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Extensions of Some Fixed Point Theorems for Weak-Contraction Mappings in Partially Ordered Modular Metric Spaces

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Abstract. The purpose of this paper is to establish fixed point results for a single mapping in a partially ordered modular metric space, and to prove a common fixed point theorem for two self-maps satisfying some weak contractive inequalities.

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1. Introduction

Many generalizations and extensions of Banach contraction principle have been studied in various settings (see [12, 21, 20, 22]). Most established results provide sufficient conditions for the existence and uniqueness of fixed points of

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certain classes of self-mappings (see [6, 7, 10, 24, 26, 27, 28]) and some of them provide iterative schemes and numerical algorithms to approximate those fixed points (see [23, 25, 29]).

In 2008, Dutta and Choudhury in [11] proved the following theorem, by using the weak contraction introduced by Alber and Guerre-Delabrere in [5].

Theorem 1.1. *Let (X,d) be a complete metric space, and let $T : X \to X$ be a self-mapping satisfying the following inequality:*

$$\psi(d(Tx,Ty)) \leq \psi(d(x,y)) - \varphi(d(x,y)), \quad \text{for all } (x,y) \in X^2$$

where $\psi, \varphi : [0,+\infty[\to [0,+\infty[$ are both continuous and nondecreasing functions with $\psi(t) = \varphi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point.

In 2012, Abkar and Choudhury in [1] proved the following theorem which is a generalization of the above result in a partially ordered metric space having the following property (P):
for each non-decreasing sequence $(x_n)_{n\in\mathbb{N}} \subset X$ that converges to some $x \in X$, we have: $x_n \preceq x$ for all $n \in \mathbb{N}$.

Theorem 1.2. *Let $(X, \preceq, d)$ be an ordered complete metric space with a partial order " $\preceq$ " and having the property (P). Let $S,T : X \to X$ be two self mappings such that for all comparable $x,y \in X$,*

$$\psi_1(d(Sx,Ty)) \leq \psi_2(M(x,y)) - \varphi(M(x,y))$$

where

$$M(x,y) = \max\{d(x,y), d(x,Sx), d(y,Ty), \frac{1}{2}(d(x,Ty) + d(y,Sx))\},$$

$\psi_1, \psi_2 : [0,+\infty[\to [0,+\infty[$ are both continuous and monotone non-decreasing functions and $\varphi : [0,+\infty[\to [0,+\infty[$ is lower semi-continuous function which satisfies $\psi_1(t) - \psi_2(t) + \varphi(t) > 0$, for all $t > 0$.
If there exists a point $x_0 \in X$ satisfying

$$x_0 \preceq Sx_0 \preceq TSx_0 \preceq STSx_0 \preceq (TS)^2x_0 \preceq ...$$

then there exists a point $u \in X$ such that $Su = Tu = u$.

On the other hand, in 2010, Chistyakov in [8] and [9] has introduced the concept of modular metric space. This is a generalization of the classical modular spaces like Orlicz spaces (see [16]). Fixed point theorems in modular function spaces, generalizing the classical Banach fixed point theorem in metric spaces, have been studied extensively (see [4, 6, 17, 18, 24]).

In recent years, there has been a great interest in the study of the fixed point property in modular metric spaces (see [2, 3, 19]). For more details on modular metric fixed point theory, the reader may consult the books [16, 8].

In this paper we prove some fixed and common fixed point theorems for a weak contractive mapping in modular metric spaces. Our results generalize and extend the above theorems in partially ordered modular metric spaces for mappings satisfying weak contraction that involves three control functions.

2. Preliminaries

Let X be a nonempty set. For a function $\omega :]0,+\infty[\times X \times X \to [0,+\infty]$, we will use the notation

$$\omega_\lambda(x,y) = \omega(\lambda, x, y), \quad \text{for all } \lambda > 0 \text{ and } x, y \in X.$$

Definition 2.1. ([9]) A function $\omega :]0,+\infty[\times X \times X \to [0,+\infty]$ is said to be modular metric on X if it satisfies the following conditions:

(i) Given $x, y \in X$, $x = y$ if and only if $\omega_\lambda(x,y) = 0$ for all $\lambda > 0$;
(ii) For all $x, y \in X$, for all $\lambda > 0$, $\omega_\lambda(x,y) = \omega_\lambda(y,x)$;
(iii) For all $x, y, z \in X$ and for all $\lambda, \mu > 0$, $\omega_{\lambda+\mu}(x,y) \le \omega_\lambda(x,z) + \omega_\mu(z,y)$.

