpha_{(\beta,\beta)}$-topological ring is richer in comparison with the structure of an $\alpha_{(\beta,\beta)}$-topological group. The theory of $\alpha_{(\beta,\beta)}$-topological rings has many common features with the theory of $\alpha_{(\beta,\beta)}$-topological groups. Formally, the theory of $\alpha_{(\beta,\beta)}$-topological abelian groups is included in the theory of $\alpha_{(\beta,\beta)}$-topological rings.

The purpose of this paper is to introduce and study the concepts of $\alpha_{(\beta,\beta)}$-topological rings and $\alpha_{(\beta,\gamma)}$-topological R-modules. we show how they may be introduced by specifying the neighborhoods of zero, and present some basic constructions. We provide fundamental concepts and basic results on $\alpha_{(\beta,\beta)}$-topological rings and $\alpha_{(\beta,\gamma)}$-topological R-modules.

Keywords: Operations, α_β-Open set, Rins, $\alpha_{(\beta,\beta)}$-Topological rings, $\alpha_{(\beta,\gamma)}$-Topological R-Modules.

2020 Mathematics subject classification: 13Jxx, 54H13

1. Introduction

Since the 1940s, systematic investigation of topological rings has been actively carried out using the frame of topological algebra. Several parts of the

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Received 26 July 2017; Accepted 27 March 2019

theory of topological rings have been exposed in mathematical texts. For example, topological fields (of real, complex, p-adic numbers, etc.) are under analysis from different points of view while taking into account complexly their algebraic, topological, metrical, ordered, and other structures.

One of the first fundamental results in the theory of topological rings was obtained by L. S. Pontryagin in the classification of locally compact skew fields and was included in his famous book [20] on topological groups. Some properties of topological rings and modules were also noted in books [3, 16]. Intensive research during the last fifty years has been carried out in the field of normed and Banach algebras as well; those algebras form one of the most important classes of topological rings (see, for example [5, 7, 8, 9, 18]). The theory of topological linear spaces [4], one of many rich chapters on functional analysis, is also a good introduction to the theory of topological modules. Another source of topological modules is the theory of topological Abelian groups, in particular, duality theory [20].

In 2013, Ibrahim [10] introduced a strong form of α-open sets called α_β-open via operation and studied some of its properties. Khalaf and Ibrahim [12, 13, 14] continued studying the properties of operations defined on the family of α-open sets introduced by Ibrahim [10].

2. Preliminaries

Let A be a subset of a topological space (G, τ). We denote the interior and the closure of a set A by $Int(A)$ and $Cl(A)$ respectively. A subset A of a topological space (G, τ) is called α-open [19] if $A \subseteq Int(Cl(Int(A)))$. By $\alpha O(G, \tau)$, we denote the family of all α-open sets of G. An operation $\beta : \alpha O(G, \tau) \to P(G)$ [10] is a mapping satisfying the condition, $V \subseteq V^\beta$ for each $V \in \alpha O(G, \tau)$. We call the mapping β an operation on $\alpha O(G, \tau)$. A subset A of G is called an α_β-open set [10] if for each point $x \in A$, there exists an α-open set U of G containing x such that $U^\beta \subseteq A$. The complement of an α_β-open set is said to be α_β-closed. We denote the set of all α_β-open sets of (G, τ) by $\alpha O(G, \tau)_\beta$. The α_β-closure [10] of a subset A of G with an operation β on $\alpha O(G)$ is denoted by $\alpha_\beta Cl(A)$ and is defined to be the intersection of all α_β-closed sets containing A. An operation β on $\alpha O(G, \tau)$ is said to be α-regular if for every α-open sets U and V of each $x \in G$, there exists an α-open set W of x such that $W^\beta \subseteq U^\beta \cap V^\beta$.

Definition 2.1. [12] Let (G, τ) be a topological space and $x \in G$, then a subset N of G is said to be α_β-neighbourhood of x, if there exists an α_β-open set U in G such that $x \in U \subseteq N$.

Definition 2.2. [14] Two subsets A and B of a topological space (G, τ) are called α_β-separated if $(\alpha_\beta Cl(A) \cap B) \cup (A \cap \alpha_\beta Cl(B)) = \phi$.

Definition 2.3. [14] A subset C of a space G is said to be α_β-disconnected if there are nonempty α_β-separated subsets A and B of G such that $C = A \cup B$, otherwise C is called α_β-connected.

Definition 2.4. [14] A set C is called maximal α_β-connected set if it is α_β-connected and if $C \subseteq D \subseteq G$ where D is α_β-connected, then $C = D$. A maximal α_β-connected subset C of a space G is called an α_β-component of G.

Definition 2.5. [10] A topological space (G, τ) with an operation β on $\alpha O(G)$ is said to be:

(1) $\alpha_\beta T_0$ if for any two distinct points $x, y \in X$, there exists an α_β-open set U such that either $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$.
(2) $\alpha_\beta T_1$ if for any two distinct points $x, y \in X$, there exist two α_β-open sets U and V containing x and y, respectively, such that $y \notin U$ and $x \notin V$.
(3) $\alpha_\beta T_2$ if for any two distinct points $x, y \in X$, there exist two α_γ-open sets U and V containing x and y, respectively, such that $U \cap V = \phi$.

Definition 2.6. [13] A function $f : (G, \tau) \to (G^{'}, \tau^{'})$ is said to be $\alpha_{(\beta,\beta')}$-open if for any α_β-open set A of (G, τ), $f(A)$ is $\alpha_{\beta'}$-open in $(G^{'}, \tau^{'})$.

Definition 2.7. [10] A mapping $f : (G, \tau) \to (G^{'}, \tau^{'})$ is said to be $\alpha_{(\beta,\beta')}$-continuous if for each x of G and each $\alpha_{\beta'}$-open set V containing $f(x)$, there exists an α_β-open set U such that $x \in U$ and $f(U) \subseteq V$.

Definition 2.8. [10] A mapping $f : (G, \tau) \to (G, \tau)$ is said to be $\alpha_{(\beta,\beta)}$-homeomorphism, if f is bijective, $\alpha_{(\beta,\beta)}$-continuous and f^{-1} is $\alpha_{(\beta,\beta)}$-continuous.

Corollary 2.9. [14] *A function $f : G \to G^{'}$ is $\alpha_{(\beta,\beta')}$-continuous if and only if $f^{-1}(V)$ is α_β-open in G, for every $\alpha_{\beta'}$-open set V in $G^{'}$.*

Some parts of the theory of topological rings were systematically investigated in a number of review papers [6, 15, 17, 21, 22, 26] as well as monographs [2, 1, 23, 24, 25, 27, 28] and most of these references contains the following definitions.

