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Lommel Matrix Functions

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Abstract. The main objective of this study is to develop a pair of Lommel matrix functions suggested by the hypergeometric matrix functions and some of their properties are studied. Some properties of the hypergeometric and Bessel matrix functions are obtained.

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1. Introduction

The pair of the Lommel functions are introduced by Eugen Von Lommel (see [17])

$$s_{\mu,\nu}(z) = \frac{z^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)}\ {}_1F_2\left(1; \frac{1}{2}(\mu-\nu+3), \frac{1}{2}(\mu+\nu+3); -\frac{1}{4}z^2\right)$$

and

$$S_{\mu,\nu}(z) = z^{\mu-1}\ {}_3F_0\left(1, \frac{1}{2}(1-\mu+\nu), \frac{1}{2}(1-\mu-\nu); -; -\frac{4}{z^2}\right).$$

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The differential equation for Lommel function is a non-homogeneous form of the Bessel differential equation (see [16, 17, 18]):

$$z^2\frac{d^2y}{dz^2}+z\frac{dy}{dz}+(z^2-\nu^2)y=z^{\mu+1}.$$

Studies of special matrix polynomials and orthogonal matrix polynomials are very important by virtue to their applications in particular areas such as physics, statistics, engineering, Lie groups theory, splines, interpolation and quadrature, and medical imaging. Because of this study, some mathematicians have demonstrated that some results in the literature can be extended to matrix functions and matrix polynomials (see for example [8, 9, 10, 11, 13, 15, 20, 21, 23, 24, 25, 26, 29]).

Motivated by the results of Sastre and Jódar [19], Çekim and Altin [2], Çekim and Erkuş-Duman [3], and Shehata [22, 27, 28], we present in this paper a new class for the pair of the Lommel matrix functions. The organization of the paper is as follows: We rephrase some results from the previous works which is used in this study in Section 1. In Section 2, we look back on briefly some known facts on the hypergeometric matrix functions and derive some of its properties, prove new interesting properties, a matrix differential equation and some matrix transformations of the hypergeometric matrix functions are obtained. We give the generalized hypergeometric matrix functions and derive some of its properties. Section 3 is devoted to the Lommel matrix functions, using hypergeometric matrix functions, and this includes a matrix differential equation of second order, matrix recurrence relations and integral representation which are satisfied by the Lommel matrix functions. We define a pair of the modified Lommel matrix functions and some properties related to these functions are also given in Section 4.

1.1. **Preliminaries.** Firstly, we will give some basic facts, lemma, definition, notation or terminology and properties of the matrix functional calculus, used in the next sections.

During this paper, for a matrix A in $\mathbb{C}^{N\times N}$, $\sigma(A)$ symbolize the spectrum of the set of all eigenvalues of A. The two-norm is described by

$$\|A\|=\sup_{x\neq 0}\frac{\|Ax\|_2}{\|x\|_2},$$

where $||x||_2=(x^Hx)^{\frac{1}{2}}$ denotes the well-known Euclidean norm of a vector x in $\mathbb{C}^N$. Furthermore, the identity matrix and the null matrix or zero matrix in $\mathbb{C}^{N\times N}$ will be symbolized by I and $\mathbf{0}$, respectively.

In this work, we symbolize by $\mu(A)$ the logarithmic norm of A, which is defined by [8, 9] as follows:

$$\mu(A)=\max\left\{z : z \text{ eigenvalue of } \frac{A+A^H}{2}\right\}, \tag{1.1}$$

where A^H denotes the conjugate transpose. We also denote by the number $\tilde{\mu}(A)$ as

$$\tilde{\mu}(A) = \min\left\{z : z \text{ eigenvalue of } \frac{A+A^H}{2}\right\}. \tag{1.2}$$

From [9], it follows that $\|e^{At}\| \leq e^{t\mu(A)}$ for $t \geq 0$. Hence we have

$$\|t^A\| = \begin{cases} t^{\mu(A)}, & \text{if } t \geq 1 \\ t^{\tilde{\mu}(A)}, & \text{if } 0 \leq t \leq 1. \end{cases} \tag{1.3}$$

Definition 1.1. Let A be a matrix in $\mathbb{C}^{N\times N}$. We say that A is a positive stable matrix if

$$Re(\mu) > 0 \quad \forall\, \mu \in \sigma(A). \tag{1.4}$$

Theorem 1.2. [7] *If $U(z)$ and $V(z)$ are holomorphic functions in an open set Ω of the complex plane, and P, Q are commutative matrices in $\mathbb{C}^{N\times N}$ with $\sigma(P) \subset \Omega$ and $\sigma(Q) \subset \Omega$, then*

$$U(P)V(Q) = V(Q)U(P).$$

Definition 1.3. [12] For a positive stable matrix A in $\mathbb{C}^{N\times N}$, the Gamma matrix function $\Gamma(P)$ is described by

$$\Gamma(P) = \int_0^\infty e^{-t} t^{P-I} dt; \quad t^{P-I} = \exp\Big((P-I)\ln t\Big). \tag{1.5}$$

Definition 1.4. For A in $\mathbb{C}^{N\times N}$, the matrix form of the Pochhammer symbol is given by

$$(A)_n = A(A+I)\dots(A+(n-1)I) = \Gamma(A+nI)\Gamma^{-1}(A),\ n \in \mathbb{N},\ (A)_0 = I, \tag{1.6}$$

where $A+nI$ is an invertible matrix for every integer $n \geq 0$ and $\Gamma(A)$ is an invertible matrix, its inverse coincides with $\Gamma^{-1}(A)$.