In this case, (X, ω) is called modular metric space.
The modular ω is said to be regular if the condition (i) holds for some $\lambda > 0$.
The modular ω is said to be convex if for all $\lambda, \mu > 0$ and $x, y, z \in X$, we have:

$$\omega_{\lambda+\mu}(x,y) \le \frac{\lambda}{\lambda+\mu}\omega_\lambda(x,z) + \frac{\mu}{\lambda+\mu}\omega_\mu(z,y).$$

Note that for a modular metric ω on a set X, and any $x, y \in X$, the function $\lambda \to \omega_\lambda(x,y)$ is non-increasing on $]0,+\infty[$. Indeed, if $0 < \mu < \lambda$, then

$$\omega_\lambda(x,y) \le \omega_{\lambda-\mu}(x,x) + \omega_\mu(x,y) = \omega_\mu(x,y).$$

Definition 2.2. ([8]) Let (X, ω) be a modular metric space. Fix $x_0 \in X$. Set

$$X_\omega = X_\omega(x_0) = \{x \in X : \omega_\lambda(x, x_0) \longrightarrow 0 \ \ as \ \ \lambda \longrightarrow \infty\},$$

and

$$X_\omega^* = X_\omega^*(x_0) = \{x \in X : \exists \lambda > 0 \ , \ \omega_\lambda(x, x_0) < \infty\}.$$

The two linear spaces X_ω and X_ω^* are said to be modular spaces (around x_0).

Note that X_ω is metrizable by the metric

$$d_\omega(x,y) = \inf\{t > 0 : \omega_t(x,y) \le t\}.$$

If ω is convex, then $X_\omega^* = X_\omega$ and we can endowed these sets with the metric d_ω^* defined by

$$d_\omega^*(x,y) = \inf\{t > 0 : \omega_t(x,y) \le 1\}.$$

Definition 2.3. ([2]) Let ω be a modular metric on X.

(1) We say that a sequence $\{x_n\} \subset X_\omega$ is $\omega-$convergent to some $x \in X_\omega$ if and only if $\lim\limits_{n\to+\infty} \omega_1(x_n, x) = 0$. We will call x the $\omega-$limit of $\{x_n\}$.
(2) We say that a sequence $\{x_n\} \subset X_\omega$ is $\omega-$Cauchy if

$$\lim_{n,m\to+\infty} \omega_1(x_n, x_m) = 0.$$

(3) We say that $M \subset X_\omega$ is $\omega-$closed if the $\omega-$limit of an $\omega-$convergent sequence of M is in M.
(4) We say that $M \subset X_\omega$ is $\omega-$complete if any $\omega-$Cauchy sequence in M is $\omega-$convergent and its $\omega-$limit belongs to M.
(5) We say that ω satisfies Fatou property if we have

$$\omega_1(x,y) \leq \liminf_{n\to+\infty} \omega_1(x_n,y)$$

for any sequence $\{x_n\} \subset X_\omega$ which $\omega-$converges to x and for any $y \in X_\omega$.

Definition 2.4. ([8]) Let ω be a modular metric on X. We say that ω satisfies the Δ_2-condition, or simply ω is Δ_2, if, given a sequence $\{x_n\} \subset X_\omega$, $x \in X_\omega$ and $\lambda > 0$ such that $\lim_{n\to+\infty} \omega_\lambda(x_n,x) = 0$, we have $\lim_{n\to+\infty} \omega_{\frac{\lambda}{2}}(x_n,x) = 0$.

Definition 2.5. Let ω be a modular metric on X. We say that ω satisfies the Δ_2-type condition if there exists a constant $K > 0$ such that

$$\omega_{\frac{\lambda}{2}}(x,y) \leq K\omega_\lambda(x,y),$$

for all $x, y \in X_\omega$ and any $\lambda > 0$.

The following results are immediate:

Lemma 2.6. *If ω satisfies the Δ_2- type condition, then ω satisfies the Δ_2-condition.*

Lemma 2.7. *Let $\{x_n\}$ be a sequence in X_ω. Let $\lambda > 0$. If ω satisfies the Δ_2- type condition, then $\{x_n\}$ is $\omega-$Cauchy if and only if $\lim_{n,m\to+\infty} \omega_\lambda(x_n,x_m) = 0$.*

Lemma 2.8. *If ω satisfies the Δ_2- type condition, then ω is regular.*

Lemma 2.9. *If ω satisfies the Δ_2- type condition, then $\omega_\lambda(x,y) < \infty$, for all $\lambda > 0$ and for all $(x,y) \in X_\omega^2$.*

Proof. Suppose that there exists $\lambda > 0$ and $x, y \in X_\omega = X_\omega(x_0)$ such that $\omega_\lambda(x,y) = \infty$. Since ω satisfies the Δ_2- type condition, then, for all $n \in \mathbb{N}^*$, $\omega_{2^n\lambda}(x,y) = \infty$. Since $\omega_{2^n\lambda}(x,y) \leq \omega_{2^{n-1}\lambda}(x,x_0) + \omega_{2^{n-1}\lambda}(y,x_0)$ and $x, y \in X_\omega(x_0)$, we have: $\lim_{n\to+\infty} \omega_{2^{n-1}\lambda}(x,x_0) = 0$ and $\lim_{n\to+\infty} \omega_{2^{n-1}\lambda}(y,x_0) = 0$. Then, $\lim_{n\to+\infty} \omega_{2^n\lambda}(x,y) = 0$. Which is a contradiction. □