Definition 2.10. A group G is an algebraic structure consisting of a nonempty set equipped with an operation on its elements that satisfies four conditions, namely closure, associativity, identity and invertibility. Moreover, if the operation is abelian then G is called an abelian group

Definition 2.11. Let G be an abelian group and $B \subseteq G$. Then B is called a subgroup, if B is a group with respect to the existing operations.

A subset C of an abelian group G is called symmetric if $-C = C$.

Definition 2.12. A ring is a set R (possibly without the unitary element) with two associative operations (addition and multiplication) such that:

(1) R is an abelian group with respect to addition.
(2) The left and right distributive laws: $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(b + c) \cdot a = b \cdot a + c \cdot a$ are satisfied for all $a, b, c \in R$.

An element a of a ring R with the unitary element 1 is called **invertible** if there exists $b \in R$ such that $a \cdot b = b \cdot a = 1$. If all non-zero elements of R are invertible, then R is called a **skew field** (a division ring). A commutative skew field is called a **field**.

Definition 2.13. By an R-module M (unless otherwise stated) we mean a left module over a ring R, that is, an abelian group M with given left multiplication by elements of R such that the following conditions are satisfied:

(1) $r \cdot (m_1 + m_2) = r \cdot m_1 + r \cdot m_2$.
(2) $(r_1 + r_2) \cdot m = r_1 \cdot m + r_2 \cdot m_2$.
(3) $r_1 \cdot (r_2 m) = (r_1 r_2) \cdot m$, for all $r_1, r_2, r \in R$ and $m_1, m_2, m \in M$ (if R is a ring with the unitary element 1, and $1 \cdot m = m$ for any $m \in M$, then M is called unitary).

Definition 2.14. Let G be an abelian group (R-module, ring) and $B \subseteq G$. Then B is called a subgroup (submodule, subring), if B is a group (R-module, ring) with respect to the existing operations.

Let R be a ring and $I \subseteq R$, then I is a called left (right) ideal if I is a subgroup of the additive group of R and $r \cdot i \in I$ ($i \cdot r \in I$) for all $i \in I$, $r \in R$.

If I is both left and right ideal of a ring, then I is called a two-sided ideal or, briefly, an ideal of the ring.

A non-empty subset S of the group G is a **subgroup** of G if $x+S = S = S+x$ for every $x \in S$. Equivalently, if for every $x, y \in S$, $x - y \in S$.

It is obvious that the group G and $\{0\}$ both are subgroups of G.

Definition 2.15. Let $n \in \mathbb{N}$, R be a ring and $A, B \subseteq R$. Let M be an R-module, $D, E \subseteq M$, and C be a subset of either R or M, then put:

(1) $A \cdot C = \{a \cdot c | a \in A, c \in C\}$.
(2) $A^{(1)} = A$ and $A^{(n)} = A \cdot A^{(n-1)}$, for $n > 1$.
(3) $AC = \{\sum_{i=1}^{k} a_i \cdot c_i | a_i \in A, c_i \in C, 1 \leq i \leq k, k \in \mathbb{N}\}$
(4) $A^n = \{\sum_{i=1}^{k} b_i | b_i \in A^{(n)}, 1 \leq i \leq k, k \in \mathbb{N}\}$.
(5) $(A : B)_R = \{r \in R | r \cdot B \subseteq A\}$.
(6) $(D : E)_R = \{r \in R | r \cdot E \subseteq D\}$.
(7) $(D : A)_M = \{m \in M | A \cdot m \subseteq D\}$.

If E is a subgroup of the group M, then $(E : D)_R$ and $(E : A)_M$ are subgroups of the groups $R(+)$ and M, respectively.

If E is a submodule of the R-module M, then $(E : D)_R$ is a left ideal of the ring R.

If E and D are submodules of the R-module M, then $(E : D)_R$ is an ideal of the ring R.

If E is a subgroup of the group M and A is a right ideal of the ring R, then $(E : A)_M$ is a submodule of the R-module M.

Definition 2.16. Let M be an R-module, $S \subseteq M$ and $Q \subseteq R$. If $Q \cdot S \subseteq S$, then the subset S is called Q-stable.

Definition 2.17. Let R be a ring. A left (right) annihilator of a subset U of R is defined by $l_R = \{a \in R | aU = 0\}$ ($r_R = \{a \in R | Ua = 0\}$).

Definition 2.18. Let $(G, +)$ be abelian group and τ be a topology on G. A triple $(G, +, \tau)$ is said to be a topological group if the following conditions are satisfied:

(1) For any two elements $a, b \in G$ and $U \in \tau$ such that $a + b \in U$, there exist $V, W \in \tau$ with $a \in V$, $b \in W$ and $V + W \subseteq U$.
(2) For any element $a \in G$ and $U \in \tau$ such that $-a \in U$, there exists $V \in \tau$ with $a \in V$ and $-V \subseteq U$.

Definition 2.19. Let $(R, +, \cdot)$ be a ring and (R, τ) be a topological space. Then, $(R, +, \cdot, \tau)$ is called a topological ring if the following conditions are satisfied:

(1) $(R, +, \tau)$ is topological group.
(2) For each elements $a, b \in R$ and $U \in \tau$ such that $a \cdot b \in U$, there exist $V, W \in \tau$ with $a \in V$, $b \in W$ and $V \cdot W \subseteq U$.

Definition 2.20. Let $(K, +, \cdot)$ be a skew field (field) and (K, τ) be a topological space. Then, $(K, +, \cdot, \tau)$ is called a topological skew field (field) if the following conditions are satisfied:

(1) $(K, +, \cdot, \tau)$ is topological ring.
(2) For any non-zero element $x \in K$ and any open set U containing x^{-1}, there exists an open set V containing the element x such that $(V \setminus \{0\})^{-1} \subseteq U$.

Definition 2.21. Let $(R, +, \cdot, \tau)$ be a topological ring. A left R-module M is called a topological left R-module if on M is specified a topology such that M is a topological abelian group and the following condition is satisfied:

For any $r \in R$ and $m \in M$ and arbitrary open set U containing the element $r \cdot m$ in M, there exist an open set V containing the element r in R and an open set W the element m in M such that $V \cdot W \subseteq U$.

We recall the following definitions and results from [11].

Definition 2.22. Let $(G, +)$ be abelian group and τ be a topology on G. A triple $(G, +, \tau)$ is said to be an $\alpha_{(\beta,\beta)}$-topological group if the following conditions are satisfied:

(1) For any two elements $a, b \in G$ and $U \in \alpha O(G, \tau)_\beta$ such that $a + b \in U$, there exist $V, W \in \alpha O(G, \tau)_\beta$ with $a \in V$, $b \in W$ and $V + W \subseteq U$.

(2) For any element $a \in G$ and $U \in \alpha O(G,\tau)_\beta$ such that $-a \in U$, there exists $V \in \alpha O(G,\tau)_\beta$ with $a \in V$ and $-V \subseteq U$.