Fact 1.1. *(Jódar and Cortés* [14]*) Let us denote the real numbers $M(A)$, $m(A)$ for A in $\mathbb{C}^{N\times N}$ such that*

$$M(A) = \max\{Re(z) : z \in \sigma(A)\},\ \textit{and}\, m(A) = \min\{Re(z) : z \in \sigma(A)\}. \tag{1.7}$$

Then we get

$$\|e^{tA}\| \leq e^{tM(A)} \sum_{k=0}^{N-1} \frac{(\|A\| N^{\frac{1}{2}} t)^k}{k!}, \quad t \geq 0 \tag{1.8}$$

and

$$\|n^A\| \leq n^{M(A)} \sum_{k=0}^{N-1} \frac{(\|A\| N^{\frac{1}{2}} \ln n)^k}{k!}, \quad n \in \mathbb{N}. \tag{1.9}$$

Theorem 1.5. *(Jódar and Cortés* [12]*) Let A be a positive stable matrix in $\mathbb{C}^{N\times N}$. Thus, one has the following property*

$$\Gamma(A) = \lim_{n\to\infty}(n-1)![(A)_n]^{-1}n^A \text{ for } \quad n \in \mathbb{N}. \tag{1.10}$$

Definition 1.6. [12] For positive stable matrices P and Q in $\mathbb{C}^{N\times N}$, the definition of $\mathbf{B}(P,Q)$ Beta matrix function is given by

$$\mathbf{B}(P,Q) = \int_0^1 t^{P-I}(1-t)^{Q-I}dt. \tag{1.11}$$

Lemma 1.7. [12] *Let P, Q and $P+Q$ be positive stable matrices in $\mathbb{C}^{N\times N}$ satisfy the conditions $PQ = QP$, and $P+nI$, $Q+nI$ and $P+Q+nI$ are invertible matrices for all eigenvalues $n \geq 0$. Then we get*

$$\boldsymbol{B}(P,Q) = \Gamma(P)\Gamma(Q)\Gamma^{-1}(P+Q). \tag{1.12}$$

Also, for a matrix $A(k,n)$ in $\mathbb{C}^{N\times N}$ for $n \geq 0$ and $k \geq 0$, the relation is given by Defez and Jódar [4]

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k,n-k). \tag{1.13}$$

Definition 1.8. Let us take A a matrix in $\mathbb{C}^{N\times N}$ such that

$$\nu \text{ is not a negative integer for every } \nu \in \sigma(A). \tag{1.14}$$

Then the Bessel matrix function $J_A(z)$ of the first kind of order A was given in [19] as follows:

$$\begin{aligned} J_A(z) &= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\Gamma^{-1}(A+(k+1)I)\left(\frac{1}{2}z\right)^{A+2kI} \\ &= \left(\frac{1}{2}z\right)^{A}\Gamma^{-1}(A+I)\ {}_0F_1\left(-;A+I;-\frac{z^2}{4}\right);\ |z|<\infty;\ |arg(z)|<\pi. \end{aligned} \tag{1.15}$$

2. Properties of hypergeometric matrix functions

In this section, we firstly give the definitions of hypergeometric matrix functions ${}_1F_2$ and ${}_3F_0$.

Definition 2.1. Let A_1, B_1 and B_2 be commutative matrices in $\mathbb{C}^{N\times N}$ satisfying the following condition

$$B_1 + kI \text{ and } B_2 + kI \quad \text{are invertible matrices } \forall k \in \mathbb{N}\cup\{0\}. \tag{2.1}$$

Then we define the first hypergeometric matrix function ${}_1F_2$ as

$${}_1F_2(A_1;B_1,B_2;z) = \sum_{k=0}^{\infty}\frac{z^k}{k!}(A_1)_k[(B_1)_k]^{-1}[(B_2)_k]^{-1}. \tag{2.2}$$

Now, we find an interesting radius of convergence R of ${}_1F_2$. For this aim, with the help of method in [20] and (2.2), then we give

$$\begin{aligned}
\frac{1}{R} &= \limsup_{k\to\infty} (\|U_k\|)^{\frac{1}{k}} = \lim_{k\to\infty}\sup\left(\left\|\frac{(A_1)_k[(B_1)_k]^{-1}[(B_2)_k]^{-1}}{k!}\right\|\right)^{\frac{1}{k}}\\
&= \limsup_{k\to\infty}\Bigg[\Bigg\|\frac{k^{-A_1}(A_1)_k}{(k-1)!}(k-1)!k^{A_1}\\
&\times \frac{k^{-B_1}}{(k-1)!}(k-1)![(B_1)_k]^{-1}k^{B_1}\frac{k^{-B_2}}{(k-1)!}(k-1)![(B_2)_k]^{-1}k^{B_2}\frac{1}{k!}\Bigg\|\Bigg]^{\frac{1}{k}} \qquad (2.3)\\
&= \limsup_{k\to\infty}\left[\left\|\Big(\Gamma^{-1}(A_1)\Gamma(B_1)\Gamma(B_2)\Big)k^{A_1}k^{-B_1}k^{-B_2}\frac{1}{(k-1)!k!}\right\|\right]^{\frac{1}{k}}\\
&\le \limsup_{k\to\infty}\left[\left\|k^{A_1}k^{-B_1}k^{-B_2}\frac{1}{(k-1)!k!}\right\|\right]^{\frac{1}{k}} \le \limsup_{k\to\infty}\left[\frac{\|k^{A_1}\|\|k^{-B_1}\|\|k^{-B_2}\|}{(k-1)!k!}\right]^{\frac{1}{k}}.
\end{aligned}$$