Lemma 2.10. *Let (X,ω) be a modular space. Let $\{x_n\}$ be a sequence in X_ω. If $\{x_n\}$ is not $\omega-$Cauchy, then there exists $\varepsilon > 0$ and two subsequences of integers $\{n_k\}$ and $\{m_k\}$ such that $n_k > m_k \geq k$, $\omega_1(x_{n_k}, x_{m_k}) \geq \varepsilon$ and $\omega_{\frac{1}{2}}(x_{n_k-1}, x_{m_k}) < \varepsilon$.*

Proof. If we suppose that $(x_n)_{n\in\mathbb{N}}$ is not a $\omega-$Cauchy, then there exists $\varepsilon > 0$ and for all $k \in \mathbb{N}$ there exists $n_k, m_k \in \mathbb{N}$ such that $n_k > m_k \geq k$ and $\omega_1(x_{n_k}, x_{m_k}) \geq \varepsilon$. Let us fix $k \in \mathbb{N}$, and consider the set

$$\mathcal{A}_k = \{h \in \mathbb{N}^* \ : \ h > m_k \geq k \ and \ \omega_1(x_h, x_{m_k}) \geq \varepsilon\}.$$

Since $n_k \in \mathcal{A}_k$, then $\mathcal{A}_k \neq \emptyset$. Let us consider the set:

$$\mathcal{B}_k = \{h \in \mathcal{A}_k \ : \ \omega_{\frac{1}{2}}(x_h, x_{m_k}) \geq \varepsilon\}.$$

One can see that $\mathcal{B}_k \subseteq \mathbb{N}^*$ and $\mathcal{B}_k \neq \emptyset$. Since

$$\omega_{\frac{1}{2}}(x_{n_k}, x_{m_k}) \geq \omega_1(x_{n_k}, x_{m_k}) \geq \varepsilon,$$

then $\mathcal{B}_k$ admits the least element n'_k that belongs to $\mathcal{A}_k$, and so $n'_k > m_k \geq k,\ \omega_1(x_{n'_k}, x_{m_k}) \geq \varepsilon$ and $\omega_{\frac{1}{2}}(x_{n'_k-1}, x_{m_k}) < \varepsilon$. □

Using the same argument as in the proof of Lemma 2.10 and applying Lemma 2.7, we have the following:

Lemma 2.11. *Let $s, t \in \mathbb{N}^*$. If ω satisfies the Δ_2- type condition and $\{x_n\}$ is not a $\omega-$Cauchy sequence, then there exists $\varepsilon > 0$ and two subsequences of integers $\{n_k\}$ and $\{m_k\}$ such that $n_k > m_k \geq k$, $\omega_{2^s}(x_{n_k}, x_{m_k}) \geq \varepsilon$ and $\omega_{\frac{1}{2^t}}(x_{n_k-1}, x_{m_k}) < \varepsilon$.*

Lemma 2.12. *Let (X, ω) be a modular space such that ω is convex and satisfies the Δ_2-condition. If $\{x_n\}$ is a sequence in X_ω such that $\lim\limits_{n\to+\infty} \omega_1(x_n, x_{n+1}) = 0$, then $\{x_n\}$ is $\omega-$Cauchy.*

Proof. Suppose that $\{x_n\}$ is not $\omega-$Cauchy, then according to Lemma 2.10, there exists $\varepsilon > 0$ and two subsequences of integers $\{n_k\}$ and $\{m_k\}$ such that $n_k > m_k \geq k$ and $\omega_1(x_{n_k}, x_{m_k}) \geq \varepsilon$ and $\omega_{\frac{1}{2}}(x_{n_k-1}, x_{m_k}) < \varepsilon$. Since ω is convex, we have:

$$\omega_1(x_{n_k}, x_{m_k}) \leq \frac{1}{2}\omega_{\frac{1}{2}}(x_{n_k-1}, x_{m_k}) + \frac{1}{2}\omega_{\frac{1}{2}}(x_{n_k-1}, x_{n_k}).$$

Then, for all $k \in \mathbb{N}$

$$\varepsilon \leq \frac{\varepsilon}{2} + \frac{1}{2}\omega_{\frac{1}{2}}(x_{n_k-1}, x_{n_k})$$

Since ω satisfies the Δ_2-type condition, then $\lim\limits_{n\to+\infty} \omega_{\frac{1}{2}}(x_{n_k-1}, x_{n_k}) = 0$. So, $\varepsilon \leq \frac{\varepsilon}{2}$. Which is a contradiction. □

Definition 2.13. Let X be a nonempty set. Then $(X, \preceq, \omega)$ is called a partially ordered modular metric space if and only if (i) (X, ω) is a modular metric space and (ii) $(X, \preceq)$ is a partially ordered set.

Definition 2.14. Let $(X, \preceq, \omega)$ be a partially ordered modular metric space. We say that ω satisfies the property (P), if a non-decreasing sequence $\{x_n\}$ $\omega-$converges to some $x \in X_\omega$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.

Definition 2.15. Let C be a nonempty subset of X_ω. A self-mapping $T : C \to C$ is said to be $\omega-$continuous, if a sequence $\{x_n\}$ $\omega-$converges to some $x \in C$, then $\{Tx_n\}$ $\omega-$converges to Tx.

