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Roman k-tuple Domination in Graphs

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Abstract. For any integer $k \geq 1$ and any graph $G = (V, E)$ with minimum degree at least $k - 1$, we define a function $f : V \to \{0, 1, 2\}$ as a Roman k-tuple dominating function on G if for any vertex v with $f(v) = 0$ there exist at least k and for any vertex v with $f(v) \neq 0$ at least $k - 1$ vertices w in its neighborhood with $f(w) = 2$. The minimum weight of a Roman k-tuple dominating function f on G is called the Roman k-tuple domination number of the graph where the weight of f is $f(V) = \sum_{v \in V} f(v)$.

In this paper, we initiate to study the Roman k-tuple domination number of a graph, by giving some tight bounds for the Roman k-tuple domination number of a garph, the Mycieleskian of a graph, and the corona graphs. Also finding the Roman k-tuple domination number of some known graphs is our other goal. Some of our results extend these one given by Cockayne and et al. [1] in 2004 for the Roman domination number.

Keywords: Roman k-tuple domination number, Roman k-tuple graph, k-Tuple domination number, k-Tuple total domination number.

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1. Introduction

All graphs considered here are finite, undirected and simple. For standard graph theory terminology not given here we refer to [9]. Let $G = (V, E)$ be a graph with the *vertex set* V of *order* $n(G)$ and the *edge set* E of *size* $m(G)$.

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The *open neighborhood* of a vertex $v \in V$ is $N_G(v) = \{u \in V \mid uv \in E\}$, while its cardinality is the *degree* of v. The *closed neighborhood* of v is defined by $N_G[v] = N_G(v) \cup \{v\}$. Similarly, the *open* and *closed neighborhoods* of a subset $X \subseteq V$ are $N_G(X) = \cup_{v\in X} N_G(v)$ and $N_G[X] = N_G(X) \cup X$, respectively. The *minimum* and *maximum degree* of G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. If $\delta = \Delta = k$, then G is called *k-regular*. We write K_n, C_n, P_n, and W_n for a *complete graph*, a *cycle*, a *path*, and a *wheel* of order n, respectively, while $K_{n_1,\ldots,n_p}$ denotes a *complete p-partite graph*. Also $G[S]$ and $\overline{G}$ denote the subgraph induced by a subset $S \subseteq V$ and the *complement* of G, respectively. Also $G \cong H$ means that two graphs G and H are isomorphic.

For any integer $k \geq 1$, the *k-join* $G \circ_k H$ of a graph G to a graph H of order at least k is the graph obtained from the disjoint union of G and H and joining each vertex of G to at least k vertices of H [5].

Domination in graphs is now well studied in graph theory and the literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi, and Slater [3, 4]. One type of domination is k-tuple domination number that was introduced by Harary and Haynes [2].

Definition 1.1. [2] *For any positive integer k, a subset $S \subseteq V$ is a* k-tuple dominating set, abbreviated kDS, *of the graph G, if $|N_G[v] \cap S| \geq k$ for every $v \in V$. The* k-tuple domination number $\gamma_{\times k}(G)$ *of G is the minimum cardinality among the k-tuple dominating sets of G.*

Henning and Kazemi in [5] introduced another type of domination called k-tuple total domination number of a graph which is an extension of the total domination number.

Definition 1.2. [5] *For any integer $k \geq 1$, a subset $S \subseteq V$ is called a* k-tuple total dominating set, *abbreviated kTDS, of G if for every vertex $v \in V$, $|N(v) \cap S| \geq k$. The* k-tuple total domination number $\gamma_{\times k,t}(G)$ *of G is the minimum cardinality of a kTDS of G.*

Note that the 1-tuple domination number (1-tuple total domination number) is the classical domination number $\gamma(G)$ (total domination number $\gamma_t(G)$). A kDS (kTDS) of minimum cardinality of a graph G is called a *min-*kDS or $\gamma_{\times k}(G)$*-set* (*min-*kTDS or $\gamma_{\times k,t}(G)$*-set*).

According to [1], Constantine the Great (Emperor of Rome) issued a decree in the 4th century A.D. for the defense of his cities. He decreed that any city without a legion stationed to secure it must neighbor another city having two stationed legions. If the first were attacked, then the second could deploy a legion to protect it without becoming vulnerable itself. The objective, of course, is to minimize the total number of legions needed. According to it, Ian Steward by an article in Scientific American, entitled Defend the Roman Empire! [8] suggested the Roman dominating function.

In [6], Kämmerling and Volkmann extended the Roman dominating function to the *Roman k-dominating function* in this way that for any vertex v with $f(v) = 0$ there are at least k vertices w in its neighborhood with $f(w) = 2$, and they defined the *Roman k-domination number* $\gamma_{kR}(G)$ of a graph G as the minimum weight of a Roman k-dominating function f on G where the *weight* of f is $f(V) = \sum_{v \in V} f(v)$.

This problem that for securing a city without a legion stationed or a city with at least one legion stationed we need at least, respectively, k or $k-1$ cities having two stationed legions, is our motivation to define the concept of Roman k-tuple domination number which is another extension of the Roman domination number.

Definition 1.3. *For any integer $k \geq 1$, a* Roman k-tuple dominating function, *abbreviated RkDF, on a graph G with minimum degree at least $k-1$ is a function $f\colon V \to \{0, 1, 2\}$ such that for any vertex v with $f(v) = 0$ there exist at least k and for any vertex v with $f(v) \neq 0$ there exist at least $k-1$ vertices w in its neighborhood with $f(w) = 2$. The* Roman k-tuple domination number *$\gamma_{\times kR}(G)$ of a graph G is the minimum weight of a RkDF f on G.*

The Roman 1-tuple domination number is the usual *Roman domination number* $\gamma_R(G)$.

A *min*-RkDF is a RkDF with the minimum weight. For a RkDF f let (V_0, V_1, V_2) be the ordered partition of V induced by f where $V_i = \{v \in V \mid f(v) = i\}$ for $i = 0, 1, 2$. Since there is a one-to-one correspondence between the function f and the ordered partitions (V_0, V_1, V_2) of V, we will write $f = (V_0, V_1, V_2)$. Figure 1 shows a min-R2DF of cycle C_{10}.

