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On Beck's Coloring for Measurable Functions

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Abstract. We study Beck-like coloring of measurable functions on a measure space Ω taking values in a measurable semigroup Δ. To any measure space Ω and any measurable semigroup Δ, we assign a graph (called a zero-divisor graph) whose vertices are labeled by the classes of measurable functions defined on Ω and having values in Δ, with two vertices f and g adjacent if $f \cdot g = 0$ a.e.. We show that, if Ω is atomic, then not only the Beck's conjecture holds but also the domination number coincides to the clique number and chromatic number as well. We also determine some other graph properties of such a graph.

Keywords: Clique number, Coloring, Domination number, Measurable function, Zero divisor graph.

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1. Introduction

Beck [10] introduced a coloring of commutative rings as follows: given a commutative ring R, he associated to R a simple graph G whose vertices are labeled by the elements of R, with two vertices adjacent (connected by an edge)

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if $x \cdot y = 0$. Such a graph is called a zero-divisor graph of R. The purpose of his idea was to establish a connection between graph theory and commutative ring theory. Beck was mainly interested in characterizing and discussing the rings which are finitely colorable, leaving aside possible applications to graph theory.

By a measure space we mean a masurable space $(\Omega, \mathcal{M})$ along with a measure μ on $(\Omega, \mathcal{M})$. The measure μ is atomless if $\mu(\{x\}) = 0$ for every $x \in \Omega$. A measurable set A is null if $\mu(A) = 0$ and conull if $\mu(\Omega \setminus A) = 0$. (See [16])

A graph is said to be embeddable in the plane, or planar, if it can be drawn in the plane so that its edges intersect only at their ends. Any graph derived from a graph G by a sequence of edge subdivisions is called a subdivision of G or a G-subdivision.

A subset D of vertices in a graph G is a dominating set if every vertex of the graph G is either an element of D or adjacent to an element of D. The domination number $\gamma(G)$ of a graph G is the minimum cardinality of a dominating set of G.

A proper coloring or simply a coloring of the vertices of a graph G is an assignment of colors to the vertices in such a way that adjacent vertices have distinct colors. $\chi(G)$ is the minimal number of colors in a coloring of the graph G. A maximal complete subgraph of a graph is a clique, and the clique number $\omega(G)$ of a graph G is the maximum order of a clique in G. It is well known that $\chi(G) \geq \omega(G)$ [11]. Beck conjectured that $\chi(G) = \omega(G)$ for the zero divisor graph of R and an arbitrary ring R. But Anderson and Naseer [5] have shown that this is not the case in general, namely they presented an example of a commutative local ring R with 32 elements for which $\chi(G) > \omega(G)$. After that lots of authors have worked on such graphs and found classes of graphs on which Beck's conjecture holds (see for instance [1, 2, 3, 6, 7, 8, 9, 14, 15, 17]).

The main aim of the present paper is to show that in fact Beck's conjecture is valid for a much wider class of relational structures, namely for measurable functions. Also, we give a condition under which such functions are not finitely colorable. We also found a relation between the domination number and chromatic number of such graphs.

2. Measurable Zero Divisor Graphs

Let $(\Omega, \mathcal{M}, \mu)$ be a measure space, Δ be a measurable semigroup with 0 and R be the set of all measurable functions from Ω into Δ. It is easy to see that $(R, \cdot)$ is a semigroup, where $\cdot$ denotes the product of functions generated by the binary operation of Δ.

For a fixed non-zero $\delta \in \Delta$ and $A \subseteq \Omega$, we define the characteristic function $\chi_{A,\delta}$ defined on Ω and taking values in Δ which sends all elements of A into δ and all other elements to 0.

Equal almost everywhere is an equivalence relation in R and we use f for both f and the class of f.

Observe that the class of the constant function on Ω which sends all elements of Ω to 0, the zero elemet of Δ is also the zero element of R, is denoted by 0.

Definition 2.1. An element f of R is a zero-divisor element if and only if $\mu(f \neq 0) > 0$ and for some $g \in R$ with $\mu(g \neq 0) > 0$, the equality $f \cdot g = 0$ a.e. holds.

One can verify that f is a zero–divisor element of R if and only if both sets $\{f = 0\}$ and $\{f \neq 0\}$ have positive measure.