Definition 2.23. A family B_x of subsets of an $\alpha_{(\beta,\beta)}$-topological abelian group G is called a basis of α_β-neighborhoods of $x \in G$ if any subset of B_x is an α_β-neighborhood of x and any α_β-neighborhood of the element x contains some subset from B_x.

Proposition 2.24. *Let a family B_0 of subsets of an $\alpha_{(\beta,\beta)}$-topological abelian group G be a basis of α_β-neighborhoods of zero in G and β be an α-regular operation on $\alpha O(G)$. Then, the following conditions are satisfied:*

(1) $0 \in \bigcap_{V \in B_0} V$.
(2) *For any subsets U and V from B_0, there exists a subset $W \in B_0$ such that $W \subseteq U \cap V$.*
(3) *For any subset $U \in B_0$, there exists a subset $V \in B_0$ such that $V + V \subseteq U$.*
(4) *For any subset $U \in B_0$, there exists a subset $V \in B_0$ such that $-V \subseteq U$.*

Besides, if $a \in G$, then $B_a = \{a + V | V \in B_0\}$ is a basis of α_β-neighborhoods of the element a.

Proposition 2.25. *Let G be an $\alpha_{(\beta,\beta)}$-topological abelian group, $a \in G$, B and C be subsets of G. Then, the following statements are true:*

(1) *The mappings $f : G \to G$ and $f_a : G \to G$, where $f(x) = -x$ and $f_a(x) = x + a$, are both $\alpha_{(\beta,\beta)}$-homeomorphisms from the topological space G onto itself.*
(2) *The following conditions are equivalent:*
   (a) *B is α_β-open (α_β-closed).*
   (b) *$-B$ is α_β-open (α_β-closed).*
   (c) *$B + a$ is α_β-open (α_β-closed).*
(3) *If the subset B is α_β-open, then $B + C$ is also an α_β-open.*

Theorem 2.26. *For any $\alpha_{(\beta,\beta)}$-topological abelian group G and β an α-regular operation on $\alpha O(G)$, the following conditions are equivalent:*

(1) *G is an $\alpha_\beta T_2$-space.*
(2) *$\{0\}$ is α_β-closed subset in G.*
(3) *If B_0 is a basis of α_β-neighborhoods of zero of G, then $\bigcap_{V \in B_0} V = \{0\}$.*
(4) *G is an $\alpha_\beta T_0$-space.*
(5) *G is an $\alpha_\beta T_1$-space.*

Theorem 2.27. *Let B be a subgroup of an $\alpha_{(\beta,\beta)}$-topological group $(G, +, \tau)$. Then $(B, +, \alpha O(G)_\beta | B)$ is a topological group.*

Proposition 2.28. *Let S be a subset of an $\alpha_{(\beta,\beta)}$-topological abelian group G with a basis B_0 of α_β-neighborhoods of zero. Then, $\alpha_\beta Cl(S) = \bigcap_{V \in B_0}(S + V)$.*

Proposition 2.29. *Let B be a subgroup of an $\alpha_{(\beta,\beta)}$-topological abelian group G. Then $\alpha_\beta Cl(B)$ is a subgroup of the $\alpha_{(\beta,\beta)}$-topological group G.*

Proposition 2.30. *For an $\alpha_{(\beta,\beta)}$-topological abelian group G, the following statements are true:*

(1) *If $a \in G$, and $C(G)$ is an α_β-component containing zero, then $C(G)+a$ is an α_β-component of a.*
(2) *If $C(G)$ is an α_β-component containing zero, then $C(G)$ is an α_β-closed subgroup.*

3. $\alpha_{(\beta,\beta)}$-Topological Ring and Modules

In this section, we give some fundamental concepts and basic results on $\alpha_{(\beta,\beta)}$-topological rings and modules. Moreover, we define and discuss the properties of submodules, subrings and ideals by using α-operations.

Definition 3.1. Let $(R, +, \cdot)$ be a ring and (R, τ) be a topological space. Then, $(R, +, \cdot, \tau)$ is called an $\alpha_{(\beta,\beta)}$-topological ring if the following conditions are satisfied:

(1) $(R, +, \tau)$ is $\alpha_{(\beta,\beta)}$-topological group.
(2) For each elements $a, b \in R$ and $U \in \alpha O(R,\tau)_\beta$ such that $a \cdot b \in U$, there exist $V, W \in \alpha O(R,\tau)_\beta$ with $a \in V$, $b \in W$ and $V \cdot W \subseteq U$.

Example 3.2. Consider the ring $(R, +, \cdot) = (Z_3, +_3, \cdot_3)$. Let τ be the discrete topology on Z_3. For each $A \in \alpha O(Z_3, \tau)$, we define β on $\alpha O(Z_3, \tau)$ by

$$A^\beta = \begin{cases} \{1,2\} & \text{if } A = \{1\}, \\ Z_3 & \text{if } A \neq \{1\}. \end{cases}$$

Then, $(Z_3, +_3, \cdot_3, \tau)$ is an $\alpha_{(\beta,\beta)}$-topological ring.

Remark 3.3. By virtue of Definition 3.1, the additive group of any $\alpha_{(\beta,\beta)}$-topological ring is an $\alpha_{(\beta,\beta)}$-topological abelian group.

Definition 3.4. Let $(K, +, \cdot)$ be a skew field (field) and (K, τ) be a topological space. Then, $(K, +, \cdot, \tau)$ is called an $\alpha_{(\beta,\beta)}$-topological skew field (field) if the following conditions are satisfied:

(1) $(K, +, \cdot, \tau)$ is $\alpha_{(\beta,\beta)}$-topological ring.
(2) For any non-zero element $x \in K$ and any α_β-open set U containing x^{-1}, there exists an α_β-open set V containing the element x such that $(V \setminus \{0\})^{-1} \subseteq U$.

Example 3.5. Consider the field $(K, +, \cdot) = (Z_5, +_5, \cdot_5)$. Let $\tau = \{\phi, Z_5, \{0\}, \{4\}, \{0, 4\}\}$. For each $A \in \alpha O(Z_5, \tau)$, we define β on $\alpha O(Z_5, \tau)$ by $A^\beta = Z_5$. Then, $(Z_5, +_5, \cdot_5, \tau)$ is an $\alpha_{(\beta,\beta)}$-topological field.

Remark 3.6. The multiplicative group of non-zero elements of the $\alpha_{(\beta,\beta)}$-topological field is an $\alpha_{(\beta,\beta)}$-topological abelian group.