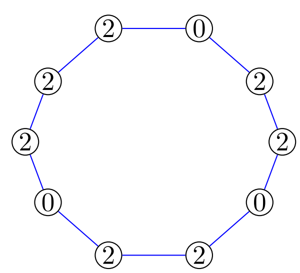

Figure 1. $\gamma_{\times 2R}(C_{10}) = 14$

In this paper, we initiate to study the Roman k-tuple domination number of a graph, by giving some tight bounds for the Roman k-tuple domination number of a garph, the Mycieleskian of a graph, and the corona graphs. Also finding the Roman k-tuple domination number of some known graphs is our other goal. Some of our results extend these one given by Cockayne and et al. [1] in 2004 for the Roman domination number.

2. General results

In this section, we state some properties of the Roman k-tuple dominating functions, and some tight bounds for the Roman k-tuple domination number of a graph.

Proposition 2.1. *For any min-RkDF $f = (V_0, V_1, V_2)$ on a graph G with $\delta(G) \geq k-1$, the following statements hold.*
(a) $\gamma_{\times kR}(G) \geq \gamma_{kR}(G)$.
(b) $V_1 \cup V_2$ *is a kDS of* G.
(c) V_2 *is a kDS of* $G[V_0 \cup V_2]$.
(d) *For* $k \geq 2$, V_2 *is a* $(k-1)$*TDS of* G.
(e) *Every vertex of degree* $k-1$ *belongs to* $V_1 \cup V_2$.
(f) $G[V_1]$ *has maximum degree* 1.
(g) *Every vertex in* V_1 *is adjacent to precisely* $k-1$ *vertices in* V_2.
(h) *Each vertex in* V_0 *is adjacent to at most two vertices in* V_1.

Proof. We omit the proofs of (a)-(e); they are clear. Let $f = (V_0, V_1, V_2)$ be any min-RkDF of G.

(f) For any $x \in V_1$, since $f' = (V_0 \cup (N(x) \cap V_1), V_1 - N(x), V_2 \cup \{x\})$ with the value $f'(V) = f(V) - d + 1$, is a RkDF on G if and only if $d \leq 1$, we conclude that $G[V_1]$ has maximum degree 1.

(g) For any $x \in V_1$, let $|N(x) \cap V_2| = d$. Then $d \geq k-1$. Since $d \geq k$, for some $x \in V_1$, implies that $f' = (V_0 \cup \{x\}, V_1 - \{x\}, V_2)$ is a RkDF on G with the value $f'(V) = f(V) - 1 = \gamma_{\times kR}(G) - 1$, we obtain $d = k-1$.

(h) For $x \in V_0$, let $|N(x) \cap V_1| = d$. Since $f' = (V_0 \cup (N(x) \cap V_1), V_1 - N(x), V_2 \cup \{x\})$ is a RkDF on G with the value $f'(V) = f(V) - d + 2 \geq f(V)$, we have $d \leq 2$. □

As a consequence of Proposition 2.1 (c),(d), we have the following result.

Corollary 2.2. *If G is a Roman k-tuple graph, that is $\gamma_{\times kR}(G) = 2\gamma_{\times k}(G)$, then*

$$2 \max\{\gamma_{\times (k-1),t}(G), \gamma_{\times k}(G)\} \leq \gamma_{\times kR}(G).$$

For any graph $G = (V, E)$ of order n and with minimum degree at least $k-1 \geq 1$, since $(\emptyset, \emptyset, V)$ is a RkDF on G, we have $\gamma_{\times kR}(G) \leq 2n$. On the other hand, since for any RkDF $f = (V_0, V_1, V_2)$, $|V_2| \geq k$, we have $\gamma_{\times kR}(G) \geq 2k$. Also, it can be easily verified that $\gamma_{\times kR}(G) = 2k$ if and only if $G = K_k$ or $G = H \circ_k K_k$ for some graph H. Therefore we have proved next theorem.

Theorem 2.3. *For any graph G of order n and with $\delta(G) \geq k-1 \geq 1$,*

$$2k \leq \gamma_{\times kR}(G) \leq 2n,$$

and $\gamma_{\times kR}(G) = 2k$ if and only if $G = K_k$ or $G = H \circ_k K_k$ for some graph H.

Theorem 2.3 characterizes graphs G with $\gamma_{\times kR}(G) = 2k$. Next proposition characterizes graphs G with $\gamma_{\times kR}(G) = 2k+1$. First we construct a graph.

Graphs $\mathcal{A}_k$. Let $n \geq k+1 \geq 3$. For $n = k+1$ let $\mathcal{A}_k$ be the complete graph K_{k+1} minus an edge, and for $n > k+1$ let $\mathcal{A}_k$ be the graph with the vertex set $V = \{v_i \mid 1 \leq i \leq n\}$ such that the induced subgraph $\mathcal{A}_k[\{v_i \mid 1 \leq i \leq k+1\}] \cong K_{k+1} - \{v_k v_{k+1}\}$, and for any $i \geq k+2$, $\{v_j \mid 1 \leq j \leq k\} \subseteq N_{\mathcal{A}_k}(v_i)$.

Proposition 2.4. *For any graph G with $\delta(G) \geq k-1 \geq 1$, $\gamma_{\times kR}(G) = 2k+1$ if and only if $G \cong \mathcal{A}_k$.*

Proof. Let G be a graph with $\delta(G) \geq k-1 \geq 1$. If $G \cong \mathcal{A}_k$, then obviousely (V_0, V_1, V_2) is a min-RkDF on G where $V_2 = \{v_i \mid 1 \leq i \leq k\}$, $V_1 = \{v_{k+1}\}$ and $V_0 = V(\mathcal{A}_k) - V_1 \cup V_2$, and so $\gamma_{\times kR}(G) = 2k+1$.