Definition 2.16. We say that a partially ordered set $(X, \preceq)$ is up-directed, if for all $(x, y) \in X^2$ there exists an element $z \in X$ such that $x \preceq z$ and $y \preceq z$.

We will use the following notations:
Let X be a nonempty set and S and T be a two self-mappings on X. We denote by $\mathfrak{F}(S)$ the fixed point set of S, i.e., $\mathfrak{F}(S) := \{x \in X : Sx = x\}$. Also, we denote by $\mathfrak{F}(S,T)$ the common fixed point set of S and T, i.e., $\mathfrak{F}(S,T) = \mathfrak{F}(S) \cap \mathfrak{F}(T)$

3. Main results

Let us consider three functions $\psi_1, \psi_2, \varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that:

(a) ψ_1,ψ_2 are continuous and φ is lower semi-continuous.
(b) ψ_1 is strictly increasing.
(c) For all $i \in \{1,2\}$, $\psi_i(0) = \varphi(0) = 0$.
(d) For all $t > 0$, $\psi_1(t) - \psi_2(t) + \varphi(t) > 0$.

3.1. Fixed point of $(\psi_1, \psi_2, \varphi)-$contraction.

In this section, we obtain fixed point results for a single mapping satisfying a $(\psi_1, \psi_2, \varphi)-$contractive condition in the framework of a partially ordered modular metric space.

Theorem 3.1. *Let $(X, \omega, \preceq)$ be a partially modular metric space. Assume that ω satisfies the Δ_2-type condition. Let C be an $\omega-$complete nonempty subset of X_ω. Let $T : C \to C$ be a non-decreasing self-mapping. If the following conditions are verified*

(i) *for all comparable elements $x, y \in C$,*

$$\psi_1(\omega_1(Tx, Ty)) \leq \psi_2(\omega_1(x, y)) - \varphi(\omega_1(x, y)) \tag{3.1}$$

(ii) *there exists an element $x_0 \in C$ such that $x_0 \preceq Tx_0$;*
(iii) *ω satisfies the property (P),*

then T has a fixed point in C. Moreover if $(C, \preceq)$ is up-directed, then the fixed point is unique.

Proof. We divide this proof into four steps.

Step.1. Consider the sequence $\{x_n\}$ defined by $x_n = T^n x_0$. One can see that

$$x_n \preceq x_{n+1}, \quad \text{for all } n \in \mathbb{N}.$$

If we suppose that there exists an integer n such that $x_n = x_{n+1}$, then T admits at least a fixed point in C. So, let us assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$.
In (i), if we take $x = x_n$ and $y = x_{n+1}$, we obtain:

$$\psi_1(\omega_1(x_{n+1}, x_{n+2})) \leq \psi_2(\omega_1(x_n, x_{n+1})) - \varphi(\omega_1(x_n, x_{n+1})) \leq \psi_1(\omega_1(x_n, x_{n+1})) \tag{3.2}$$

Then the sequence $\{\omega_1(x_n, x_{n+1})\}$ is decreasing and bounded below. So, it converges to some nonnegative real r. If we suppose that $r > 0$, then taking the limit superior as $n \to +\infty$ in the first part of the above inequality, we obtain

$$\limsup_{n\to+\infty} \psi_1(\omega_1(x_{n+1}, x_{n+2})) \leq \limsup_{n\to+\infty} \psi_2(\omega_1(x_n, x_{n+1})) - \liminf_{n\to+\infty} \varphi(\omega_1(x_n, x_{n+1})).$$

Using the continuity of ψ_1 and ψ_2 and the lower semi-continuity of φ and applying the condition (d), we obtain

$$\psi_1(r) \leq \psi_2(r) - \varphi(r) < \psi_1(r),$$

a contradiction. Therefore, $r = 0$.

Step.2. Let us prove that the sequence $\{x_n\}$ is $\omega-$Cauchy. Suppose that $\{x_n\}_{n\in\mathbb{N}}$ is not $\omega-$Cauchy. Then, according to Lemma 2.11 for ω_4 and $\omega_{\frac{1}{2}}$, there exists $\varepsilon > 0$ and subsequences of integers $\{m_k\}$ and $\{n_k\}$ such that

$$m_k > n_k \geq k,\ \ \omega_4(x_{m_k}, x_{n_k}) \geq \varepsilon \ \text{ and } \ \omega_{\frac{1}{2}}(x_{m_k-1}, x_{n_k}) < \varepsilon.$$

In (i), if we take $x = x_{m_k-1}$ and $y = x_{n_k-1}$, and since $x_{n_k-1} \preceq x_{m_k-1}$, we obtain:

$$\begin{aligned}\psi_1(\varepsilon) \leq \psi_1(\omega_4(x_{m_k}, x_{n_k})) &\leq \psi_1(\omega_1(x_{m_k}, x_{n_k})) \\ &\leq \psi_2(\omega_1(x_{m_k-1}, x_{n_k-1})) - \varphi(\omega_1(x_{m_k-1}, x_{n_k-1})) \quad (3.3)\end{aligned}$$