Definition 3.7. Let $(R, +, \cdot, \tau)$ be an $\alpha_{(\beta,\beta)}$-topological ring. A left R-module M is called an $\alpha_{(\beta,\gamma)}$-topological left R-module if on M is specified a topology such that M is an $\alpha_{(\gamma,\gamma)}$-topological abelian group and the following condition is satisfied:

For any $r \in R$ and $m \in M$ and arbitrary α_γ-open set U containing the element $r \cdot m$ in M, there exist an α_β-open set V containing the element r in R and an α_γ-open set W the element m in M such that $V \cdot W \subseteq U$.

Example 3.8. Consider the ring $(R, +, \cdot) = (\mathbb{R}, +, \cdot)$, where $\mathbb{R}$ is the set of all real numbers. Let τ be the indiscrete topology on $\mathbb{R}$ and $\tau_1 = \{\phi, \{0\}\}$ be a topology on the ring $(\{0\}, +, \cdot)$. For each $A \in \alpha O(\mathbb{R}, \tau)$, we define β on $\alpha O(\mathbb{R}, \tau)$ by $A^\beta = A$ and for each $B \in \alpha O(\{0\}, \tau_1)$, we define γ on $\alpha O(\{0\}, \tau_1)$ by $B^\gamma = \{0\}$. Then, left $\mathbb{R}$-module $\{0\}$ is an $\alpha_{(\beta,\gamma)}$-topological left $\mathbb{R}$-module.

Remark 3.9. In a similar way it is possible to investigate $\alpha_{(\gamma,\beta)}$-topological right R-modules over an $\alpha_{(\beta,\beta)}$-topological ring. Any $\alpha_{(\beta,\beta)}$-topological ring R is both an $\alpha_{(\beta,\beta)}$-topological left R-module and an $\alpha_{(\beta,\beta)}$-topological right R-module.

Proposition 3.10. *Let R be an $\alpha_{(\beta,\beta)}$-topological ring, M an $\alpha_{(\beta,\gamma)}$-topological R-module, $r \in R$, $a \in M$, and Q a subset in R, B a subset in M. Then the following statements are true:*

(1) *The mapping $f_r : M \to M$, where $f_r(x) = r \cdot x$, $x \in M$, is an $\alpha_{(\gamma,\gamma)}$-continuous mapping of the topological space M into itself.*
(2) *The mapping $f_a : R \to M$, where $f_a(x) = x \cdot a$, $x \in R$, is an $\alpha_{(\beta,\gamma)}$-continuous mapping of the topological space R to the topological space M.*
(3) $\alpha_\gamma Cl(Q \cdot B) \supseteq \alpha_\beta Cl(Q) \cdot \alpha_\gamma Cl(B)$.

Proof. (1) Let $x \in M$ and $r \in R$, then $f_r(x) = r \cdot x$. Let U be any α_γ-open set of M containing $r \cdot x$, then by Definition 3.7, there exist α_β-open set V in R containing r and α_γ-open set W in M containing x, such that $V \cdot W \subseteq U$. This gives that $f_r(W) = r \cdot W \subseteq V \cdot W \subseteq U$. This proves that f_r is an $\alpha_{(\gamma,\gamma)}$-continuous mapping.

(2) Let $x \in R$ and $a \in M$, then $f_a(x) = x \cdot a$. Let U be any α_γ-open set of M containing $x \cdot a$, then by Definition 3.7, there exist α_β-open set V in R containing x and α_γ-open set W in M containing a, such that $V \cdot W \subseteq U$. This gives that $f_a(V) = V \cdot a \subseteq V \cdot W \subseteq U$. This proves that f_a is an $\alpha_{(\beta,\gamma)}$-continuous mapping.

(3) Let $y \in \alpha_\beta Cl(Q) \cdot \alpha_\gamma Cl(B)$ and let U be an α_γ-open set containing the element y. Then, $y = b \cdot c$, where $b \in \alpha_\beta Cl(Q)$ and $c \in \alpha_\gamma Cl(B)$, and, hence, there exist α_β-open set V in R containing b and α_γ-open set W in M containing c, such that $V \cdot W \subseteq U$. By virtue of the fact that $V \cap Q \neq \phi$ and $W \cap B \neq \phi$, elements $b_1 \in V \cap Q$ and $c_1 \in W \cap B$

can be found. Thus, $b_1 \cdot c_1 \in Q \cdot B$ and $b_1 \cdot c_1 \in V \cdot W \subseteq U$, that is $(Q \cdot B) \cap U \neq \phi$. Consequently, $\alpha_\gamma Cl(Q \cdot B) \supseteq \alpha_\beta Cl(Q) \cdot \alpha_\gamma Cl(B)$.

□

The proof of the following corollary is obvious and hence omitted.

Corollary 3.11. *Let R be an $\alpha_{(\beta,\beta)}$-topological ring, $a \in R$, and let B and C be subsets in R. Then, the following statements are true:*

(1) *The mappings $R_a : R \to R$ and $L_a : R \to R$, where $R_a(x) = x \cdot a$ and $L_a(x) = a \cdot x$, for $x \in R$, are $\alpha_{(\beta,\beta)}$-continuous mappings of the topological space R into itself.*
(2) $\alpha_\beta Cl(B \cdot C) \supset \alpha_\beta Cl(B) \cdot \alpha_\beta Cl(C)$.

Proposition 3.12. *Let R be an $\alpha_{(\beta,\beta)}$-topological ring with the unitary element and M be an $\alpha_{(\beta,\gamma)}$-topological R-module. Let $a \in R$ be an invertible element, then:*

(1) *The mapping $f_a : M \to M$ is $\alpha_{(\gamma,\gamma)}$-homeomorphism.*
(2) *The mappings $R_a : R \to R$ and $L_a : R \to R$ are $\alpha_{(\beta,\beta)}$-homeomorphisms.*

Proof. Let B be an α_γ-open subset of M, and $b_1 \in f_a(B)$. Then $b_1 = f_a(b) = a \cdot b$ for some $b \in B$. From $b = a^{-1} \cdot b_1$ and Definition 3.7, follows the existence of an α_γ-open set U_1 of the element b_1 in M such that $a^{-1} \cdot U_1 \subseteq B$. Then, $U_1 \subseteq a \cdot B = f_a(B)$ and, hence, $f_a(B)$ is α_γ-open containing the element b_1 in M, that is, $f_a(B)$ is an α_γ-open subset of M. Hence, f_a is $\alpha_{(\gamma,\gamma)}$-open mapping.

In the same manner it can be proved that R_a and L_a are $\alpha_{(\beta,\beta)}$-open mappings too.