Substitute from (1.8) and (1.9) into (2.3), one gets

$$\begin{aligned}
\frac{1}{R} &\le \limsup_{k\to\infty}\Bigg\{\frac{1}{(k-1)!k!}k^{M(A_1)}\left(\sum_{j=0}^{N-1}\frac{(\| A_1 \| N^{\frac{1}{2}}\ln k)^j}{j!}\right)\\
&\times k^{-m(B_1)}\left(\sum_{s=0}^{N-1}\frac{(\| B_1 \| N^{\frac{1}{2}}\ln k)^s}{s!}\right)k^{-m(B_2)}\left(\sum_{r=0}^{N-1}\frac{(\| B_2 \| N^{\frac{1}{2}}\ln k)^r}{r!}\right)\Bigg\}^{\frac{1}{k}}.
\end{aligned} \qquad (2.4)$$

By using (1.8), we can write

$$\sum_{j=0}^{N-1}\frac{(\| A_1 \| N^{\frac{1}{2}}\ln k)^j}{j!} \le (N\ln k)^{N-1}\sum_{j=0}^{N-1}\frac{(\| A_1 \|)^j}{j!} = (N\ln k)^{N-1}e^{\|A_1\|}.$$

Then we have

$$\frac{1}{R} \le \limsup_{k\to\infty}\left\{\frac{k^{M(A_1)}k^{-m(B_1)}k^{-m(B_2)}}{\sqrt{2\pi(k-1)}(\frac{k-1}{e})^{k-1}\sqrt{2\pi\, k}(\frac{k}{e})^k}e^{\|A_1\|}e^{\|B_1\|}e^{\|B_2\|}(N\ln k)^{3N-3}\right\}^{\frac{1}{k}} = 0.$$

Thus, the power series (2.2) is convergent for all complex numbers z. That is, the function ${}_1F_2$ is an entire function under the condition in (2.1) of the definition (2.2).

On the other hand, we take in consideration the differential operator $\theta = z\frac{d}{dz}$, $D_z = \frac{d}{dz}$, $\theta z^k = kz^k$. Thus we have

$$\begin{aligned}
&\theta\ (\theta\ I + B_1 - I)(\theta\ I + B_2 - I)\ {}_1F_2\\
&= \sum_{k=1}^{\infty}\frac{z^k}{(k-1)!}(A_1)_k[(B_1)_{k-1}]^{-1}[(B_2)_{k-1}]^{-1}.
\end{aligned}$$

Replace k by $k+1$, we have

$$\begin{aligned}&\theta\,(\theta\,I+B_1-I)(\theta\,I+B_2-I)\ {}_1F_2\\&=\sum_{k=0}^{\infty}\frac{z^{k+1}}{k!}(A_1)_{k+1}[(B_1)_k]^{-1}[(B_2)_k]^{-1}=z(\theta\,I+A_1)\ {}_1F_2.\end{aligned}$$

Thus, the next result has been obtained:

Theorem 2.2. *Let A_1, B_1 and B_2 be commutative matrices in $\mathbb{C}^{N\times N}$ satisfying the spectral condition (2.1). Then the function ${}_1F_2$ is a solution of the following matrix differential equation of the three order*

$$\Big[\theta\,(\theta\,I+B_1-I)(\theta\,I+B_2-I)-z(\theta\,I+A_1)\Big]\ {}_1F_2=\mathbf{0}. \tag{2.5}$$

Definition 2.3. Let us define the second hypergeometric matrix function ${}_3F_0$ as

$$ {}_3F_0(A_1,A_2,A_3;-;z)=\sum_{k=0}^{\infty}\frac{z^k}{k!}(A_1)_k(A_2)_k(A_3)_k \tag{2.6}$$

where A_1, A_2 and A_3 are commutative matrices in $\mathbb{C}^{N\times N}$.

Similarly, for ${}_3F_0$, we have

$$\theta\ {}_3F_0=\sum_{k=1}^{\infty}\frac{k\ z^k}{k!}(A_1)_k(A_2)_k(A_3)_k=\sum_{k=1}^{\infty}\frac{z^k}{(k-1)!}(A_1)_k(A_2)_k(A_3)_k.$$

Replace k by $k+1$, thus we have

$$\begin{aligned}\theta\ {}_3F_0&=\sum_{k=0}^{\infty}\frac{z^{k+1}}{k!}(A_1)_{k+1}(A_2)_{k+1}(A_3)_{k+1}\\&=z(\theta\,I+A_1)(\theta\,I+A_2)(\theta\,I+A_3)\ {}_3F_0.\end{aligned}$$

This result is summarized below.

Theorem 2.4. *Suppose that A_1, A_2 and A_3 are commutative matrices in $\mathbb{C}^{N\times N}$. Then the function ${}_3F_0$ satisfy the matrix differential equation of the three order*

$$\Big[\theta\,I-z(\theta\,I+A_1)(\theta\,I+A_2)(\theta\,I+A_3)\Big]\ {}_3F_0=\mathbf{0}. \tag{2.7}$$

To derive interesting relations, we will benefit from the following corollary given by Defez and Jódar in [5].

Theorem 2.5. [1, 5] *Let A and B be matrices in $\mathbb{C}^{N\times N}$ such that A and $B-A$ are positive stable matrices with $AB=BA$ and $B+nI$ is an invertible matrix for $n\in\mathbb{N}\cup\{0\}$. Then the following identity holds*

$$ {}_2F_1\Big(-nI,A;B;1\Big)=(B-A)_n[(B)_n]^{-1}. \tag{2.8}$$

Now, we demonstrate that well-known hypergeometric matrix functions ${}_2F_1$ and ${}_0F_1$ satisfy various transformation formulaes.