Motivated by the definition of zero divisor graph of semigroups [13] we define measurable zero-divisor graph of Ω and Δ as follows.

Definition 2.2. Let $(\Omega, \mathcal{M}, \mu)$ be a measure space, Δ be a measurable semigroup with 0 and R be the set of all measurable functions from Ω into Δ. The measurable zero-divisor graph of Ω is a graph G with the set of all classes of zero divisor elements of R as the vertex set and two vertices f and g are adjacent if and only if $f \cdot g = 0$ a.e.. We will denote it by $ZD(\Omega, \Delta)$.

One can easily see that the definition of zero- divisor graph of measurable functions is well-defined, i.e., for f' and g' in R with $f' = f$ a.e. and $g' = g$ a.e., $f \cdot g = 0$ a.e., implies $f' \cdot g' = 0$ a.e..

For the rest of the paper we assume that $(\Omega, \mathcal{M}, \mu)$ is a measure space, Δ is a measurable semigroup with zero element, s is some fixed non-zero element of Δ and $G = ZD(\Omega, \Delta)$.

Example 2.3. G is a complete graph if and only if Ω is atomic and contains one or two atoms.

Example 2.4. G is a star graph if and only if Ω is atomic and contains one or two atoms.

Example 2.5. If G has a positive nonatomic part, or if G is atomic with more than two atoms, then G is neither vertex-transitive, nor edge-transitive.

First of all we present some equiavalent conditions for the distance between vertices of G.

Theorem 2.6. *The distance of two vertices f and g of G can be calculated as follows:*

(i) $\operatorname{supp} f \cap \operatorname{supp} g$ *is null if and only if* $d(f, g) = 1$.
(ii) *Neither* $\operatorname{supp} f \cap \operatorname{supp} g$ *is null nor* $\operatorname{supp} f \cup \operatorname{supp} g$ *is conull if and only if* $d(f, g) = 2$.
(iii) $\operatorname{supp} f \cap \operatorname{supp} g$ *is not null but* $\operatorname{supp} f \cup \operatorname{supp} g$ *is conull if and only if* $d(f, g) = 3$.

Proof. (i) It is easy to see that the following equalities hold.

$$\{f \cdot g \neq 0\} = \{f \neq 0\} \cap \{g \neq 0\} = \operatorname{supp} f \cap \operatorname{supp} g.$$

Therefore, we can say

$$\begin{aligned} d(f,g) = 1 \quad &\Longleftrightarrow \quad f \text{ is adjacent to } g \\ &\Longleftrightarrow \quad f \cdot g = 0 \quad a.e. \\ &\Longleftrightarrow \quad \mu(\{f \cdot g \neq 0\}) = 0 \\ &\Longleftrightarrow \quad \mu(\operatorname{supp} f \cap \operatorname{supp} g) = 0. \end{aligned}$$

(ii) Assume that $\mu(\operatorname{supp} f \cap \operatorname{supp} g) \neq 0$ and $\mu((\operatorname{supp} f \cup \operatorname{supp} g)^c) \neq 0$. By (i) and the first assumption, we can conclude that f is not adjacent to g, i.e. $d(f,g) > 2$. Now define $h := \chi_{(suppf \cup suppg)^c, \delta}$, by the assumption $(suppf \cup suppg)^c$ has positive measure, thus we can say that h is a measurable function with $\mu(h \neq 0) > 0$.

We also have

$$\operatorname{supp} h \cap \operatorname{supp} f = \emptyset$$

yields

$$\mu(\operatorname{supp} h \cap \operatorname{supp} f) = 0.$$

By (i) we can say that f is adjacent to h.

Similar argument shows that g is also adjacent to h. Thus $f \sim h \sim g$ is a path from f to g, hence $d(f,g) \leq 2$, i.e., $d(f,g) = 2$.