In view of the fact that all the mappings f_a, R_a and L_a are bijections, the proposition is proved completely. □

Corollary 3.13. *Let R be an $\alpha_{(\beta,\beta)}$-topological ring with the unitary element, $a \in R$ be an invertible element and $x \in R$. Then, the following statements are equivalent:*

(1) *U is an α_β-neighborhood of the element x in R.*
(2) *$U \cdot a$ is an α_β-neighborhood of the element $x \cdot a$ in R.*
(3) *$a \cdot U$ is an α_β-neighborhood of the element $a \cdot x$ in R.*

Proof. (1) ⇒ (2). Obvious, since $R_a : R \to R$ is an $\alpha_{(\beta,\beta)}$-homeomorphism (Proposition 3.12)and $x \cdot a = R_a(x)$ and $U \cdot a = R_a(U)$.

(2) ⇒ (3). The mapping $\theta_a : R \to R$, where $\theta_a(z) = a \cdot (z \cdot a^{-1})$ for $z \in R$, is the composition of the $\alpha_{(\beta,\beta)}$-homeomorphism mappings R_a and L_a (Proposition 3.12), hence, it is an $\alpha_{(\beta,\beta)}$-homeomorphism. Since $\theta_a(x \cdot a) = a \cdot x$ and $\theta_a(U \cdot a) = a \cdot U$.

(3) ⇒ (1). The equality $L_{a^{-1}}(a \cdot x) = x$ and $L_{a^{-1}}(a \cdot U) = U$ are obtained by considering the $\alpha_{(\beta,\beta)}$-homeomorphism $L_{a^{-1}} : R \to R$. Then, from Proposition 3.12, it follows that U is an α_β-neighborhood of x. □

The proof of the following results are clear, so it is omitted.

Corollary 3.14. *Let a be an invertible element of an $\alpha_{(\beta,\beta)}$-topological ring R with the unitary element. Then, the following statements are equivalent:*

(1) *U is an α_β-neighborhood of 0 in R.*
(2) *$U \cdot a$ is an α_β-neighborhood of 0 in R.*
(3) *$a \cdot U$ is an α_β-neighborhood of 0 in R.*

Corollary 3.15. *Let a be an invertible element of an $\alpha_{(\beta,\beta)}$-topological ring R with the unitary element and let $B \subseteq R$. Then, the following conditions are equivalent:*

(1) *B is α_β-open (α_β-closed).*
(2) *$a \cdot B$ is α_β-open (α_β-closed).*
(3) *$B \cdot a$ is α_β-open (α_β-closed).*

An element $a \in (R, \tau)$ is called an α_β-accumulation point (an α_β-limit) of a sequence $a_1, a_2, ...$ in (R, τ) if for any α_β-neighborhood V of a and any $n \in \mathbb{N}$ (for some $n \in \mathbb{N}$) we get that $a_i \in V$ for some $i > n$ (for all $i > n$).

Proposition 3.16. *Let K be an $\alpha_{(\beta,\beta)}$-topological skew field and let $0 \neq a \in K$. If element a is an α_β-accumulation point (an α_β-limit) of a sequence of non-zero elements $a_1, a_2, ... \in K$, then the element a^{-1} is an α_β-accumulation point (an α_β-limit) of the sequence $a_1^{-1}, a_2^{-1}, ...$ in the skew field K.*

Proof. Let U be an α_β-neighborhood of the element a^{-1}, and let V be an α_β-neighborhood of the element a such that $(V \setminus \{0\})^{-1} \subseteq U$. By virtue of the fact that a is an α_β-accumulation point (an α_β-limit) of the sequence $a_1, a_2, ...$, we get that for any $n \in N$ (there exists $n \in N$) there exists $i > n$ (for any $i > n$) such that $a_i \in V$. Since $a_i^{-1} \neq 0$, then $a_i^{-1} \in (V \setminus \{0\})^{-1} \subseteq U$, that is, a^{-1} is an α_β-accumulation point (an α_β-limit) of the sequence $a_1^{-1}, a_2^{-1}, ...$ □

Proposition 3.17. *Let K be an $\alpha_{(\beta,\beta)}$-topological skew field. Then, the mapping $\theta : K \setminus \{0\} \to K \setminus \{0\}$, where $\theta(x) = x^{-1}$ for $x \neq 0$, is an $\alpha_{(\beta,\beta)}$-homeomorphism of the topological subspace $K \setminus \{0\}$ onto itself.*

Proof. By Definition 3.4, θ is an $\alpha_{(\beta,\beta)}$-continuous mapping. Since $\theta = \theta^{-1}$, then θ is an $\alpha_{(\beta,\beta)}$-homeomorphism. □

Proposition 3.18. *Let B_0 be a basis of α_β-neighborhoods of zero of an $\alpha_{(\beta,\beta)}$-topological ring R and β be an α-regular operation on $\alpha O(R)$. Then, the following conditions are satisfied:*

(1) $0 \in \bigcap_{V \in B_0} V$.
(2) *For any subsets U and V from B_0, there exists a subset $W \in B_0$ such that $W \subseteq U \cap V$.*
(3) *For any subset $U \in B_0$, there exists a subset $V \in B_0$ such that $V + V \subseteq U$.*

(4) *For any subset* $U \in B_0$*, there exists a subset* $V \in B_0$ *such that* $-V \subseteq U$.
(5) *For any subset* $U \in B_0$*, there exists a subset* $V \in B_0$ *such that* $V \cdot V \subseteq U$.
(6) *For any subset* $U \in B_0$ *and any element* $a \in R$*, there exists a subset* $V \in B_0$ *such that* $a \cdot V \subseteq U$ *and* $V \cdot a \subseteq U$.

Proof. Since B_0 is a basis of α_β-neighborhoods of zero of the additive $\alpha_{(\beta,\beta)}$-topological group $R(+)$, the fulfillment of conditions (1) − (4) follows from Proposition 2.24. The fulfillment of conditions (5) and (6) results from definition of $\alpha_{(\beta,\beta)}$-topological ring with regard to $0 \cdot 0 = 0$ and $0 \cdot a = a \cdot 0 = 0$ for any $a \in R$. □

Proposition 3.19. *Let* R *be an* $\alpha_{(\beta,\beta)}$*-topological ring,* B_0 *be a basis of* α_γ*-neighborhoods of zero of an* $\alpha_{(\beta,\gamma)}$*-topological* R*-module* M *and* γ *be an* α*-regular operation on* $\alpha O(M)$*. Then conditions* (1) *to* (4) *of Proposition 2.24, are satisfied together with the following conditions:*

(1) *For any subset* $U \in B_0$*, there exist a subset* $V \in B_0$ *and an* α_β*-neighborhood* W *of zero in* R *such that* $W \cdot V \subseteq U$.
(2) *For any subset* $U \in B_0$ *and any element* $r \in R$*, there exists a subset* $V \in B_0$ *such that* $r \cdot V \subseteq U$.
(3) *For any subset* $U \in B_0$ *and any element* $a \in M$*, there exists an* α_β*-neighborhood* W *of zero in* R *such that* $W \cdot a \subseteq U$.