And we have

$$\varepsilon \leq \omega_4(x_{m_k}, x_{n_k}) \leq \omega_1(x_{m_k-1}, x_{n_k-1}) + \omega_1(x_{m_k-1}, x_{m_k}) + \omega_2(x_{n_k-1}, x_{n_k}),$$

and

$$\omega_1(x_{m_k-1}, x_{n_k-1}) \leq \omega_{\frac{1}{2}}(x_{m_k-1}, x_{n_k}) + \omega_{\frac{1}{2}}(x_{n_k-1}, x_{n_k}).$$

Thus

$$\varepsilon - \omega_1(x_{m_k-1}, x_{m_k}) - \omega_2(x_{n_k-1}, x_{n_k}) \leq \omega_1(x_{m_k-1}, x_{n_k-1}) \leq \varepsilon + \omega_{\frac{1}{2}}(x_{n_k-1}, x_{n_k}).$$

Since ω satisfies the Δ_2-type condition, then

$$\lim_{k\to+\infty} \omega_{\frac{1}{2}}(x_{n_k-1}, x_{n_k}) = \lim_{k\to+\infty} \omega_2(x_{n_k-1}, x_{n_k}) = 0.$$

So $\lim_{k\to+\infty} \omega_1(x_{m_k-1}, x_{n_k-1}) = \varepsilon$. And by letting $k \to +\infty$, in the inequality (3.3), we obtain

$$\psi_1(\varepsilon) \leq \psi_2(\varepsilon) - \varphi(\varepsilon) < \psi_1(\varepsilon)$$

which is a contradiction, then $\{x_n\}$ is $\omega-$Cauchy in C. Since C is $\omega-$complete, there exists $x \in C$ such that $\lim_{n\to+\infty} \omega_1(x_n, x) = 0$.

Step.3. Let us prove that x is a fixed point of T. Since $x_n \preceq x$, for all $n \in \mathbb{N}^*$, we have:

$$\psi_1(\omega_1(x_n, Tx)) \leq \psi_2(\omega_1(x_{n-1}, x)) - \varphi(\omega_1(x_{n-1}, x)) \leq \psi_1(\omega_1(x_{n-1}, x)).$$

Then $0 \leq \omega_1(x_n, Tx) \leq \omega_1(x_{n-1}, x)$ and by passing to limit, we obtain

$$\lim_{n\to+\infty} \omega_1(x_n, Tx) = 0.$$

And as

$$\omega_2(x,Tx) \le \omega_1(x_n,x) + \omega_1(x_n,Tx),$$

then $\omega_2(x,Tx) = 0$ and since ω is regular, we obtain $Tx = x$.

Step.4. Assume that $(C,\preceq)$ is up-directed. Let $y \in C$ be another fixed point of T. There exists $z \in C$ such that $x \preceq z$ and $y \preceq z$. As T is non-decreasing, we have for all $n \in \mathbb{N}$, $x \preceq T^n z = z_n$ and $y \preceq T^n z = z_n$. Then

$$\psi_1(\omega_1(x,z_{n+1})) \le \psi_2(\omega_1(x,z_n)) - \varphi(\omega_1(x,z_n)) \le \psi_1(\omega_1(x,z_n)).$$

Thus the sequence $\{\omega_1(x,z_n)\}$ is decreasing and bounded below. So, it converges to some $l \ge 0$ and by tending n to $+\infty$ in the above inequality, we obtain $l = 0$. By the same argument we prove that $\lim_{n\to+\infty} \omega_1(y,z_n) = 0$. By passing to limit in the following inequality

$$\omega_2(x,y) \le \omega_1(x,z_n) + \omega_1(y,z_n),$$

we obtain $\omega_2(x,y) = 0$. Since ω is regular, we have $x = y$. Then, T admits a unique fixed point in C. □

In what follows, we prove that Theorem 3.1 is still valid if we neglect the property (P) and assume that T is $\omega-$continuous.

Theorem 3.2. *Let $(X,\preceq,\omega)$ be a partially modular metric space. Assume that ω satisfies the Δ_2-type condition. Let C be an $\omega-$complete nonempty subset of X_ω. Let $T : C \to C$ be an $\omega-$continuous non-decreasing self-mapping. If the following conditions are verified*

(i) *for all comparable elements $x,y \in C$,*

$$\psi_1(\omega_1(Tx,Ty)) \le \psi_2(\omega_1(x,y)) - \varphi(\omega_1(x,y)) \tag{3.4}$$

(ii) *there exists an element $x_0 \in C$ such that $x_0 \preceq Tx_0$,*

then T has a fixed point in C. Moreover if $(C,\preceq)$ is up-directed, then the fixed point is unique.

Proof. Following the proof of Theorem 3.1 we only have to check that $Tx = x$. As $\{x_n\}$ $\omega-$converges to x and T is $\omega-$continuous, then the sequence $\{x_{n+1}\} = \{Tx_n\}$ is $\omega-$convergent to Tx. And since the regularity of ω implies the uniqueness of limit, we obtain $Tx = x$. □

If we take $\psi_1 = \psi_2 = \psi$, and if we define ω by:

$$\omega_\lambda(x,y) = \tfrac{d(x,y)}{\lambda}, \text{ for all } (x,y) \in X^2$$

we obtain the following result proved, in 2010, by Harjani and Sadarangani in the setting of metric spaces [13, Theorem 2.1 and Theorem 2.2]).