Conversely, let $f = (V_0, V_1, V_2)$ be a min-RkDF on G with weight $2k+1$. Hence $|V_2| = k$ and $|V_1| = 1$. If $V_2 = \{v_i \mid 1 \leq i \leq k\}$ and $V_1 = \{v_{k+1}\}$, then $\gamma_{\times kR}(G) = 2k+1$ implies that there exists a vertex in V_2, say v_k, which is not adjacent to v_{k+1}, that is, $G \cong \mathcal{A}_k$. □

Note that if $k \geq 2$ and G is $(k-1)$-regular, then $\gamma_{\times kR}(G) = 2n$. We will show that its converse holds only for $k = 2$. For $k \geq 3$, for example, if G is a graph which is obtained by the complete bipartite graph $K_{k,k}$ minus a matching of size $k-1$, then $\gamma_{\times kR}(G) = 4k$ while G is not $(k-1)$-regular.

Proposition 2.5. *For any graph G of order n and without isolate vertex, $\gamma_{\times 2R}(G) = 2n$ if and only if $G = \ell K_2$ for some $\ell \geq 1$.*

Proof. Let $G = (V, E)$ be a graph of order n and without isolate vertex, and let $\gamma_{\times 2R}(G) = 2n$. Since $deg(w) \geq 2$, for some vertex w, implies that the function $(\{w\}, \emptyset, V - \{w\})$ is a R2DF on G with weight less than $2n$, we conclude $G = \ell K_2$ for some $\ell \geq 1$. Since the proof of inverse case is trivial, we have completed our proof. □

Cockayne and et al. in [1] proved that for any graph G,

$$\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G). \tag{2.1}$$

As an extension of inequality (2.1), next theorem improves the lower bound $2k$ given in Theorem 2.3 for $k \geq 2$.

Theorem 2.6. *For any graph G with $\delta(G) \geq k-1 \geq 1$,*

$$\gamma_{\times k}(G) + k \leq \gamma_{\times kR}(G) \leq 2\gamma_{\times k}(G),$$

and the lower bound is tight.

Proof. Since for any min-kDS S of $G = (V, E)$, the function $f = (V - S, \emptyset, S)$ is a RkDF on G, we have $\gamma_{\times kR}(G) \leq 2|S| = 2\gamma_{\times k}(G)$. On the other hand, since for any min-RkDF $f = (V_0, V_1, V_2)$ on G, $V_1 \cup V_2$ is a kDS of G, we have

$$\gamma_{\times kR}(G) = 2|V_2| + |V_1| \geq \gamma_{\times k}(G) + |V_2| \geq \gamma_{\times k}(G) + k.$$

For any graph H of order k, the lower bound is tight for $G = \overline{K_k} \circ_{*(k-1)} K_k$. Because $(\emptyset, V(\overline{K_k}), V(K_k))$ is a min-RkDF on G and $V(K_k)$ is a min-kDS of G. □

Following E. J. Cockayne, P. A. Dreyer Jr., S. M. Hedetniemi and S. T. Hedetniemi [1], we will say that a graph G is a *Roman k-tuple graph* if $\gamma_{\times kR}(G) = 2\gamma_{\times k}(G)$. Next proposition characterizes the Roman k-tuple graphs.

Proposition 2.7. *A graph G with $\delta(G) \geq k-1$ is a Roman k-tuple graph if and only if it has a min-RkDF $f = (V_0, \emptyset, V_2)$, that is, V_2 is a min-kDS of G.*

Proof. Let G be a Roman k-tuple graph, and let S be a min-kDS of G. Since $f = (V - S, \emptyset, S)$ is a RkDF on G with weight $f(V) = 2|S| = 2\gamma_{\times k}(G) = \gamma_{\times kR}(G)$, we conclude that f is a min-RkDF.

Conversely, if $f = (V_0, \emptyset, V_2)$ is a min-RkDF on G, then $\gamma_{\times kR}(G) = 2|V_2|$, and V_2 is a kDS of G. Hence $\gamma_{\times k}(G) \leq |V_2| = \gamma_{\times kR}(G)/2$. Applying Theorem 2.6 implies $\gamma_{\times kR}(G) = 2\gamma_{\times k}(G)$, that is, G is a Roman k-tuple graph. □

Corollary 2.8. [1] *A graph G is a Roman graph if and only if it has a min-RDF $f = (V_0, \emptyset, V_2)$.*

3. Complete bipartite graphs, paths, cycles and wheels

Here, we calculate the Roman k-tuple domination number of a complete bipartite graph, a cycle, a path, and a wheel.

Proposition 3.1. *For any integer $n \geq m \geq k-1 \geq 1$,*

$$\gamma_{\times kR}(K_{n,m}) = \begin{cases} 3k - 3 + n & \text{if } n \geq m = k-1, \\ 4k - 2 & \text{if } n \geq m = k, \\ 4k - 1 & \text{if } n = m = k+1, \\ 4k & \text{if } n > m \geq k+1. \end{cases}$$

Proof. Assume that $V(K_{n,m})$ is partitioned to the independent sets X and Y such that $|X| = n$ and $|Y| = m$. Since the Roman k-tuple dominating functions given in each of the following cases have minimum weight, our proof is completed.

- $n \geq m = k-1$. Consider $f = (\emptyset, \emptyset, X \cup Y)$ when $n = m$, and consider $f = (\emptyset, V_1, V_2)$ when $n > m$ in which $Y \subseteq V_2$, $|V_2 \cap X| = k-1$ and $V_1 = X - V_2$.
- $n \geq m = k$. Consider $f = (V_0, \emptyset, V_2)$ where $|V_2 \cap Y| = k$, $|V_2 \cap X| = k-1$ and $V_0 = X \cup Y - V_2$.

- $n = m = k+1$. Consider $f = (V_0, V_1, V_2)$ where $|V_2 \cap Y| = k$, $|V_2 \cap X| = k-1$, $V_1 = Y - V_2$ and $V_0 = X \cup Y - V_1 \cup V_2$.
- $n > m \geq k$. Consider $f = (V_0, \emptyset, V_2)$ where $|V_2 \cap X| = |V_2 \cap Y| = k$ and $V_0 = X \cup Y - V_2$.