Theorem 2.6. *Let A be a matrix in $C^{N\times N}$ where $A+nI$ and $(2A-I)+nI$ are invertible matrices for $n\in\mathbb{N}\cup\{0\}$. Then we get*

$$ {}_2F_1\bigg(-nI, I-A-nI; A; 1\bigg) = (2A-I)_{2n}[(A)_n]^{-1}[(2A-I)_n]^{-1}. \tag{2.9}$$

Proof. Taking $A\to I-A-nI$ and $B\to A$ in (2.8), we have

$$\begin{aligned} {}_2F_1(-nI, I-A-nI; A; 1) =&(2A+(n-1)I)_n[(A)_n]^{-1}\\ =&\Gamma(2A+(2n-1)I)\Gamma^{-1}(2A-I)\Gamma(2A-I) \\ &\times\Gamma(2A+(n-1)I)\Gamma(A)\Gamma^{-1}(A+nI). \end{aligned}\tag{2.10}$$

By (1.3), we can rewrite the formula

$$\begin{aligned} &\Gamma(2A+(2n-1)I)\Gamma^{-1}(2A-I) = (2A-I)_{2n},\\ &\Gamma(2A-I)\Gamma^{-1}(2A+(n-1)I) = [(2A-I)_n]^{-1},\\ &\Gamma(A)\Gamma^{-1}(A+nI) = [(A)_n]^{-1}. \end{aligned}\tag{2.11}$$

From (2.11) into (2.10), we get (2.6). □

Theorem 2.7. *If A is a matrix in $C^{N\times N}$ providing the conditions $A+kI$ and $(2A-I)+kI$ are invertible matrices for $k\in\mathbb{N}\cup\{0\}$, then we get*

$$ {}_0F_1\bigg(-;A;z\bigg)\,{}_0F_1\bigg(-;A;z\bigg) = {}_1F_2\bigg(\frac{1}{2}(2A-I); A, 2A-I; 4z\bigg). \tag{2.12}$$

Proof. From (1.13) and (2.9), we have

$$\begin{aligned} {}_0F_1\bigg(-;A;z\bigg)\,{}_0F_1\bigg(-;A;z\bigg) &= \sum_{m=0}^{\infty}\sum_{n=0}^{m}\frac{[(A)_{m-n}]^{-1}[(A)_n]^{-1}z^m}{n!(m-n)!}\\ =\sum_{m=0}^{\infty}\sum_{n=0}^{m}&\frac{(I-A-mI)_n[(A)_n]^{-1}(-mI)_n}{n!}\frac{[(A)_m]^{-1}}{m!}z^m\\ =\sum_{m=0}^{\infty}&{}_2F_1\bigg(-mI, I-A-mI; A; 1\bigg)\frac{[(A)_m]^{-1}}{m!}z^m\\ =\sum_{m=0}^{\infty}&2^{2m}\bigg(\frac{1}{2}(2A-I)\bigg)_m[(A)_m]^{-1}[(2A-I)_m]^{-1}\frac{z^m}{m!}\\ ={}&{}_1F_2\bigg(\frac{1}{2}(2A-I); A, 2A-I; 4z\bigg). \end{aligned}$$

□

Theorem 2.8. *If A is a matrix in $C^{N\times N}$ providing the conditions $A+kI$, $A+(k+1)I$ and $2A+kI$ are invertible matrices for every integer $k\geq 0$, then* *we get*

$$ {}_0F_1\left(-;A;z\right)\,{}_0F_1\left(-;A+I;z\right)= {}_1F_2\left(\frac{1}{2}(2A+I);A+I,2A;4z\right). \quad (2.13) $$

Proof. From (1.13) and (2.9), we have

$$ \begin{aligned} &{}_0F_1\left(-;A;z\right)\,{}_0F_1\left(-;A+I;z\right)=\sum_{m=0}^{\infty}\sum_{n=0}^{m}\frac{[(A)_{m-n}]^{-1}[(A+I)_n]^{-1}z^m}{n!(m-n)!}\\ &=\sum_{m=0}^{\infty}\sum_{n=0}^{m}\frac{(I-A-mI)_n[(A+I)_n]^{-1}(-mI)_n}{n!}\frac{[(A)_m]^{-1}}{m!}z^m\\ &=\sum_{m=0}^{\infty}{}_2F_1\left(-mI,I-A-mI;A+I;1\right)\frac{[(A)_m]^{-1}}{m!}z^m\\ &=\sum_{m=0}^{\infty}(2A)_{2m}[(A+I)_m]^{-1}[(2A)_m]^{-1}\frac{[(A)_m]^{-1}}{m!}z^m\\ &=\sum_{m=0}^{\infty}2^{2m}\left(\frac{1}{2}(2A+I)\right)_m[(A+I)_m]^{-1}[(2A)_m]^{-1}\frac{z^m}{m!}\\ &={}_1F_2\left(\frac{1}{2}(2A+I);A+I,2A;4z\right). \end{aligned} $$

□

Theorem 2.9. *For A is a matrix in $\mathbb{C}^{N\times N}$ providing the conditions $A\mp kI$ and $(2A-I)+kI$ are invertible matrices for $k\in\mathbb{N}\cup\{0\}$ and $|arg(z)|<\pi$, then the product of two Bessel matrix functions hold the following feature*

$$ \begin{aligned} J_A(z)J_{A+I}(z)=&\left(\frac{z}{2}\right)^{2A+I}\Gamma^{-1}(A+I)\Gamma^{-1}(A+2I)\\ &\times\ {}_1F_2\left(A+\frac{3}{2}I;A+2I,2A+2I;-z^2\right). \end{aligned} \quad (2.14) $$

Proof. From (1.15) and (2.13), we obtain (2.14). □

3. Lommel's matrix functions: Definitions and Properties

In this section, the pair of Lommel's matrix functions with the help of hypergeometric matrix functions are introduced.