Now suppose that $d(f,g) = 2$. Thus by part (i) we can conclude that $\mu(\operatorname{supp} f \cap \operatorname{supp} g) > 0$. We also have a vertex h which is adjacent to both f and g. Therefore, we have the following statements:

$$f \sim h \Longrightarrow \mu(\operatorname{supp} f \cap \operatorname{supp} h) = 0,$$

$$g \sim h \Longrightarrow \mu(\operatorname{supp} g \cap \operatorname{supp} h) = 0.$$

Therefore, we imply that

$$\mu((\operatorname{supp} f \cup \operatorname{supp} g) \cap \operatorname{supp} h) = 0.$$

The fact $\mu(h = 0) > 0$ implies that

$$\mu((\operatorname{supp} f \cup \operatorname{supp} g)^c \cap \operatorname{supp} h) > 0,$$

therefore,

$$\mu((\operatorname{supp} f \cup \operatorname{supp} g)^c) > 0.$$

(iii) If $\mu(\operatorname{supp} f \cap \operatorname{supp} g) \neq 0$ and $\mu((\operatorname{supp} f \cup \operatorname{supp} g)^c) \neq 0$, Then by parts (i) and (ii), $d(f,g) > 2$.

$f \neq g$, implies that $\mu(f \neq g) > 0$. By the assumption in (iii), we have $\mu(\operatorname{supp} f \setminus \operatorname{supp} g) > 0$.

Now, define $h := \chi_{\operatorname{supp} f \setminus \operatorname{supp} f, \delta}$ and $k := \chi_{(\operatorname{supp} f)^c, \delta}$.

Therefore $f \sim k \sim h \sim g$ is also a path, i.e., $d(f,g) \leq 3$.

The inverse is deduced by (i) and (ii). □

Applying Theorem 2.6, we will have the following corollary.

Corollary 2.7. *With the notation of Theorem 2.6, the girth of G is 3 if and only if $\mathcal{M}$ contains at least 3 disjoint measurable sets with positive measure.*

Theorem 2.8. *For an atomic measure space Ω with the set of atoms $\mathcal{A}$, we have*

$$\gamma(G) = \omega(G) = \chi(G) = |\mathcal{A}|.$$

Proof. Let $D = \{\chi_{A,\delta} : \ A \in \mathcal{A}\}$. It is easily seen that D is a clique in G. Thus we can calculate that

$$\chi(G) \geq \omega(G) \geq |D| = |\mathcal{A}|.$$

If we set $|\mathcal{A}|$ disjoint colors to all elements of $\mathcal{A}$, then we can color the graph G with these colors in the sense that, designate the color of some of atoms of $\operatorname{supp} f$ to any vertex f. If f is adjacent to g, then by Theorem 2.6,

$$\mu(\operatorname{supp} f \cap \operatorname{supp} f) = 0.$$

Therefore, atoms of $\operatorname{supp} f$ and $\operatorname{supp} g$ and thus colors of f and g are different. Hence we can color the graph G with $|\mathcal{A}|$ colors, i.e., $\chi(G) \leq |\mathcal{A}|$. Finally we have

$$|\mathcal{A}| \leq \omega(G) \leq \chi(G) \leq |\mathcal{A}|.$$

We also prove that D is a dominating set which has the minimum cardinality among dominatng sets. For a vertex f of G, f is a zero divisor element, thus $\mu(f = 0) > 0$, hence the set $\{f = 0\}$ contains an atom A. Therefore $f = \chi_{A,\delta}$, i.e., D is a dominating set. Now, suppose that D' is a dominating set with $|D'| < |D|$.

Set

$$E' = \{\operatorname{supp} f : \ f \in D'\}.$$

For every $W \in E$, there is an atom Z such that $Z \subseteq W$.

For every $Z \in A$, there is some $W \in E$ which contains Z, otherwise for some $Z \in A$, there is no $W \in E$ which contains Z. Since Z is an atom, we can say

$$\forall W \in E' \quad \mu(W \cap Z) = 0,$$

with the fact that every element of E has positive measure, yields

$$\forall W \in E' \quad \mu(W \cap Z^c) > 0.$$

Therefore,

$$\forall [f] \in D' \quad [\chi_{Z^c}] \cdot f \neq 0,$$

which is a contradiction with the condition that D is a dominating set.

Now by the axiom of choice, we can define a function $\phi : \mathcal{A} \longrightarrow E'$ which sends every element $Z \in \mathcal{A}$ to some $W \in E'$ where $Z \subseteq W$.