□

Corollary 3.2. *If $n > m \geq k+1 \geq 3$, then $\gamma_{\times kR}(K_{n,m}) = k\gamma_R(K_{n,m})$.*

Proof. It is sufficient to consider

$$\gamma_{\times R}(K_{n,m}) = \begin{cases} 2 & \text{if } n \geq m = 1, \\ 3 & \text{if } n \geq m = 2, \\ 4 & \text{if } n \geq m \geq 3. \end{cases}$$

□

In the next two propositions, we will calculate $\gamma_{\times 2R}(C_n)$ and $\gamma_{\times 2R}(P_n)$ (notice $\gamma_{\times 3R}(C_n) = 2n$ by Proposition 2.1).

Proposition 3.3. *For any cycle C_n of order $n \geq 3$, $\gamma_{\times 2R}(C_n) = 2\lceil \frac{2n}{3} \rceil$.*

Proof. Let C_n be a cycle with $V(C_n) = \{1, 2, ..., n\}$ and $E(C_n) = \{ij \mid j \equiv i+1 \pmod{n},\ 1 \leq i \leq n\}$. Since $(V_0, \emptyset, V(C_n) - V_0)$ is a R2DF on C_n where $V_0 = \{3t+1 \mid 0 \leq t \leq \lfloor \frac{n}{3} \rfloor - 1\}$, we have $\gamma_{\times 2R}(C_n) \leq 2\lceil \frac{2n}{3} \rceil$.

On the other hand, since in each R2DF every three consecutive vertices have at least weight 4, we have $\gamma_{\times 2R}(C_n) \geq \lceil \frac{4n}{3} \rceil$. Since $\lceil \frac{4n}{3} \rceil = 2\lceil \frac{2n}{3} \rceil$ where $n \not\equiv 2 \pmod 3$, we consider $n \equiv 2 \pmod 3$. Then $\lceil \frac{4n}{3} \rceil = 2\lceil \frac{2n}{3} \rceil - 1$. Now let $f = (V_0, V_1, V_2)$ be a min-R2DF on C_n. Since every vertex in V_2 is adjacent to at least one vertex in V_2 and f has minimum weight, we conclude $i-1, i \in V_2$ implies $i+1 \in V_0$ as possible as. Therefore for $0 \leq t \leq \lfloor \frac{n}{3} \rfloor - 1$, $f(3t+1) = 0$ and $f(3t) = f(3t+2) = 2$. This implies $f(n-2) = f(n-1) = 2$, and so $\gamma_{\times 2R}(C_n) = f(V(C_n)) = \lceil \frac{4n}{3} \rceil + 1 = 2\lceil \frac{2n}{3} \rceil$. □

Proposition 3.4. *For any path P_n of order $n \geq 2$,*

$$\gamma_{\times 2R}(P_n) = \begin{cases} 2\lceil \frac{2n}{3} \rceil & \text{if } n \equiv 1, 2 \pmod 3, \\ 2\lceil \frac{2n}{3} \rceil + 1 & \text{if } n = 3, \\ 2\lceil \frac{2n}{3} \rceil + 2 & \text{otherwise.} \end{cases}$$

Proof. Let P_n be a path with $V(P_n) = \{1, 2, ..., n\}$ and let $E(P_n) = \{ij \mid j = i+1,\ 1 \leq i \leq n-1\}$. Since $(\emptyset, \emptyset, V(P_n))$ is the only min-R2DF on P_2 and $(\emptyset, \{1\}, \{2, 3\})$ is a min-R2DF on P_3, we consider $n \geq 4$. Let $f = (V_0, V_1, V_2)$ be a min-R2DF on P_n. Then $f(1) = f(n) = 1$, and $f(2) = f(3) = f(n-2) = f(n-1) = 2$, which implies $\gamma_{\times 2R}(P_4) = 6$, $\gamma_{\times 2R}(P_5) = 8$, $\gamma_{\times 2R}(P_6) = 10$, as desired. So, we assume $n \geq 7$. Let $\mathcal{L} = V(P_n) - \{1, 2, 3, n-2, n-1, n\}$. Since every three consecutive vertices in $\mathcal{L}$ have at least weight 4 and every two consecutive vertices in it have at least weight 2, we conclude (V_0, V_1, V_2)

is a min-R2DF on P_n where $V_0 = \{3t+1 \mid 1 \leq t \leq \lfloor \frac{n-1}{3} \rfloor - 1\}$, $V_1 = \{1, n\}$, $V_2 = V(P_n) - V_0 \cup V_1$, and this completes our proof. □

Since it can be easily verified that for any $n \geq 3$,

$$\gamma_{\times 2}(C_n) = \begin{cases} \lceil \frac{2n}{3} \rceil & \text{if } n \text{ is odd,} \\ \lfloor \frac{2n}{3} \rfloor & \text{if } n \text{ is even,} \end{cases}$$

and for any $n \geq 2$,

$$\gamma_{\times 2}(P_n) = \begin{cases} \lceil \frac{2n}{3} \rceil & \text{if } n \equiv 0, 2, 5, 8 \pmod 9, \\ \lceil \frac{2n}{3} \rceil + 1 & \text{otherwise,} \end{cases}$$

by Propositions 3.3 and 3.4, the next result charactrizes cycles and paths which are Roman 2-tuple graph.

Proposition 3.5. *i. Any cycle C_n is a Roman 2-tuple graph if and only if $n \not\equiv 2, 4 \pmod 6$.*

ii. Any path P_n is a Roman 2-tuple graph if and only if $n \neq 3$ and $n \not\equiv 0, 1, 4, 7 \pmod 9$.

Finally, we consider wheels. We recall that W_n denotes a wheel of order $n \geq 4$ with $V(W_n) = \{v_0, v_1, \cdots, v_{n-1}\}$ such that $deg(v_0) = n-1$ and $deg(v_i) = 3$ for $1 \leq i \leq n-1$. Here, we calculate $\gamma_{\times kR}(W_n)$ when $1 \leq k \leq 4$ (because $\delta(W_n) = 3 \geq k-1$). Since $\gamma_R(W_n) = 2$ and $\gamma_{\times 4R}(W_n) = 2n$, we consider $k = 2, 3$. First we recall a result from [6].