Definition 3.1. Let A and B be matrices in $\mathbb{C}^{N\times N}$ so

$$ Re(\mu) \text{ is not an odd negative integer for all } \mu\in\sigma(A\pm B). \quad (3.1) $$

Then, we define the Lommel matrix functions $s_{A,B}(z)$ of the first kind in the form:

$$
\begin{aligned}
s_{A,B}(z) =& z^{A+I}(A-B+I)^{-1}(A+B+I)^{-1} \\
&\times \ {}_1F_2\left(I; \frac{1}{2}(A-B+3I), \frac{1}{2}(A+B+3I); -\frac{1}{4}z^2\right) \\
=& \frac{z^{A+I}}{4}\sum_{k=0}^{\infty}(-1)^k\left(\frac{z}{2}\right)^{2k}\Gamma\left(\frac{1}{2}(A-B+I)\right)\Gamma\left(\frac{1}{2}(A+B+I)\right) \\
&\times \Gamma^{-1}\left(\frac{1}{2}(A-B+3I+2kI)\right)\Gamma^{-1}\left(\frac{1}{2}(A+B+3I+2kI)\right),
\end{aligned} \tag{3.2}
$$

where $A \pm B + I$ and $A \pm B + 3I$ are invertible matrices, $AB = BA$ and $|arg(z)| < \pi$.

Thus, the Lommel matrix function $s_{A,B}(z)$ is an entire function under the condition (3.1).

Definition 3.2. Let A and B be matrices in $\mathbb{C}^{N\times N}$ so

$$
Re(\mu) \text{ is an odd positive integer for all } \mu \in \sigma(A \pm B), \text{ and } AB = BA. \tag{3.3}
$$

Then, we define the Lommel matrix function $s_{A,B}(z)$ of the second kind in the form

$$
S_{A,B}(z) = z^{A-I}\ {}_3F_0\left(I, \frac{1}{2}(I-A+B), \frac{1}{2}(I-A-B); -; -\frac{4}{z^2}\right), \tag{3.4}
$$

where $|arg(z)| \leq \pi - \delta$, $\delta > 0$ for $|z| \longrightarrow \infty$.

Theorem 3.3. *Let A, $A-I$, B and $B-I$ be matrices in $\mathbb{C}^{N\times N}$ providing the restriction (3.3), and $AB = BA$. The matrix recurrence relations and difference differential equations for Lommel's matrix functions $S_{A,B}(z)$ are*

$$
S_{A+2I,B}(z) = z^{A+I} - [(A+I)^2 - B^2]S_{A,B}(z), \tag{3.5}
$$

$$
S'_{A,B}(z) + \frac{B}{z}S_{A,B}(z) = (A+B-I)S_{A-I,B-I}(z), \tag{3.6}
$$

$$
S'_{A,B}(z) - \frac{B}{z}S_{A,B}(z) = (A-B-I)S_{A-I,B+I}(z), \tag{3.7}
$$

$$
\frac{2}{z}B\ S_{A,B}(z) = (A+B-I)S_{A-I,B-I}(z) - (A-B-I)S_{A-I,B+I}(z) \tag{3.8}
$$

and

$$
2S'_{A,B}(z) = (A+B-I)S_{A-I,B-I}(z) + (A-B-I)S_{A-I,B+I}(z), \tag{3.9}
$$

where $|arg(z)| \leq \pi - \delta$, $\delta > 0$ for $|z| \longrightarrow \infty$.

Proof. From (3.4), we have

$$S_{A+2I,B}(z) = z^{A+I} \sum_{k=0}^{\infty} \frac{(-1)^k (I)_k \left(\frac{1}{2}(I-A-2I+B)\right)_k \left(\frac{1}{2}(I-A-2I-B)\right)_k}{k!} \left(\frac{4}{z^2}\right)^k$$

$$= z^{A+I}\left[I + \sum_{k=1}^{\infty} \frac{(-1)^k (I)_k \left(\frac{1}{2}(I-A-2I+B)\right)_k \left(\frac{1}{2}(I-A-2I-B)\right)_k}{k!} \left(\frac{4}{z^2}\right)^k\right]$$

$$= z^{A+I}\left[I - \frac{1}{z^2}((A+I)-B)(A+I+B)\right.$$

$$\left.\times \sum_{k=0}^{\infty} \frac{(-1)^k (I)_k \left(\frac{1}{2}(I-A+B)\right)_k \left(\frac{1}{2}(I-A-B)\right)_k}{k!} \left(\frac{2}{z}\right)^{2k}\right]$$

$$= z^{A+I} - [(A+I)^2 - B^2] z^{A-I} \sum_{k=0}^{\infty} \frac{(-1)^k (I)_k \left(\frac{1}{2}(I-A+B)\right)_k \left(\frac{1}{2}(I-A-B)\right)_k}{k!} \left(\frac{2}{z}\right)^{2k}$$

$$= z^{A+I} - [(A+I)^2 - B^2] S_{A,B}(z).$$

Multiplying (3.4) by z^B and taking the derivative with respect to z, we have

$$\frac{d}{dz}\left[z^B S_{A,B}(z)\right] = z^{A+B-2I}$$

$$\times \sum_{k=0}^{\infty} \frac{(-1)^k (I)_k (A+B-2kI-I) \left(\frac{1}{2}(I-A+B)\right)_k \left(\frac{1}{2}(I-A-B)\right)_k}{k!} \left(\frac{2}{z}\right)^{2k}$$

$$= 2z^{A+B-2I} \sum_{k=0}^{\infty} \frac{(-1)^{k+1} (I)_k \left(\frac{1}{2}(I-A+B)\right)_k \left(\frac{1}{2}(I-A-B)\right)_{k+1}}{k!} \left(\frac{2}{z}\right)^{2k}$$

$$= (A+B-I) z^{A+B-2I} \sum_{k=0}^{\infty} \frac{(-1)^k (I)_k \left(\frac{1}{2}(I-A+B)\right)_k \left(\frac{1}{2}(I-A-B)+I\right)_k}{k!} \left(\frac{2}{z}\right)^{2k}$$