Corollary 3.3. *Let $(X,\preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X,d) is a complete metric space. Let $T : X \to X$ be a non-decreasing self-mapping. If the following conditions are verified*

(i) *for all comparable elements* $x, y \in C$,

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \varphi(d(x, y)) \tag{3.5}$$

(ii) *there exists an element* $x_0 \in C$ *such that* $x_0 \preceq Tx_0$;

(iii) X *satisfies the property (P) or* T *is* $\omega-$*continuous,*

then T *has a unique fixed point in* C *provided that* $(C, \preceq)$ *is up-directed.*

3.2. **Common fixed point of generalized** $(\psi_1, \psi_2, \varphi)-$**contraction.** In this section, we obtain a common fixed point for a pair of mappings satisfying a generalized $(\psi_1, \psi_2, \varphi)-$contractive condition in the framework of a partially ordered convex modular metric space. We set

$$M(x, y) = \max\{\omega_1(x, y), \omega_1(x, Sx), \omega_1(y, Ty), \omega_2(x, Ty) + \omega_2(y, Sx)\}.$$

Theorem 3.4. *Let* $(X, \preceq, \omega)$ *be a partially ordered modular metric space where* ω *is convex and satisfies the* Δ_2-*type condition. Let* C *be an* $\omega-$*complete nonempty subset of* X_ω *and* $T, S : C \to C$ *be two self-mappings. If the following conditions are verified:*

(i) *for all comparable elements* $x, y \in C$,

$$\psi_1(\omega_1(Sx, Ty)) \leq \psi_2(M(x, y)) - \varphi(M(x, y)); \tag{3.6}$$

(ii) ω *satisfies Fatou property or* T *is* $\omega-$*continuous;*

(iii) *there exists an element* $x_0 \in C$ *such that*

$$x_0 \preceq Sx_0 \preceq TSx_0 \preceq STSx_0 \preceq (TS)^2x_0 \preceq S(TS)^2x_0 \preceq ...$$

(iv) ω *satisfies the property (P),*

then S *and* T *have a common fixed point in* C *and* $\mathfrak{F}(S, T) = \mathfrak{F}(S) = \mathfrak{F}(T)$. *Particulary, if* $\mathfrak{F}(S, T)$ *is totally ordered, then* T *and* S *have a unique fixed point.*

Proof. Consider the sequence $\{x_n\}$ defined by

$$x_{2n+1} = Sx_{2n} \text{ and } x_{2n+2} = Tx_{2n+1}, \text{ for all } n \in \mathbb{N}.$$

The condition (ii) insures that $\{x_n\}$ is non-decreasing. If there exists an integer n such that

$$x_{2n} = x_{2n+1} = x_{2n+2},$$

then x_{2n} is a common fixed point of S and T. Otherwise, suppose that

$$x_{2n} \neq x_{2n+1} \text{ or } x_{2n} \neq x_{2n+2}, \text{ for all } n \in \mathbb{N}.$$

Let $n \in \mathbb{N}$. From $x_{2n} \preceq x_{2n+1}$ and applying the inequality (3.6) for $x = x_{2n}$ and $y = x_{2n+1}$, we obtain

$$\psi_1(\omega_1(x_{2n+1}, x_{2n+2})) \leq \psi_2(M(x_{2n}, x_{2n+1})) - \varphi(M(x_{2n}, x_{2n+1})) \tag{3.7}$$

where

$$M(x_{2n},x_{2n+1}) = \max\{\omega_1(x_{2n},x_{2n+1}),\omega_1(x_{2n+1},x_{2n+2}),\omega_2(x_{2n},x_{2n+2})\}.$$

Since ω is convex, we have

$$\omega_2(x_{2n},x_{2n+2}) \leq \frac{1}{2}(\omega_1(x_{2n},x_{2n+1}) + \omega_1(x_{2n+1},x_{2n+2})).$$

Then

$$M(x_{2n},x_{2n+1}) = \max\{\omega_1(x_{2n},x_{2n+1}),\omega_1(x_{2n+1},x_{2n+2})\}.$$

If we suppose that there exists an integer n such that:

$$\omega_1(x_{2n},x_{2n+1}) \leq \omega_1(x_{2n+1},x_{2n+2}),$$

then

$$M(x_{2n},x_{2n+1}) = \omega_1(x_{2n+1},x_{2n+2}).$$

Thus

$$\begin{aligned}\psi_1(\omega_1(x_{2n+1},x_{2n+2})) &\leq \psi_2(\omega_1(x_{2n+1},x_{2n+2})) - \varphi(\omega_1(x_{2n+1},x_{2n+2}))\\ &< \psi_1(\omega_1(x_{2n+1},x_{2n+2})),\end{aligned}$$

a contradiction. Hence, for all $n \in \mathbb{N}$, $\omega_1(x_{2n+1},x_{2n+2}) < \omega_1(x_{2n},x_{2n+1})$.
By the same argument, if we take, in the inequality (3.6), $x = x_{2n-1}$ and $y = x_{2n}$ we obtain

$$\omega_1(x_{2n},x_{2n+1}) < \omega_1(x_{2n-1},x_{2n}), \text{ for all } n \in \mathbb{N}^*.$$