Lemma 3.6. [6] *For any wheel W_n of order $n \geq 4$ and any integer k,*

$$\gamma_{kR}(W_n) = \begin{cases} 2 & \text{if } k = 1, \\ \lceil \frac{2n+4}{3} \rceil & \text{if } k = 2, \\ n & \text{if } k \geq 3. \end{cases}$$

Proposition 3.7. *For any wheel W_n of order $n \geq 4$,*

$$\gamma_{\times kR}(W_n) = \begin{cases} \lceil \frac{2n+4}{3} \rceil & \text{if } k = 2, \\ 2n - 2\lfloor \frac{n-1}{3} \rfloor & \text{if } k = 3. \end{cases}$$

Proof. We prove in the following two cases.

- $k = 2$. Let $X = \{v_{3t+1} \mid 0 \leq t \leq \lfloor \frac{n-1}{3} \rfloor - 1\} \cup \{v_0\}$ and let $V_0 = V(W_n) - (V_1 \cup V_2)$ in a R2DF (V_0, V_1, V_2) on W_n. Since

$$f = (V_0, V_1, V_2) = \begin{cases} (V_0, \emptyset, X \cup \{v_{n-2}\}) & \text{if } n \equiv 0 \pmod 3, \\ (V_0, \emptyset, X) & \text{if } n \equiv 1 \pmod 3, \\ (V_0, \{v_{n-2}\}, X) & \text{if } n \equiv 2 \pmod 3, \end{cases}$$

is a R2DF on W_n with weight $\lceil \frac{2(n-1)}{3} \rceil + 2 = \lceil \frac{2n+4}{3} \rceil$, we obtain $\gamma_{\times 2R}(W_n) = \lceil \frac{2n+4}{3} \rceil$, by Proposition 2.1-(a) and Lemma 3.6.

- $k = 3$. Let $f = (V_0, V_1, V_2)$ be a minimal R3DF on W_n. Since every vertex, except probably v_0, has degree 3 and v_0 is adjacent to all other $n-1$ vertices, we conclude $v_0 \in V_2$. Also, we know $v_i \in V_2$, for some $1 \leq i \leq n-1$, implies $v_{i-1}, v_{i+1} \in V_2$. By considering these facts and the minimality of the weight of f, we obtain $V_1 = \emptyset$ and $|V_0| \leq \lfloor \frac{n-1}{3} \rfloor$. Hence $\gamma_{\times 3R}(W_n) \geq 2|V_2| = 2n - 2|V_0| \geq 2n - 2\lfloor \frac{n-1}{3} \rfloor$. On the other hand, since $(\{v_{3t+1} \mid 0 \leq t \leq \lfloor \frac{n-1}{3} \rfloor - 1\}, \emptyset, V(W_n) - V_0)$ is a R3DF on W_n with weight $2n - 2\lfloor \frac{n-1}{3} \rfloor$, we obtain $\gamma_{\times 3R}(W_n) = 2n - 2\lfloor \frac{n-1}{3} \rfloor$.

□

4. Mycieleskian of a Graph

In this section, we give some shap bounds for the Roman k-tuple domination number of the Mycieleskian of a graph in terms of the same number of the graph and k. Also we present the Roman k-tuple domination number of the Mycieleskian of complete graphs. First we define the Mycieleskian of a graph.

Definition 4.1. [9] The *Mycieleskian* $M(G)$ of a graph $G = (V, E)$ is a graph with vertex set $V \cup U \cup \{w\}$, and edge set $E \cup \{u_j v_i \mid v_j v_i \in E \text{ and } u_j \in U\} \cup \{u_j w \mid u_j \in U\}$ where $U = \{u_j \mid v_j \in V\}$ and $(\{w\} \cup U) \cap V = \emptyset$.

Figure 2 shows the Myclieskian of K_5.

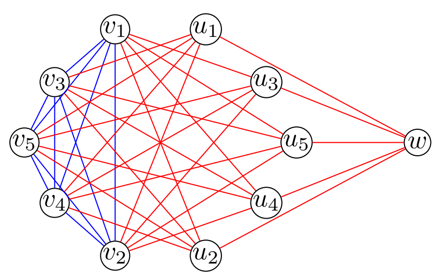

Figure 2. The Myclieskian of K_5

Theorem 4.2. *For any graph G with $\delta(G) \geq k-1 \geq 1$,*

$$\gamma_{\times kR}(G) + \min\{k-1, 2\} \leq \gamma_{\times kR}(M(G)) \leq \gamma_{\times kR}(G) + 2k.$$

Proof. Let G be a graph with $\delta(G) \geq k-1 \geq 1$ and vertex set $V = \{v_i \mid 1 \leq i \leq n\}$. Since for any min-R$k$DF $f = (V_0, V_1, V_2)$ on G, the function $g = (W_0, W_1, W_2)$ is a RkDF on $M(G)$ with weight $\gamma_{\times kR}(G) + 2k$ where $W_2 = V_2 \cup U' \cup \{w\}$ for some subset $U' \subseteq U$ of cardinality $k-1$, $W_1 = V_1$ and $W_0 = V_0 \cup (U - U')$, we obtain $\gamma_{\times kR}(M(G)) \leq \gamma_{\times kR}(G) + 2k$.

Now let $f = (V_0, V_1, V_2)$ be a min-RkDF on $M(G)$ such that $|V_1 \cap U|$ and $|V_2 \cap U|$ is as possible as minimum. Let $L = \{i \mid u_i \in V_1\}$, $L' = \{i \mid v_i \in V_1\}$,

$T = \{i \mid u_i \in V_2\}$, and $T' = \{i \mid v_i \in V_2\}$ where $|T| = t \geq k-1$(because of $N_{M(G)}(w) = U$), $|T'| = t'$, $|L| = \ell$, and $|L'| = \ell'$. By proving $\gamma_{\times kR}(M(G)) \geq \gamma_{\times kR}(G)+\min\{k-1, 2\}$ in the following three cases, our proof will be completed.