$$= (A+B-I) z^B S_{A-I,B-I}(z).$$

Thus, we have

$$S'_{A,B}(z) + \frac{B}{z} S_{A,B}(z) = (A+B-I) S_{A-I,B-I}(z).$$

On the other hand, we can similarly write

$$\begin{aligned}
&\frac{d}{dz}\left[z^{-B}S_{A,B}(z)\right] = 2z^{A-B-2I}\\
&\times\sum_{k=0}^{\infty}\frac{(-1)^{k+1}(I)_k\left(\frac{1}{2}(I-A+B)+kI\right)\left(\frac{1}{2}(I-A+B)\right)_k\left(\frac{1}{2}(I-A-B)\right)_k}{k!}\left(\frac{2}{z}\right)^{2k}\\
=&2z^{A-B-2I}\sum_{k=0}^{\infty}\frac{(-1)^{k+1}(I)_k\left(\frac{1}{2}(I-A+B)\right)_{k+1}\left(\frac{1}{2}(I-A-B)\right)_k}{k!}\left(\frac{2}{z}\right)^{2k}\\
=&(A-B-I)z^{A-B-2I}\sum_{k=0}^{\infty}\frac{(-1)^{k}(I)_k\left(\frac{1}{2}(I-A+B)+I\right)_k\left(\frac{1}{2}(I-A-B)\right)_k}{k!}\left(\frac{2}{z}\right)^{2k}\\
=&(A-B-I)z^{-B}S_{A-I,B+I}(z).
\end{aligned}$$

Thus, we get

$$S'_{A,B}(z)-\frac{B}{z}S_{A,B}(z)=(A-B-I)S_{A-I,B+I}(z).$$

With the help of subtracting and adding operation on these results, we have the following formulaes

$$\begin{aligned}
\frac{2}{z}B\ S_{A,B}(z) &= (A+B-I)S_{A-I,B-I}(z)-(A-B-I)S_{A-I,B+I}(z),\\
2S'_{A,B}(z) &= (A+B-I)S_{A-I,B-I}(z)+(A-B-I)S_{A-I,B+I}(z).
\end{aligned}$$

□

Theorem 3.4. *Let A, $A-I$, B and $B-I$ be matrices in $\mathbb{C}^{N\times N}$ providing the restriction (3.1) and $AB=BA$. Then the $s_{A,B}(z)$ satisfy the following matrix recurrence relations and difference differential equations*

$$s_{A+2I,B}(z)=z^{A+I}-[(A+I)^2-B^2]s_{A,B}(z), \tag{3.10}$$

$$s'_{A,B}(z)+\frac{B}{z}s_{A,B}(z)=(A+B-I)s_{A-I,B-I}(z), \tag{3.11}$$

$$s'_{A,B}(z)-\frac{B}{z}s_{A,B}(z)=(A-B-I)s_{A-I,B+I}(z), \tag{3.12}$$

$$\frac{2}{z}B\ s_{A,B}(z)=(A+B-I)s_{A-I,B-I}(z)-(A-B-I)s_{A-I,B+I}(z) \tag{3.13}$$

and

$$2s'_{A,B}(z)=(A+B-I)s_{A-I,B-I}(z)+(A-B-I)s_{A-I,B+I}(z), \tag{3.14}$$

where $|arg(z)|<\pi$.

Proof. The proof is similar to Theorem 3.1. □

Corollary 3.5. *Let A, B and $-B$ be matrices in $\mathbb{C}^{N\times N}$ providing (3.3), $AB = BA$. Then the $S_{A,B}(z)$ satisfy the following relation*

$$S_{A,B}(z) = S_{A,-B}(z), \tag{3.15}$$

where $|arg(z)| \leq \pi - \delta$, $\delta > 0$.

Proof. Taking $-B$ instead of B in (3.4), we obtain (3.15). □

Corollary 3.6. *Let A, B and $-B$ be matrices in $\mathbb{C}^{N\times N}$ providing the restriction (3.1) and $AB = BA$. Then the $s_{A,B}(z)$ satisfy the following relation*

$$s_{A,B}(z) = s_{A,-B}(z), \tag{3.16}$$

where $|arg(z)| < \pi$.

Proof. The proof is similar to Corollary 3.1. □

Theorem 3.7. *Let A and B be matrices in $\mathbb{C}^{N\times N}$ satisfying the conditions (3.3), $A \pm B - I$ are invertible matrices and $AB = BA$. Then the $s_{A,B}(z)$ is a solution of the Lommel matrix differential equation of two order*

$$\left[z^2 \frac{d^2}{dz^2} I + z \frac{d}{dz} I + (z^2 I - B^2)\right] S_{A,B}(z) = z^{A+I}, \tag{3.17}$$

where $|arg(z)| \leq \pi - \delta$, $\delta > 0$.

Proof. From (3.5), we get

$$S_{A-2I,B}(z) = \left[(A-I)^2 - B^2\right]^{-1} \left[z^{A-I} - S_{A,B}(z)\right]. \tag{3.18}$$

Replace A by $A - I$ and B by $B - I$ in (3.7), we get

$$S'_{A-I,B-I}(z) - \frac{B-I}{z} S_{A-I,B-I}(z) = (A - B - I) S_{A-2I,B}(z). \tag{3.19}$$

Using (3.7) and (3.18), we have

$$\begin{aligned} S'_{A-I,B-I}(z) = \frac{B-I}{z}(A+B-I)^{-1}\left[S'_{A,B}(z) + \frac{B}{z} S_{A,B}(z)\right] \\ + (A-B-I)[(A-I)^2 - B^2]^{-1}\left[z^{A-I} - S_{A,B}(z)\right]. \end{aligned} \tag{3.20}$$

Differentiating of (3.6) with respect to z, we get

$$S''_{A,B}(z) + \frac{B}{z} S'_{A,B}(z) - \frac{B}{z^2} S_{A,B}(z) = (A + B - I) S'_{A-I,B-I}(z). \tag{3.21}$$

Taking into account (3.20) in (3.21), we obtain (3.17). □

In a similar manner, we can give the next result.