Since $|\mathcal{A}| > |E'|$, ϕ can not be a one to one function, hence for two distinct elements of $\mathcal{A}$ such as Z and Z' we have $\phi(Z) = \phi(Z')$, thus for some $W \in E'$, $Z \in W$ and $Z' \in W$ and there is no $W' \in E'$ with $W' = Z$. Thus we can say

$$\forall W' \in E' \quad \mu(W' \setminus Z) > 0.$$

Now if we set $g = \chi_{\Omega \setminus Z,\delta}$, we have

$$\forall f \in D,\ g \cdot f \neq 0,$$

since

$$\operatorname{supp} f \cap \operatorname{supp} g = (\Omega \setminus Z) \cap (W' \setminus Z) = W' \setminus Z,$$

where $W' = \operatorname{supp} f$ and $\mu(W' \setminus Z) \neq 0$.

Therefore, f is not adjacent to any element of D' which says D' is not a dominating set. A contradiction with the hypothesis $|D'| < |D|$. □

Theorem 2.9. *If Ω has an atomless subset, then*

$$\gamma(G) = \omega(G) = \chi(G) = \infty.$$

Proof. Suppose that A is an atomless subset, thus there is a decreasing sequence of measurable sets

$$A \supset A_1 \supset A_2 \cdots$$

where

$$\mu(A) > \mu(A_1) > \mu(A_2) > \cdots$$

By defining the difference of the sets we will have a sequence of disjoint sets $B_1, B_2, \cdots$ with positive measures. One can see that the set

$$\{\chi_{B_i,\delta} :\ i \in \mathbb{N}\}$$

is a clique with infinite cardinality, therefore, $\omega(G) = \chi(G) = \infty$.

If D is a finite dominating set with cardinality n, we can suppose that $D = \{f_1, f_2, \cdots, f_n\}$ is a dominating set. Let $A_1 = \operatorname{supp} f_1$ and for $1 < i \leq n$, define $A_i = A_{i-1} \cap \operatorname{supp} f_i$ if $\mu(A_{i-1} \cap \operatorname{supp} f_i) > 0$, otherwise let $A_i = A_{i-1}$. Thus $\mu(A_n) > 0$. Ω is atomless, thus we can find $B \subset A_n$, where $\mu(B) > 0$ and $\mu(B) < \mu(A_n)$.

One can easily see that $\chi_{B^c,\delta}$ is a vertex different with all elements of D which is adjacent to non of elements of D, a contradiction. Hence domination number of G is infinite. □

Halas in [14] proved that if the clique number of a zero divisor graph for a poset $(P, \leq)$ is finite, then the number of all minimal prime ideals of P is finite. In the following, we present some similar but a little different properties of G.

Lemma 2.10. *$M \subset R$ is a minimal ideal if and only if M is an ideal and for some atom A, all non zero elements f of M have A as their support.*

Proof. Assume that for some $f \in M$ we have $\operatorname{supp} f = B$ which is not an atom or contains two atoms. Thus we can find another positive measure C which is a pure subset of B. Now it is enough to define

$$N = \{f \in M : \operatorname{supp} f \subseteq C\},$$

which is a non-zero ideal of R. The converse is obvious. □

Lemma 2.10 and Theorem 2.8 conclude the following theorem.

Theorem 2.11. *The following are equivalent:*

(1) *the clique number of G is finite.*
(2) *the chromatic number of G is finite.*
(3) *the dominating number of G is finite.*
(4) *the number of minimal ideals of R is finite.*
(5) *Ω is atomic with finite number of atoms.*

Now we can determine all ends of the graph G and their corollaries.

Theorem 2.12. *Every vertex f of G is either an end or is contained in a triangle. The first case may occur only if Ω has an atom.*

Proof. Suppose that $f = \chi_{A^c,\delta}$ where A is an atom. The only vertex which is adjacent to f is $g = \chi_{A,\delta}$, therefore f is an end of G.

Now assume that Ω is atomless, thus B has positive measure, where $B = (\operatorname{supp} f)^c$ for a vertex f of G. Thus $B = C \cup D$, where C and D are disjoint with positive measures. One can see that f, $\chi_{C,\delta}$ and $\chi_{D,\delta}$ is a triangle. □

The core of a graph G is the largest subgraph of G in which every edge is the edge of a cycle in G [12].

The following are corollaries immediately concluded from Theorem 2.12.

Corollary 2.13. *The core of G contains all vertices of G except for the vertices f in which $\mu(\operatorname{supp} f \setminus A) = 0$ for some atom A.*

Corollary 2.14. *G has a cut vertex if and only if Ω has an atom.*

We also present some conditions for neighbors of the graph G.