Case 1. $w \in V_0$. Then $t \geq k$ and

$$|N_{M(G)}(v_i) \cap V_2 \cap V| \begin{cases} = k-1 & \text{if } i \in L, \\ \geq k-1 & \text{if } i \in T, \\ \geq k & \text{if } i \notin L \cup T. \end{cases}$$

Let

$$L_0 = \{v_i \in V_0 \mid i \in L\} \cup \{v_i \in V_0 \mid i \in T, \text{ and } |N_{M(G)}(v_i) \cap V_2 \cap V| = k-1\}$$

be a set of cardinality ℓ_0. Then $\ell \leq \ell_0 \leq \ell + t$. By choosing $V_2' = V_2 \cap V$, $V_1' = (V_1 \cap V) \cup L_0$, $V_0' = V - (V_1' \cup V_2')$, since $f' = (V_0', V_1', V_2')$ is a RkDF on G, we have

$$\begin{aligned} \gamma_{\times kR}(G) &\leq f'(V) \\ &= \gamma_{\times kR}(M(G)) + \ell_0 - \ell - 2t. \end{aligned}$$

Hence

$$\begin{aligned} \gamma_{\times kR}(M(G)) &\geq \gamma_{\times kR}(G) + 2t + \ell - \ell_0 \\ &\geq \gamma_{\times kR}(G) + t \\ &\geq \gamma_{\times kR}(G) + k. \end{aligned}$$

Case 2. $w \in V_1$. Then $t = k-1$, and $\ell \leq 1$. Because if $\ell \geq 2$, then by choosing $V_1' = V_1 \cap V$, $V_2' = V_2 \cup \{w\}$, $V_0' = V(M(G)) - V_1' \cup V_2'$ the function $f' = (V_0', V_1', V_2')$ is a RkDF on $M(G)$, and so

$$\begin{aligned} \gamma_{\times kR}(M(G)) &\leq f'(V) \\ &= 2(|V_2| + 1) + (|V_1| - 1) - |U \cup V_1| \\ &= \gamma_{\times kR}(M(G)) + 1 - \ell, \end{aligned}$$

implying that $\ell \leq 1$. Hence

$$|N_{M(G)}(v_i) \cap V_2 \cap V| \begin{cases} \geq k-1 & \text{if } i \in T \cup L, \\ \geq k & \text{if } i \notin L \cup T. \end{cases}$$

Let

$$L_1 = \{v_i \in V_0 \mid i \in T \cup L, \text{ and } |N_{M(G)}(v_i) \cap V_2 \cap V| = k-1\}$$

be a set of cardinality ℓ_1. Hence $\ell_1 \leq k$. Then $f' = (V_0', V_1', V_2')$ is a RkDF on G where $V_2' = V_2 \cap V$, $V_1' = (V_1 \cap V) \cup L_1$, $V_0' = V - (V_1' \cup V_2')$, and so

$$\begin{aligned} \gamma_{\times kR}(G) &\leq f'(V) \\ &= 2|V_2| + |V_1| - 2k + 1 + \ell_1 - \ell \\ &= \gamma_{\times kR}(M(G)) - 2k + 1 + \ell_1 - \ell. \end{aligned}$$

Hence

$$\begin{aligned} \gamma_{\times kR}(M(G)) &\geq \gamma_{\times kR}(G) + 2k - \ell_1 + \ell - 1 \\ &\geq \gamma_{\times kR}(G) + k - 1. \end{aligned}$$

Case 3. $w \in V_2$. (Notice that we may assume that there is no min-RkDF g on $M(G)$ with $g(w) \neq 2$.) Then

$$|N_{M(G)}(v_i) \cap V_2 \cap V| \begin{cases} \geq k-2 & \text{if } i \in T \cup L, \\ \geq k-1 & \text{if } i \notin L \cup T. \end{cases}$$

- **Subcase 3.1.** $T \cap T' = \emptyset$. Then the function $f' = (V_0', V_1', V_2')$ is a RkDF on G where $V_2' = (V_2 \cap V) \cup \{v_i \mid i \in T\}$, $V_1' = (V_1 \cap V) - \{v_i \mid i \in T,\ v_i \in V_1\}$ and $V_0' = V - (V_1' \cup V_2')$, and so

$$\begin{aligned} \gamma_{\times kR}(G) &\leq f'(V) \\ &= 2|V_2| + |V_1| - f(U) - f(w) + 2t - |T \cap L'| \\ &= \gamma_{\times kR}(M(G)) - \ell - 2 - |T \cap L'| \\ &\leq \gamma_{\times kR}(M(G)) - 2, \end{aligned}$$

  which implies $\gamma_{\times kR}(M(G)) \geq \gamma_{\times kR}(G) + 2$.

- **Subcase 3.2.** $T \cap T' \neq \emptyset$. Let f'' be a function which is obtained from f' in Subcase 3.1 by adding some needed vertices from $N_G[v_i]$ to V_2' or V_1' if

$$|N_G(v_i) \cap V_2| < \begin{cases} k & \text{if } f'(v_i) = 0, \\ k-1 & \text{if } f'(v_i) \neq 0 \end{cases}$$

  (this is possible because $|N_G[v_i]| \geq k$).Then f'' is a RkDF on G, and so

$$\begin{aligned} \gamma_{\times kR}(G) &\leq f''(V) \\ &= \gamma_{\times kR}(M(G)) - f(U) - f(w) + 2|T - T'| - |L' \cap T| + p \\ &= \gamma_{\times kR}(M(G)) - \ell - 2 - 2|T \cap T'| - |T \cap L'| + p \\ &\leq \gamma_{\times kR}(M(G)) - 2, \end{aligned}$$

  where $f''(V(G)) - f'(V(G)) = p$. The last inequality is obtained from the facts $p \leq 2t$, $|T \cap T'| + |T \cap L'| \leq |T| = t$, and $|T \cap T'| \leq t$. Hence $\gamma_{\times kR}(M(G)) \geq \gamma_{\times kR}(G) + 2$.

□

By $\gamma_{\times kR}(K_n) = 2k$, the next theorem states that the upper bound given in Theorem 4.2 is tight.