Then $\omega_1(x_{n+1},x_{n+2}) < \omega_1(x_n,x_{n+1})$, for all $n \in \mathbb{N}$. Thus, the sequence $\{\omega_1(x_n,x_{n+1})\}$ is decreasing and bounded below. Therefore, it $\omega-$converges to some $r \geq 0$. By passing to upper limit in the inequality (3.7), we obtain

$$\limsup_{n\to+\infty}\psi_1(\omega_1(x_{2n+1},x_{2n+2})) \leq \limsup_{n\to+\infty}\psi_2(M(x_{2n},x_{2n+1})) - \liminf_{n\to+\infty}\varphi(M(x_{2n},x_{2n+1}))$$

Since

$$\lim_{n\to+\infty} M(x_{2n},x_{2n+1}) = \lim_{n\to+\infty}\max\{\omega_1(x_{2n},x_{2n+1}),\omega_1(x_{2n+1},x_{2n+2})\} = r,$$

and using the continuity of ψ_1 and ψ_2 and the lower semi-continuity of φ, we get $\psi_1(r) \leq \psi_2(r) - \varphi(r)$, which implies that $r = 0$. Thus, $\lim\limits_{n\to+\infty}\omega_1(x_n,x_{n+1}) = 0$. Since ω is convex and satisfies the Δ_2-condition and according to lemma 2.12, then $\{x_n\}$ is $\omega-$Cauchy sequence in C. Thus, from the completeness, $\{x_n\}$ is $\omega-$convergent to some $x \in C$.

If T is $\omega-$continuous, then $\{x_{n+1}\}=\{Tx_n\}$ is $\omega-$convergent to some x and from the uniqueness of the limit we have $Tx = x$.

If ω satisfies Fatou property, we have $\omega_1(x,Tx)) \leq \liminf\limits_{n\to+\infty}\omega_1(x_{2n+1},Tx)$. Following to the condition (P), $x_{2n} \preceq x$, for all $n \in \mathbb{N}$. Then

$$\psi_1(\omega_1(x_{2n+1},Tx)) \leq \psi_2(M(x_{2n},x)) - \varphi(M(x_{2n},x)) \tag{3.8}$$

where

$$M(x_{2n},x)=\max\{\omega_1(x_{2n},x),\omega_1(x_{2n},x_{2n+1}),\omega_1(x,Tx),\omega_2(x_{2n},Tx)+\omega_2(x,x_{2n+1})\}.$$

From

$$\omega_2(x_{2n},Tx)\leq\omega_1(x_{2n},x)+\omega_1(x,Tx),$$

we obtain

$$\lim_{n\to+\infty}M(x_{2n},x)=\omega_1(x,Tx).$$

Then

$$\begin{aligned}\psi_1(\omega_1(x,Tx))&\leq\psi_1(\liminf_{n\to+\infty}\omega_1(x_{2n+1},Tx))\\&\leq\limsup_{n\to+\infty}\psi_1(\omega_1(x_{2n+1},Tx))\\&\leq\limsup_{n\to+\infty}(\psi_2(M(x_{2n},x)))-\liminf_{n\to+\infty}(\varphi(M(x_{2n},x)))\\&\leq\psi_2(\omega_1(x,Tx))-\varphi(\omega_1(x,Tx)).\end{aligned}$$

Which implies that $\omega_1(x,Tx)=0$. The regularity insures that $Tx=x$.
Now, let us prove that $Sx=x$. If we take $y=x$ into the inequality (3.6), we obtain

$$\psi_1(\omega_1(Sx,x))\leq\psi_2(M(x,x))-\varphi(M(x,x)).$$

Since $M(x,x)=\omega_1(Sx,x)$, then

$$\psi_1(\omega_1(Sx,x))\leq\psi_2(\omega_1(Sx,x))-\varphi(\omega_1(Sx,x)),$$

which implies that $\omega_1(Sx,x)=0$. So, from the regularity of ω, we conclude that $Sx=x$.
Let us suppose that there exits another common fixed point y of T and S. If we assume that $\mathfrak{F}(T,S)$ is totally ordered, then x and y are comparable and according to (3.6), we have

$$\psi_1(\omega_1(x,y))\leq\psi_2(\omega_1(x,y))-\varphi(\omega_1(x,y)),$$

which insures that $\omega_1(x,y)=0$ and so $x=y$. Therefore, the uniqueness of the common fixed point of S and T. □

Remark 3.5. If we define ω by:

$$\omega_\lambda(x,y)=\frac{d(x,y)}{\lambda},\ \text{for all } (x,y)\in X^2,$$

we obtain Theorem 1.2 established by Abkar and Choudhury in [1].