Theorem 3.8. *Let A and B be matrices in $\mathbb{C}^{N\times N}$ satisfying the conditions (3.1), $A \pm B - I$ are invertible matrices and $AB = BA$. Then the $s_{A,B}(z)$ is a solution of the Lommel matrix differential equation of two order*

$$\left[z^2 \frac{d^2}{dz^2} I - z\frac{d}{dz} - (B^2 - z^2 I)\right] s_{A,B}(z) = z^{A+I}, \tag{3.22}$$

where $|arg(z)| < \pi$.

Naturally, we are ready to give the new integral representations for the Lommel matrix functions.

Theorem 3.9. *Let A and B be matrices in $\mathbb{C}^{N\times N}$ satisfying (3.3) and $AB = BA$. Then the integral representation for Lommel matrix function is*

$$S_{A,B}(z) = z^A \int_0^\infty e^{-zt}\, {}_2F_1\left(\frac{1}{2}(B - A + I), \frac{1}{2}(I - B - A); \frac{1}{2}I; -t^2\right) dt, \tag{3.23}$$

where $Re(z) > 0$, $|arg(z)| \leq \pi - \delta$, $\delta > 0$ for $|z| \longrightarrow \infty$.

Proof. From (1.5), one gets the following integral

$$\frac{\Gamma(2k+1)}{z^{2k+1}} = \int_0^\infty e^{-z\, t} t^{2k} dt, z \neq 0. \tag{3.24}$$

Using (3.24) in the left hand side of (3.23), we have the desired result. □

Theorem 3.10. *Let A and B be matrices in $\mathbb{C}^{N\times N}$ providing the restriction (3.1), $A \pm B + I$ and $A \pm B + 3I$ are invertible matrices and $AB = BA$. Then we have the integral representation for Lommel matrix function $s_{A,B}(z)$*

$$\begin{aligned} s_{A,B}(z) = &\int_0^1 (1-t)^{\frac{1}{2}(A-B-I)}\, {}_0F_1\left(-; \frac{1}{2}(A + B + 3I); -\frac{1}{4}z^2 t\right) dt \\ &\times z^{A+I}(A - B + I)^{-1}(A + B + I)^{-1}\left(\frac{1}{2}(A - B + I)\right) \end{aligned} \tag{3.25}$$

and

$$\begin{aligned} s_{A,B}(z) = &\int_0^1 (1-t)^{\frac{1}{2}(A+B-I)}\, {}_0F_1\left(-; \frac{1}{2}(A - B + 3I); -\frac{1}{4}z^2 t\right) dt \\ &\times z^{A+I}(A - B + I)^{-1}(A + B + I)^{-1}\left(\frac{1}{2}(A + B + I)\right), \end{aligned} \tag{3.26}$$

where $|arg(z)| < \pi$.

Proof. By using (1.11) and (1.12), one can obtain the desired result. □

The next theorem can be proved using the similar technique as in Theorem 3.6.

Theorem 3.11. *If A and B are matrices in $\mathbb{C}^{N\times N}$ providing the conditions $Re(\mu)$ is not an odd positive integer for all $\mu \in \sigma(A \pm B)$ and $AB = BA$. Then the integral representations for the Lommel's matrix function $S_{A,B}(z)$ are given as*

$$\begin{aligned}
S_{A,B}(z) =& z^{A-I}\int_0^\infty e^{-t}\;{}_2F_0\left(\frac{1}{2}(I-A+B),\frac{1}{2}(I-A-B);-;-\frac{4}{z^2}t\right)dt,\\
=& z^{A-I}\Gamma^{-1}\left(\frac{1}{2}(I-A+B)\right)\int_0^\infty e^{-t}t^{\frac{1}{2}(I-A+B)-I}\\
&\times\;{}_2F_0\left(I,\frac{1}{2}(I-A-B);-;-\frac{4}{z^2}t\right)dt,\\
=& z^{A-I}\Gamma^{-1}\left(\frac{1}{2}(I-A-B)\right)\int_0^\infty e^{-t}t^{\frac{1}{2}(I-A-B)-I}\\
&\times\;{}_2F_0\left(I,\frac{1}{2}(I-A+B);-;-\frac{4}{z^2}t\right)dt,
\end{aligned}$$

where $I-A\pm B$ are invertible matrices, $|arg(z)| \leq \pi-\delta$, $\delta > 0$ for $|z| \longrightarrow \infty$.

Corollary 3.12. *Let A be a matrix in $\mathbb{C}^{N\times N}$ providing the restriction (3.3). Then the integral representation for Lommel matrix function is*

$$s_{A,A}(z) = z^A\int_0^1 sin(zt)(1-t^2)^{A-\frac{1}{2}I}dt, \tag{3.27}$$

where $|arg(z)| < \pi$.

Proof. This can be proved by using the beta matrix function in (1.12) and Maclaurian series of functions $\sin(zt)$. □

4. Modified Lommel Matrix Functions

With the help of the Lommel matrix functions, the modified Lommel matrix functions are defined:

$$t_{A,B}(z) = -i^{I-A}s_{A,B}(iz), \tag{4.1}$$

where A and B are matrices in $\mathbb{C}^{N\times N}$ providing the conditions (3.1), $AB = BA$, $|arg(z)| < \pi$, and

$$T_{A,B}(z) = -i^{I-A}S_{A,B}(iz), \tag{4.2}$$

where A and B are matrices in $\mathbb{C}^{N\times N}$ providing the conditions (3.3), $AB = BA$ and $|arg(z)| \leq \pi-\delta$, $\delta > 0$ for $|z| \longrightarrow \infty$.