Theorem 4.3. *For any $n \geq k \geq 2$, $\gamma_{\times kR}(M(K_n)) = 4k$.*

Proof. Let $V(K_n) = \{v_i \mid 1 \leq i \leq n\}$, and let $V(M(K_n)) = V \cup U \cup \{w\}$. Let $f = (V_0, V_1, V_2)$ be a min-RkDF on $M(K_n)$. We show that $f(V(M(K_n))) \geq 4k$. Since $N_{M(K_n)}(w) = U$ and $N_{M(K_n)}(u_i) \subseteq V \cup \{w\}$ for each $u_i \in U$, we have $|V_2 \cap U| \geq k-1$ and $|V_2 \cap V| \geq k-1$. Let $V_2 \cap V = \{v_i \mid i \in I\}$ and $V_2 \cap U = \{u_i \mid i \in J\}$ for some $I, J \subseteq \{1, 2, ..., n\}$. Then

$$f(V(M(K_n))) = 2(|I| + |J|) + f(w) + f(U - V_2) + f(V - V_2),$$

and we continue our proof in the following two cases.

- $|J| = k-1$. Then $w \in V_1 \cup V_2$.
  - $|I| = k-1$. Then $U - V_2 \subseteq V_1$, and so

$$\begin{array}{rcl} f(V(M(K_n))) & \geq & 4(k-1)+1+2(n-k+1) \\ & = & 4k-3+2(n-k+1). \end{array}$$

Since $n \leq 2k-3$ implies $u_i, v_i \in V_2$ for some $i \in J$, and so $|N_{M(K_n)}(u_i) \cap V_2| < k-1$, we have $n \geq 2k-2$. Then, since

$$\begin{array}{rcl} f(V(M(K_n))) & \geq & 4k-3+2(n-k+1) \\ & \geq & 4k-3+2(k-1) \\ & = & 6k-5 \\ & \geq & 4k, \end{array}$$

when $k \geq 3$, we assume $k=2$. Since $\gamma_{\times 2R}(M(K_2)) = 2\lceil \frac{10}{3} \rceil = 8 = 4k$ by Proposition 3.3, we assume $n \geq 3$ (and so $n-k+1 \geq 2$) which implies

$$\begin{array}{rcl} f(V(M(K_n))) & \geq & 4k-3+2(n-k+1) \\ & \geq & 4k+1. \end{array}$$

  - $|I| \geq k$. Since f has minimum weight, we have $|I| = k$, and so

$$f(V(M(K_n))) = 2k + 2(k-1) + f(w) + f(V_1 \cap U).$$

If $V_1 \cap U = \emptyset$, then $U \cap V_0 = U - V_2$. Since every vertex in $U \cap V_0$ must be adjacent to all vertices in $V_2 \cap V$, we have $V_2 \cap V \subseteq \{v_i \mid i \in J\}$, which is not possible. Therefore $V_1 \cap U \neq \emptyset$, and so $f(V(M(K_n))) \geq 4k$.

- $|J| \geq k$. Then

$$f(V(M(K_n))) \geq 2(|I|+|J|) + f(U - V_2) + f(V - V_2) + f(w).$$

Since $|J| \geq k+1$ or $|I| \geq k$ impily $f(V(M(K_n))) \geq 4k$, we assume $|J| = k$ and $|I| = k-1$. This implies $I \cap J = \emptyset$, and so $n \geq 2k-1$. On the other hand, $|I| = k-1$ implies $U - V_2 \subseteq V_1$, and so $f(U - V_2) \geq |U| - k = n-k \geq k-1$. Therefore

$$\begin{array}{rcl} f(V(M(K_n))) & \geq & 2(2k-1)+k-1 \\ & = & 5k-3. \end{array}$$

Since $5k-3 \geq 4k$ when $k \geq 3$, we assume $k=2$. But then $\{v_i \mid i \in J\} \subseteq V_1$, which implies $f(V(M(K_n))) \geq 5k-3+2 \geq 4k$.

Finally, by choosing a subset $W_2 \subseteq V(M(K_n))$ with this property that $|W_2 \cap V| = |W_2 \cap U| = k$, and $W_0 = V(M(K_n)) - W_2$, the function $(W_0, \emptyset, W_2)$ is a RkDF on $M(K_n)$ with weight $4k$, implying that $\gamma_{\times kR}(M(K_n)) = 4k$. Figure 3 shows some min-R3DFs for K_5 and $M(K_5)$.

□

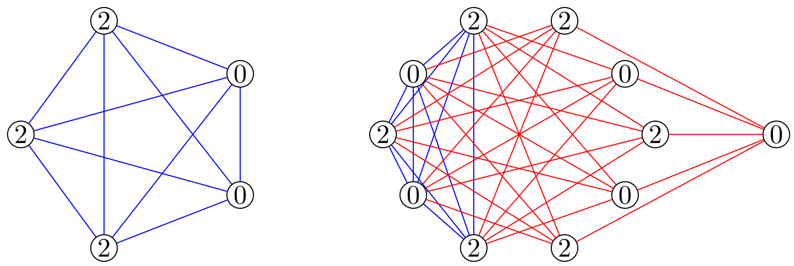

FIGURE 3. $\gamma_{\times 3R}(K_5) = 6$ (left), and $\gamma_{\times 3R}(M(K_5)) = 12$ (right)

5. The Corona Graphs

Here, we study Roman k-tuple domination number of corona graphs. We recall that for any graphs G and H of orders n and m, respectively, the *corona graph* $cor(G, H)$ is a graph obtained from G and H by taking one copy of G and n copies of H and joining with an edge each vertex from the i-th copy of H with the i-th vertex of G. Hereafter, in $cor(G, H)$ we will denote the set of vertices of G by $V = \{v_1, v_2, \cdots, v_n\}$ and the i-th copy of H by $H_i = (W_i, E_i)$.

First we give some bounds for the k-tuple domination number of a corona graph.

Theorem 5.1. *For any graphs G and H with $\delta(H) \geq k - 2 \geq 0$,*

$$k|V(G)| \leq \gamma_{\times k}(cor(G, H)) \leq (|V(H)| + 1)|V(G)|,$$

and these bounds are tight, and $\gamma_{\times k}(cor(G, H)) = k|V(G)|$ if and only if $H = K_{k-1}$ or $H = F \circ_{k-1} K_{k-1}$ for some graph F.