EXAMPLE 3.6. Consider the space $X=[0,1]$ ordered by "$\preceq$" which is the reverse of the usual order between the reals ($x\preceq y\Leftrightarrow x\geq y$) and endowed with the modular metric defined for all $\lambda>0$ as follows:

$$\omega_\lambda(x,y)=\begin{cases}\frac{x+y}{\lambda} & if \quad x\neq y\\ 0 & if \quad x=y\end{cases}$$

Consider the two self-mappings S and T defined as follows:

$$Sx = \frac{x}{4}, \text{ for all } x \in [0,1] \quad \text{and} \quad Tx = \begin{cases} \frac{x}{8} & if \quad 0 \le x < 1 \\ \frac{1}{2} & if \quad x = 1 \end{cases}$$

Consider the three functions defined for $t, s \in [0, +\infty[$ as follows:

$$\psi_1(t) = t, \psi_2(t) = \frac{3t}{4}, \ \varphi(t) = \frac{t}{8}.$$

We can see that the functions ψ_1, ψ_2 and φ satisfy all conditions described in the top of the section 3. It's easy to verify that

(i) $X_\omega = X$ is an $\omega-$complete modular metric space;
(ii) ω is convex and satisfies the Δ_2-type condition and Fatou property;
(iii) $1 \preceq S1 \preceq TS1 \preceq STS1 \preceq (TS)^2 1 \preceq S(TS)^2 1 \preceq ...$
(iv) $(X, \preceq)$ satisfies the property (P);

Let x and y be two comparable elements in X. Let us show that

$$\psi_1(\omega_1(Sx, Ty)) \le \psi_2(M(x,y)) - \varphi(M(x,y)). \tag{3.9}$$

i.e,

$$\omega_1(\frac{x}{4}, Ty) \le \frac{5}{8} M(x,y).$$

Case.1. If $x = y = 1$, then $\omega_1(\frac{1}{4}, \frac{1}{2}) = \frac{3}{4}$ and $M(1,1) = \frac{3}{2}$.
Case.2. If $x = 1$ and $y \in [0, 1[$, then

$$\omega_1(\tfrac{1}{4}, \tfrac{y}{8}) = \tfrac{2+y}{8} \text{ and } M(1,y) = \max\{1+y, \tfrac{5}{4}\}.$$

- If $y < \frac{1}{4}$, $M(1,y) = \frac{5}{4}$.
- If $y \ge \frac{1}{4}$, $M(1,y) = 1 + y$.

Case.3. If $y = 1$ and $x \in [0,1[$, then

$$\omega_1(\tfrac{x}{4}, \tfrac{1}{2}) = \tfrac{2x+4}{8} \text{ and } M(x,1) = \max\{1+x, \tfrac{3}{2}\}.$$

- If $x < \frac{1}{2}$, $M(x,1) = \frac{3}{2}$.
- If $x \ge \frac{1}{2}$, $M(x,1) = 1 + x$.

Case.4. If $x, y \in [0,1[$ and $x = y$, then $\omega_1(\frac{x}{4}, \frac{y}{8}) = \frac{3x}{8}$ and $M(x,y) = \frac{5x}{4}$.
Case.5. If $x, y \in [0,1[$ and $x > y$, then

$$\omega_1(\tfrac{x}{4}, \tfrac{y}{8}) = \tfrac{2x+y}{8} \text{ and } M(x,y) = \max\{\tfrac{5x}{4}, x+y\}.$$

- If $y \ge \frac{x}{4}$, then $M(x,y) = x + y$.
- If $y < \frac{x}{4}$, then $M(x,y) = \frac{5x}{4}$.

Case.6. If $x, y \in [0,1[$ and $x < y$, then $M(x,y) = \max\{x+y, \frac{9y}{8}\}$.

- If $x = \frac{y}{2}$, then $\omega_1(\frac{x}{4}, \frac{y}{8}) = 0$.
- If $x \in]\frac{y}{8}, \frac{y}{2}[\cup]\frac{y}{2}, y[$, then $\omega_1(\frac{x}{4}, \frac{y}{8}) = \frac{2x+y}{8}$ and $M(x,y) = x + y$.
- If $x \le \frac{y}{8}$, then $\omega_1(\frac{x}{4}, \frac{y}{8}) = \frac{2x+y}{8}$ and $M(x,y) = \frac{9y}{8}$.

One can easily see that (3.9) holds in all cases. Hence S and T verify all conditions of Theorem 3.4 and have a unique common fixed point which is 0.

4. Conclusion

The results in this paper,

(1) extend the work of Dutta and Choudhury in [11] from a metric to a partially ordered modular metric space.
(2) extend Theorem 2.2 of Abkar and Choudhury in [1] from a partially ordered metric space to a partially ordered modular metric space.
(3) extend Theorem 2.1 and Theorem 2.2 established by Harjani and Sadarangani in [13] from a metric space to a partially ordered modular metric space.

Remark 4.1. Recently, Jleli et al.[14] and Khamsi [15] replaced the Δ_2-condition by a weaker condition. One wonders if one can weaken the Δ_2-condition in the case of weak contractions defined on metric modular spaces.

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