Proof. To prove these conditions, it is necessary to use Proposition 2.24, condition of Definition 3.7, and to take account of $0 \cdot a = r \cdot 0 = 0$ for any $r \in R$ and $a \in M$. □

Proposition 3.20. *Let* B_0 *be a basis of* α_β*-neighborhoods of zero of an* $\alpha_{(\beta,\beta)}$*-topological skew field* K *and* β *be an* α*-regular operation on* $\alpha O(K)$*, then conditions (1) to (6) of Proposition 3.18, are satisfied together with the following condition:*

- *For any* $U \in B_0$*, there exists* $V \in B_0$ *such that* $((1+V)\setminus\{0\})^{-1} \subseteq 1+U$.

Proof. The conditions (1) to (6) of Proposition 3.18 are satisfied since we have an $\alpha_{(\beta,\beta)}$-topological ring.
For the last condition,let $U \in B_0$, then $1 + U$ is an α_β-neighborhood of the unitary element on the strength of Proposition 2.25. Since $1^{-1} = 1$, then there exists an α_β-neighborhood W of the element 1 such that $(W \setminus \{0\})^{-1} \subseteq 1+U$. On the strength of Proposition 2.24, the family $B_1 = \{1+V|V \in B_0\}$ of subsets of the skew field K is a basis of α_β-neighborhoods of 1. Consequently, there exists $V \in B_0$ such that $1 + V \subseteq W$. Thus, $((1 + V) \setminus \{0\})^{-1} \subseteq (W \setminus \{0\})^{-1} \subseteq 1 + U$, concluding the proof. □

The proof of the following corollary is obvious and hence omitted.

Corollary 3.21. Let R be an $\alpha_{(\beta,\beta)}$-topological ring, M be an $\alpha_{(\beta,\gamma)}$-topological R-module, $a \in R$, $m \in M$, $S \subseteq R$, $N \subseteq M$, β an α-regular operation on $\alpha O(R)$ and γ an α-regular operation on $\alpha O(M)$. Let also $B_0(R)$ be a basis of α_β-neighborhoods of zero in R, and $B_0(M)$ be a basis of α_γ-neighborhoods of zero in M. Then, the following statements are true:

(1) R has a basis of α_β-neighborhoods of zero consisting of symmetric α_β-open neighborhoods.
(2) M has a basis of α_γ-neighborhoods of zero consisting of symmetric α_γ-open neighborhoods.
(3) R has a basis of α_β-neighborhoods of zero consisting of symmetric α_β-closed neighborhoods.
(4) M has a basis of α_γ-neighborhoods of zero consisting of symmetric α_γ-closed neighborhoods.
(5) The element a has a basis of α_β-neighborhoods consisting of α_β-open neighborhoods.
(6) The element m has a basis of α_γ-neighborhoods consisting of α_γ-open neighborhoods.
(7) The element a has a basis of α_β-neighborhoods consisting of α_β-closed neighborhoods.
(8) The element m has a basis of α_γ-neighborhoods consisting of α_γ-closed neighborhoods.
(9) $\alpha_\beta Cl(S) = \bigcap_{U \in B_0(R)}(S + U)$.
(10) $\alpha_\gamma Cl(N) = \bigcap_{V \in B_0(M)}(N + V)$.
(11) The subset $\bigcap_{U \in B_0(R)} U$ is α_β-closed in R.
(12) The subset $\bigcap_{V \in B_0(M)} V$ is α_γ-closed in M.

Proof. The proof is clear. □

Proposition 3.22. *Let a be an invertible element of an $\alpha_{(\beta,\beta)}$-topological ring R with the unitary element, $B_x(R)$ be a basis of α_β-neighborhoods of the element $x \in R$. Then, $\{a \cdot U | U \in B_x(R)\}$ and $\{U \cdot a | U \in B_x(R)\}$ are bases of α_β-neighborhoods of the elements $a \cdot x$ and $x \cdot a$, respectively. In particular, if $x = 0$, then, $\{a \cdot U | U \in B_0(R)\}$ and $\{U \cdot a | U \in B_0(R)\}$ are bases of α_β-neighborhoods of zero.*

Proof. The proof results from Corollary 3.13 and Corollary 3.14. □

Corollary 3.23. *Let β be an α-regular operation on $\alpha O(R)$, then for any $\alpha_{(\beta,\beta)}$-topological ring R, then the following statements are equivalent:*

(1) *R is an α_β-T_2-space.*
(2) *$\{0\}$ is α_β-closed subset in R.*
(3) *If B_0 is a basis of α_β-neighborhoods of 0 of R, then $\bigcap_{V \in B_0} V = \{0\}$.*
(4) *R is an α_β-T_0-space.*
(5) *R is an α_β-T_1-space.*

Proof. The proof is similar to the proof of Theorem 2.26. □

Definition 3.24. Let R be an $\alpha_{(\beta,\beta)}$-topological ring, M be an $\alpha_{(\beta,\gamma)}$-topological R-module. A subset Q of the ring R (a subset N of R-module M) is called a subring of the $\alpha_{(\beta,\beta)}$-topological ring R (a submodule of the $\alpha_{(\beta,\gamma)}$-topological R-module M) if Q is a subring of R (if N is a submodule of R-module M) and ring Q (R-module N) is endowed with the family $\alpha O(R)_\beta|Q$ ($\alpha O(M)_\gamma|N$) induced by the $\alpha O(R)_\beta$ ($\alpha O(M)_\gamma$).

Theorem 3.25. *Let Q be a subring of an $\alpha_{(\beta,\beta)}$-topological ring $(R,+,\cdot,\tau)$. Then $(Q,+,\cdot,\alpha O(R)_\beta|Q)$ is a topological ring.*

Proof. Due to Theorem 2.27, $(Q,+,\alpha O(R)_\beta|Q)$ is a topological abelian group. Let x,y be elements of Q and $U\in\alpha O(R)_\beta|Q$ containing $x\cdot y$, then $U=U_1\cap Q$, where $U_1\in\alpha O(R)_\beta$ containing $x\cdot y$. Let V_1 and W_1 be α_β-open sets containing x and y respectively such that $V_1\cdot W_1\subseteq U_1$. Then $V=V_1\cap Q$ and $W=W_1\cap Q$ are in $\alpha O(R)_\beta|Q$ containing x and y respectively, besides,

$$V\cdot W=(V_1\cap Q)\cdot(W_1\cap Q)\subseteq(V_1\cdot W_1)\cap Q\subseteq U_1\cap Q=U.$$

Thus, $(Q,+,\cdot,\alpha O(R)_\beta|Q)$ is a topological ring. □

Remark 3.26. Let N be a submodule of an $\alpha_{(\beta,\gamma)}$-topological R-module M. Then $(N,+,\cdot,\alpha O(M)_\gamma|N)$ is a topological R-module.