Proof. Since for any kDS S of $cor(G, H)$ and any vertex w in H_i, $|N_{cor(G,H)}[w] \cap S| \geq k$, and since $V(cor(G, H))$ is a kDS of $cor(G, H)$, we have

$$k|V(G)| \leq \gamma_{\times k}(cor(G, H)) \leq (|V(H)| + 1)|V(G)|.$$

Obviously, $\gamma_{\times k}(cor(G, H)) = k|V(G)|$ if and only if $H = K_{k-1}$ or $H = F \circ_{k-1} K_{k-1}$ for some graph F. For the upper bound, if H is a $(k-2)$-regular graph, then $\gamma_{\times k}(cor(G, H)) = (|V(H)| + 1)|V(G)|$. □

Theorem 5.2. *For any graphs G and H with $\delta(H) \geq k - 1 \geq 1$,*

$$2k|V(G)| \leq \gamma_{\times kR}(cor(G, H)) \leq 2\gamma_{\times k}(cor(G, H)).$$

Proof. By Theorem 2.6, it is sufficient to prove the lower bound. Let $f = (V_0, V_1, V_2)$ be a RkDF on $cor(G, H)$ and let v_i be a vertex of G. We continueour proof in the following cases. Recall that for any subset $T \subseteq V$, $f(T) = \sum_{v \in T} f(v)$.

- $f(v_i) = 0$. If there exists a vertex $v \in W_i \cap V_0$, then $|N_{W_i}(v) \cap V_2| \geq k$, and so $f(W_i \cup \{v_i\}) \geq 2k$. If there exists a vertex $v \in W_i \cap V_1$, then $|N_{W_i}(v) \cap V_2| \geq k-1$. Now $k \geq 2$ implies that there exists a vertex $v' \in N_{W_i}(v) \cap V_2$, and so $|N_{W_i}(v') \cap V_2| \geq k-1$. Therefore $|(N_{W_i}(v) \cup N_{W_i}(v')) \cap V_2| \geq k$ which implies $f(W_i \cup \{v_i\}) \geq 2k+1$. Finally, if $f(v') = 2$ for any $v' \in W_i$, then $f(W_i \cup \{v_i\}) \geq 2k$.

- $f(v_i) = 1$. If there exists a vertex $v \in W_i \cap V_0$, then $|N_{W_i}(v) \cap V_2| \geq k$, and so $f(W_i \cup \{v_i\}) \geq 2k+1$. If there exists a vertex $v \in W_i \cap V_1$, then $|N_{W_i}(v) \cap V_2| \geq k-1$. Now $k \geq 2$ implies that there exists a vertex $v' \in N_{W_i}(v) \cap V_2$, and so $|N_{W_i}(v') \cap V_2| \geq k-1$. Therefore $|(N_{W_i}(v) \cup N_{W_i}(v')) \cap V_2| \geq k$ which implies $f(W_i \cup \{v_i\}) \geq 2k+2$. Finally, if $f(v') = 2$ for any $v' \in W_i$, then $f(W_i \cup \{v_i\}) \geq 2k+1$.
- $f(v_i) = 2$. If there exists a vertex $v \in W_i \cap V_0$, then $|N_{W_i}(v) \cap V_2| \geq k-1$, and so $f(W_i \cup \{v_i\}) \geq 2k$. If there exists a vertex $v \in W_i \cap V_1$, then $|N_{W_i}(v) \cap V_2| \geq k-2$, and so $f(W_i \cup \{v_i\}) \geq 2k-1$. Since $f(W_i \cup \{v_i\}) = 2k-1$ if and only if $H = K_{k-1}$, we obtain $f(W_i \cup \{v_i\}) \geq 2k$. Finally, if $f(v') = 2$ for any $v' \in W_i$, then $f(W_i \cup \{v_i\}) \geq 2k$.

□

The following theorem is obtained by Theorems 5.1 and 5.2.

Theorem 5.3. *For any graphs G and H with $\delta(H) \geq k-2 \geq 0$, $\gamma_{\times kR}(cor(G,H)) = 2k|V(G)|$ if and only if $H = K_{k-1}$ or $H = F \circ_{k-1} K_{k-1}$ for some graph F.*

6. Some Questions and Problems

Finally, we end our paper with some useful questions and problems.

Question 6.1. *Is $M(G)$ a Roman k-tuple graph if G is a Roman k-tuple graph?*

Question 6.2. *For any Roman k-tuple graph G, is there a Roman k-tuple graph H such that $G = M(H)$?*

Question 6.3. *Find graphs G whose Roman k-tuple domination number achieves the bounds in Theorem 4.2?*

Question 6.4. *For any graph G, whether $\gamma_{\times 2R}(G) \geq 2\gamma_R(G)$?*

Problem 6.5. *Find $\gamma_{\times kR}(M(C_n))$ for $2 \leq k \leq 4$ and $\gamma_{\times kR}(M(P_n))$ for $2 \leq k \leq 3$.*

Problem 6.6. *Find the Roman k-tuple domatic number of a graph.*

Problem 6.7. *Characterize graphs G with $\gamma_{\times 2R}(G) = \gamma_R(G)$.*

Problem 6.8. *Characterize graphs G with $\gamma_{\times kR}(G) = \gamma_{kR}(G)$.*

In [7], the authors have defined the *total Roman dominating function* on a graph G as a Roman domination function $f = (V_0, V_1, V_2)$ on it with this additional property that the induced subgraph $G[V_1 \cup V_2]$ has no isolated vertex, and in a similar way, they have defined the *total Roman domination number* $\gamma_{tR}(G)$ of G. Since $\gamma_{tR}(G) \leq \gamma_{\times kR}(G) \leq \gamma_{\times(k+1)R}(G)$ for any $k \geq 2$, we have

$$\gamma_{tR}(G) \leq \gamma_{\times 2R}(G). \tag{6.1}$$

So, the fnext problem is natural to appear.

Problem 6.9. *Find graphs G satisfying $\gamma_{\times 2R}(G) = \gamma_{tR}(G)$.*

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