Theorem 2.15. $\operatorname{supp} f \cup \operatorname{supp} g$ *is conull if and only if each vertex of $N(f)$ is adjacent to each vertex of $N(g)$.*

Proof. Suppose that $\operatorname{supp} f \cup \operatorname{supp} g$ is conull, $h \in N(f)$ and $k \in N(g)$. By definition we will have

$$\operatorname{supp} h \subset (\operatorname{supp} f)^c,$$

and

$$\operatorname{supp} k \subset (\operatorname{supp} g)^c.$$

With the assumption, we can conclude $\operatorname{supp} h \cap \operatorname{supp} k = 0$, i.e., h is adjacent to k. For the converse, assume that $\operatorname{supp} f \cup \operatorname{supp} g$ is not conull, hence there exists a measurable set A with positive measure which has a zero measure intersection with both $\operatorname{supp} f$ and $\operatorname{supp} g$. It is enough to consider the function $\chi_{A,\delta}$ which is obviously in both $N(f)$ and $N(g)$. $\square$

Theorem 2.16. *Ω is atomic with finite number of atoms if and only if the ACC condition holds for neighborhoods of G.*

Proof. Suppose that there exists an infinite chain of neighborhoods

$$N(f_1) \subset N(f_2) \subset N(f_3) \subset \cdots$$

Let $A_i = (\operatorname{supp} f_i)^c$ for every natural i. Hence we have

$$A_1 \subset A_2 \subset A_3 \subset \cdots$$

and for $B_i = A_i \setminus A_{i-1}$, and $B_1 = A_1$ we will have

$$\mu(B_1) < \mu(B_2) < \mu(B_3) < \cdots$$

Therefore, Ω is not atomic or has an infinite atoms.

For the converse, suppose that Ω is not atomic or has an infinite atoms. In both cases we can find a sequence of positive measure sets

$$A_1 \subset A_2 \subset A_3 \subset \cdots$$

where

$$\mu(A_2 \setminus A_1) < \mu(A_3 \setminus A_2) < \cdots$$

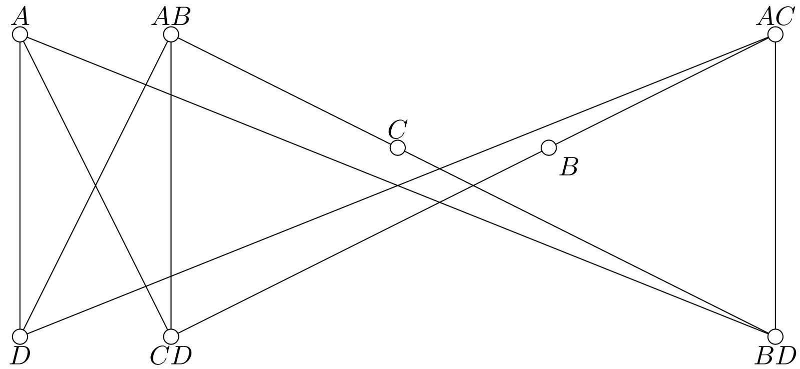
Now if we define $f_i = \chi_{A_i^c,\delta}$, we derive an infinite chain of neighborhoods of $f_i s$. $\square$

It is well known that no planar graph can contain a subdivision of either K_5 or $K_{3,3}$ (see [12]). A fundamental theorem due to Kuratowski (1930) states that every nonplanar graph necessarily contains a copy of a subdivision of either K_5 or $K_{3,3}$ as well.

Theorem 2.17. *G is planar if and only if Ω is atomic and does not contain more than three atoms.*

Proof. If Ω is atomic containing less than four atoms, a simple drawing shows that G is planar.

Now suppose that Ω contains a subset $\{a, b, c, d\}$ of atoms. Let $A := \chi_{a,\delta}$, $B := \chi_{b,\delta}$, $C := \chi_{c,\delta}$, $D := \chi_{d,\delta}$ and similarly for more than one alphabets (for instance $AB := \chi_{ac,\delta}$). Then the following diagram shows that G contains a subdivision of $K_{3,3}$. If Ω is not atomic, then there are at least four mutually disjoint measurable sets a, b, c, d with positive measure and the latter proof can be done. $\square$



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