Definition 3.27. Let K be an $\alpha_{(\beta,\beta)}$-topological skew field (field). A subset H of K is called a skew subfield (a subfield) of the $\alpha_{(\beta,\beta)}$-topological skew field (field) K, if H is a skew subfield (subfield) of K and H is endowed with the family $\alpha O(K)_\beta|H$ induced by the $\alpha O(K)_\beta$.

Proposition 3.28. *Let H be a skew subfield (subfield) of an $\alpha_{(\beta,\beta)}$-topological skew field (field) $(K,+,\cdot,\tau)$. Then, $(H,+,\cdot,\alpha O(K)_\beta|H)$ is a topological skew field (field).*

Proof. By Theorem 3.25, $(H,+,\cdot,\alpha O(K)_\beta|H)$ is a topological ring.

Let $0\neq x\in H$ and $U^{'}\in\alpha O(K)_\beta|H$ containing the element x^{-1}. Then, there exists an α_β-open set U containing x^{-1} in (K,τ) such that $U\cap H=U^{'}$. Since K is an $\alpha_{(\beta,\beta)}$-topological skew field (field), then it is possible to find an α_β-open set V containing x in (K,τ) such that $(V\setminus\{0\})^{-1}\subseteq U$. Then, $V\cap H\in\alpha O(K)_\beta|H$ containing x, besides,

$$((V\cap H)\setminus\{0\})^{-1}=[(V\setminus\{0\})\cap(H\setminus\{0\})]^{-1}$$
$$=(V\setminus\{0\})^{-1}\cap(H\setminus\{0\})^{-1}\subseteq(V\setminus\{0\})^{-1}\cap H\subseteq U\cap H=U^{'}.$$

This completes the proof. □

Proposition 3.29. *Let Q be a subset of an $\alpha_{(\beta,\beta)}$-topological ring R, and N be a subset of an $\alpha_{(\beta,\gamma)}$-topological R-module M. If N is a Q-stable subset, then $\alpha_\gamma Cl(N)$ is an $\alpha_\beta Cl(Q)$-stable subset.*

Proof. Since N is a Q-stable subset, then $Q \cdot N \subseteq N$. Then, due to Proposition 3.10, $\alpha_\beta Cl(Q) \cdot \alpha_\gamma Cl(N) \subseteq \alpha_\gamma Cl(Q \cdot N) \subseteq \alpha_\gamma Cl(N)$, that is, $\alpha_\gamma Cl(N)$ is an $\alpha_\beta Cl(Q)$-stable subset. □

Proposition 3.30. *Let R be an $\alpha_{(\beta,\beta)}$-topological ring and M be an $\alpha_{(\beta,\gamma)}$-topological R-module. Let Q be a subring of the ring R, and N be a Q-submodule of R-module M, then:*

(1) *$\alpha_\beta Cl(Q)$ is a subring of the $\alpha_{(\beta,\beta)}$-topological ring R.*
(2) *$\alpha_\gamma Cl(N)$ is an $\alpha_\beta Cl(Q)$-module.*

Proof. By Proposition 2.29, $\alpha_\beta Cl(Q)$ and $\alpha_\gamma Cl(N)$ are subgroups of $\alpha_{(\beta,\beta)}$-topological abelian group R and $\alpha_{(\gamma,\gamma)}$-topological abelian group M respectively. Since Q is a Q-stable subset of the $\alpha_{(\beta,\beta)}$-topological R-module R and N is a Q-stable subset of the $\alpha_{(\beta,\gamma)}$-topological R-module M, then, due to Proposition 3.29:

(1) $\alpha_\beta Cl(Q)$ is an $\alpha_\beta Cl(Q)$-stable subset of the $\alpha_{(\beta,\beta)}$-topological R-module R, that is, $\alpha_\beta Cl(Q)$ is a subring of R.
(2) $\alpha_\gamma Cl(N)$ is an $\alpha_\beta Cl(Q)$-stable subset of the $\alpha_{(\beta,\gamma)}$-topological R-module M, and since $\alpha_\beta Cl(Q)$ is a subring of R, then $\alpha_\gamma Cl(N)$ is an $\alpha_\beta Cl(Q)$-module.

□

The proof of the following results are clear, so it is omitted.

Corollary 3.31. *Let R be an $\alpha_{(\beta,\beta)}$-topological ring and M be an $\alpha_{(\beta,\gamma)}$-topological R-module.*

(1) *Let Q be a subring of a ring R, $\alpha_\beta Cl(Q) = R$ and N be a Q-submodule of an $\alpha_{(\beta,\gamma)}$-topological R-module M. Then, $\alpha_\gamma Cl(N)$ is a submodule of the $\alpha_{(\beta,\gamma)}$-topological R-module.*
(2) *Let N be a submodule of R-module M. Then, $\alpha_\gamma Cl(N)$ is a submodule of the $\alpha_{(\beta,\gamma)}$-topological R-module.*

Corollary 3.32. *Let R be an $\alpha_{(\beta,\beta)}$-topological ring and I be a left (right, two-sided) ideal of the ring R. Then, $\alpha_\beta Cl(I)$ is a left (right, two-sided) ideal of the ring R.*

Corollary 3.33. *Let $B_0(M)$ be a basis of α_γ-neighborhoods of zero of an $\alpha_{(\beta,\gamma)}$-topological R-module M, then $M_0 = \bigcap_{V \in B_0(M)} V$ is the smallest α_γ-closed submodule of M.*

Proof. Due to Proposition 2.28, $M_0 = \alpha_\gamma Cl(\{0\})$, then, according to the Corollary 3.31 (2), M_0 is an α_γ-closed submodule of M. Let N be an α_γ-closed submodule of M, then, from $\{0\} \subseteq N$ results that $M_0 = \alpha_\gamma Cl(\{0\}) \subseteq \alpha_\gamma Cl(N) = N$, that is, M_0 is the smallest α_γ-closed submodule of M. □

Corollary 3.34. *Let $B_0(R)$ be a basis of α_β-neighborhoods of zero of the $\alpha_{(\beta,\beta)}$-topological ring R, then $R_0 = \bigcap_{V\in B_0(R)} V$ is the smallest α_β-closed two-sided ideal of R.*

Proof. The result follows from Corollary 3.33, considering R as a left and right $\alpha_{(\beta,\gamma)}$-topological R-module. □

Remark 3.35. If β is an α-regular operation on $\alpha O(R)$, then in view of Theorem 2.26, it is easy to see that in an $\alpha_{(\beta,\beta)}$-topological ring R ($\alpha_{(\beta,\gamma)}$-topological R-module M) the smallest α_β-closed ideal (smallest α_γ-closed submodule) equals zero if and only if R (R-module M) is α_β-T_2-space (α_γ-T_2-space).

The proof of the following result is clear, so it is omitted.