Theorem 4.1. *The functions $t_{A,B}(z)$ and $T_{A,B}(z)$, respectively, are the solution of the modified Lommel matrix differential equation of the two orders as following*

$$\left[z^2\frac{d^2}{dz^2}I + z\frac{d}{dz} - (B^2+z^2I)\right]t_{A,B}(z) = z^{A+I}, \tag{4.3}$$

where $|arg(z)| < \pi$*, and*

$$\left[z^2 \frac{d^2}{dz^2} I + z \frac{d}{dz} - (B^2 + z^2 I)\right] T_{A,B}(z) = z^{A+I}, \tag{4.4}$$

where $|arg(z)| \leq \pi - \delta$*,* $\delta > 0$ *for* $|z| \longrightarrow \infty$*.*

Proof. By using (4.1), (4.2), (3.17) and (3.22), we have the desired final result. □

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References

1. R.S. Batahan, Generalized form of Hermite matrix polynomials via the hypergeometric matrix function, *Advances in Linear Algebra and Matrix Theory*, **4**, (2014), 134-141.
2. B. Çekim, and A. Altin, Matrix analogues of some properties for Bessel matrix functions, *Journal of Mathematical Sciences, The University of Tokyo*, **22 (2)**, (2015), 519-530.
3. B. Çekim, and E. Erkuş-Duman, Integral represantations for Bessel matrix functions, *Gazi University Journal of Science*, **27 (1)**, (2014), 663-667.
4. E. Defez, and L. Jódar, Some applications of the Hermite matrix polynomials series expansions, *Journal of Computational and Applied Mathematics.* **V99**, (1998), 105-117.
5. E. Defez, and L. Jódar, Chebyshev matrix polynomials and second order matrix differential equations, *Utilitas Mathematica*, **61**, (2002), 107-123.
6. D.J. Dickinson, On Lommel and Bessel polynomials, *Proceedings of the American Mathematical Society*, **5**, (1954) 946-956.
7. N. Dunford, and J.T. Schwartz, *Linear Operators, Part I, General Theory*, Interscience Publishers, INC. New York, 1957.
8. T.M. Flett, *Differential Analysis*, Cambridge University Press, 1980.
9. I. Higueras, and B. Garcia-Celayeta, Logarithmic norms for matrix pencils, *SIAM Journal on Matrix Analysis and Applications*, **20 (3)**, (1999), 646-666.
10. L. Jódar, R. Company, and E. Navarro, Bessel matrix functions: Explicit solution of coupled Bessel type equations, *Utilitas Mathematics*, **46**, (1994), 129-141.
11. L. Jódar, R. Company, and E. Navarro, Solving explicitly the Bessel matrix differential equation, without increasing problem dimension, *Congressus Numerantium*, **92**, (1993), 261-276.
12. L. Jódar, and J.C. Cortés, Some properties of Gamma and Beta matrix functions, *Applied Mathematics Letters*, **11**, (1998), 89-93.
13. L. Jódar, and J.C. Cortés, On the hypergeometric matrix function, *Journal of Computational and Applied Mathematics*, **99**, (1998), 205-217.
14. L. Jódar, and J.C. Cortés, Closed form general solution of the hypergeometric matrix differential equation, *Mathematical and Computer Modelling*, **32**, (2000), 1017-1028.
15. L. Kargin, and V. Kurt, Chebyshev-type matrix polynomials and integral transforms, *Hacettepe Journal of Mathematics and Statistics*, **44 (2)**, (2015), 341-350.
16. E. Lommel, Ueber eine mit den Bessel'schen functionen verwandte function, *Mathematische Annalen*, **9 (3)**, (1875), 425-444.
17. E. Lommel, Zur Theorie der Bessel'schen funktionen IV, *Mathematische Annalen*, **16 (2)**, (1880), 183-208.

18. Y.L. Luke, Integrals of Bessel Functions, (MGH, 1962), McGRAW-Hill Book,Company, INC. New York, Toronto London.
19. J. Sastre, and L. Jódar, Asymptotics of the modified Bessel and incomplete Gamma matrix functions, *Applied Mathematics Letters*, **16 (6)**, (2003), 815-820.
20. K.A.M. Sayyed, *Basic Sets of Polynomials of two Complex Variables and Convergence Propertiess*. Ph. D. Thesis, Assiut University, Egypt, 1975.
21. K.A.M. Sayyed, M.S. Metwally, and M.T. Mohammed, Certain hypergeometric matrix function, *Scientiae Mathematicae Japonicae*, **69**, (2009), 315-321.
22. A. Shehata, A new extension of Bessel matrix functions, *Southeast Asian Bulletin of Mathematics*, **40 (2)**, (2016), 265-288.
23. A. Shehata, Some relations on Humbert matrix polynomials, *Mathematica Bohemica*, **141 (4)**, (2016), 407-429.
24. A. Shehata, Some relations on the generalized Bessel matrix polynomials, *Asian Journal of Mathematics and Computer Research*, **17 (1)**, (2017), 1-25.
25. A. Shehata, A note on Two-variable Lommel matrix functions, *Asian-European Journal of Mathematics (AEJM)*, **11 (1)**, (2018), 1850041(14 pages).
26. A. Shehata, Some relations on generalized Rice's matrix polynomials, *An International Journal Applications and Applied Mathematics*, **12 (1)**, (2017), 367-391.
27. A. Shehata, Extended biorthogonal matrix polynomials, *Mathematica Moravica*, **21 (2)**, (2017), 27–50.
28. A. Shehata, Some properties associated with the Bessel matrix functions, *Konuralp Journal of Mathematics (KJM)*, **5 (2)**, (2017), 24-35.
29. A. Shehata, Some new results for Chebyshev matrix polynomials of first kind. *International Bulletin of Mathematical Research*, **4 (4)**, (2017), 50-55.