Corollary 3.36. *If a subgroup of the additive group of an $\alpha_{(\beta,\beta)}$-topological ring R ($\alpha_{(\beta,\gamma)}$-topological R-module M) is α_β-open (α_γ-open), then it is also α_β-closed (α_γ-closed). In particular, any α_β-open subring, any α_β-open left (right, two-sided) ideal of the ring R or α_γ-open submodule of any R-module M is α_β-closed (or α_γ-closed).*

Remark 3.37. An α_β-connected $\alpha_{(\beta,\beta)}$-topological ring R has no α_β-open subgroups of additive group, in particular α_β-open subrings, α_β-open left (right, two sided) ideals different from the R.

Remark 3.38. An α_γ-connected $\alpha_{(\beta,\gamma)}$-topological module M does not contain α_γ-open subgroups of the additive group, in particular α_γ-open submodules, different from M.

Corollary 3.39. *The α_γ-component containing zero of an $\alpha_{(\beta,\gamma)}$-topological R-module M is an α_γ-closed submodule, and the α_β-component containing zero of an $\alpha_{(\beta,\beta)}$-topological ring R is an α_β-closed two-sided ideal.*

Proof. Let $C(M)$ be the α_γ-component containing zero of an R-module M. Then, due to Proposition 2.30, $C(M)$ is an α_γ-closed subgroup of the additive group of M. Let $r \in R$, then the mapping $f_r : M \to M$, where $f_r(m) = r \cdot m$ for $m \in M$, is an $\alpha_{(\gamma,\gamma)}$-continuous mapping of the topological space M to itself. Then $r \cdot C(M) = f_r(C(M))$ is an α_γ-connected subset in M and besides, $0 \in r \cdot C(M)$. Therefore, $r \cdot C(M) \subseteq C(M)$, that is, $C(M)$ is an α_γ-closed submodule of the module M.

Considering R as left and right $\alpha_{(\beta,\beta)}$-topological R-modules, we obtain that the α_β-component $C(R)$ of the $\alpha_{(\beta,\beta)}$-topological ring R is an α_β-closed two-sided ideal of R. □

Proposition 3.40. *Let R be an $\alpha_{(\beta,\beta)}$-topological ring, N be an α_γ-closed non-empty subset of an $\alpha_{(\beta,\gamma)}$-topological R-module M, $x \in M$, $a \in R$ and γ be an α-regular operation on $\alpha O(M)$. Then, the subset $(N : x)_R$ is α_β-closed in R, and the subset $(N : a)_M$ is α_γ-closed in M.*

Proof. Let $B_0(M)$ be a basis of α_γ-neighborhoods of zero of M, $r \in \alpha_\beta Cl((N : x)_R)$, $m \in \alpha_\gamma Cl((N : a)_M)$ and $U \in B_0(M)$. Due to conditions Proposition 3.19, there exist α_β-neighborhood of zero V_u in R and α_γ-neighborhood of zero W_u in M such that $V_u \cdot x \subseteq U$ and $a \cdot W_u \subseteq U$. We can choose elements $r_u \in (N : x)_R$ and $m_u \in (N : a)_M$ such that $r - r_u \in V_u$, and $m - m_u \in W_u$. Then $r \cdot x - r_u \cdot x = (r - r_u) \cdot x \in V_u \cdot x \subseteq U$ and $a \cdot m - a \cdot m_u = a \cdot (m - m_u) \in a \cdot W_u \subseteq U$. Since $r_u \cdot x \in N$ and $a \cdot m_u \in N$, then $r \cdot x \in r_u \cdot x + U \subseteq N + U$, and analogously $a \cdot m \in N + U$. Hence,

$$r \cdot x, a \cdot m \in \bigcap_{U \in B_0(M)} (N + U) = \alpha_\gamma Cl(N) = N,$$

that is, $r \in (N : x)_R$ and $m \in (N : a)_M$. Thus, $(N : x)_R$ is α_β-closed in the $\alpha_{(\beta,\beta)}$-topological ring R and $(N : a)_M$ is α_γ-closed in the $\alpha_{(\beta,\gamma)}$-topological module M. □

Corollary 3.41. *Let R be an $\alpha_{(\beta,\beta)}$-topological ring, N be an α_γ-closed non-empty subset of an $\alpha_{(\beta,\gamma)}$-topological R-module M, $X \subseteq M$, $A \subseteq R$, β be an α-regular operation on $\alpha O(R)$ and γ be an α-regular operation on $\alpha O(M)$. Then, the following statements are true:*

(1) *$(N : X)_R$ is an α_β-closed subset in R, and $(N : A)_M$ is an α_γ-closed subset in M.*
(2) *If N is a subgroup of the additive group M, then $(N : X)_R$ is an α_β-closed subgroup of the additive group of R and $(N : A)_M$ is an α_γ-closed subgroup of the additive group of M.*
(3) *If N is a subgroup of the additive group of M and A is a right ideal of R, then $(N : A)_M$ is an α_γ-closed submodule of M.*
(4) *If N is a submodule of M, then $(N : X)_R$ is an α_β-closed left ideal of R.*
(5) *If X and N are submodules of M, then $(N : X)_R$ is an α_β-closed two-sided ideal of R.*

Proof. Since $(N : X)_R = \bigcap_{x \in X} (N : x)_R$ and $(N : A)_M = \bigcap_{a \in A} (N : a)_M$, then $(N : X)_R$ is α_β-closed and $(N : A)_M$ is α_γ-closed by Proposition 3.40. Thus, the statement (1) is proved. The statements (2)-(5) result from (1) and from the corresponding statements of Definition 2.15. □

Corollary 3.42. *Let R be an $\alpha_{(\beta,\beta)}$-topological ring, $A \subseteq R$, and let X be a subset of an $\alpha_\gamma T_2$ $\alpha_{(\beta,\gamma)}$-topological R-module M. Let β be an α-regular operation on $\alpha O(R)$ and γ be an α-regular operation on $\alpha O(M)$. Then, the following statements are true:*

(1) *$(0 : X)_R$ is an α_β-closed left ideal of R.*
(2) *If X is a submodule of M, then $(0 : X)_R$ is an α_β-closed two-sided ideal of R.*
(3) *If A is a right ideal of R, then $(0 : A)_M$ is an α_γ-closed submodule of M.*

Proof. The proof is similar to Corollary 3.41. □

Corollary 3.43. *In an $\alpha_\beta T_2$ $\alpha_{(\beta,\beta)}$-topological ring R a left annihilator $(0 : A)_R$ of any non-empty subset $A \subseteq R$ is an α_β-closed left ideal of R, where β is an α-regular operation on $\alpha O(R)$.*

Proof. The proof is similar to Corollary 3.41. □

Acknowledgments

The author would like to thank the anonymous referees for their constructive comments.

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