

Copresented Dimension of Modules

M. Amini*, F. Hassani

Department of Mathematics, Payame Noor University, Tehran, Iran.

E-mail: mostafa.amini@pnu.ac.ir

E-mail: Hassani@pnu.ac.ir

ABSTRACT. In this paper, a new homological dimension of modules, copresented dimension, is defined. We study some basic properties of this homological dimension. Some ring extensions are considered, too. For instance, we prove that if $S \geq R$ is a finite normalizing extension and S_R is a projective module, then for each right S -module M_S , the copresented dimension of M_S does not exceed the copresented dimension of $\text{Hom}_R(S, M)$.

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1. INTRODUCTION

Throughout this paper, R is an associative ring with identity and all modules are unitary. First we recall some known notions and facts needed in the sequel. Let R be a ring, n a non-negative integer and M an R -module. Then

- (1) M is said to be *finitely cogenerated* [1] if for every family $\{V_k\}_J$ of submodules of M with $\bigcap_J V_k = 0$, there is a finite subset $I \subset J$ with $\bigcap_I V_k = 0$.
- (2) M is said to be *n-copresented* [14] if there is an exact sequence of R -modules $0 \rightarrow M \rightarrow E^0 \rightarrow E^1 \rightarrow \cdots \rightarrow E^n$, where each E^i is a finitely cogenerated injective module.

*Corresponding Author

- (3) R is called *right co-coherent* [17] if every finitely cogenerated factor module of a finitely cogenerated injective R -module is finitely copresented.
- (4) R is called *n -cocoherent* [14] in case every n -copresented R -module is $(n+1)$ -copresented. It is easy to see that R is cocoherent if and only if it is 1-cocoherent. Recall that a ring R is called *right conoethrian* [4] if every factor module of a finitely cogenerated R -module is finitely cogenerated. By [4, Proposition 17], a ring R is co-noethrian if and only if it is 0-cocoherent.
- (5) M is said to be *n -presented* [5] if there is an exact sequence of R -modules $F_n \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$, where each F_i is a finitely generated free module.
- (6) R is called *coherent* [18] in case every 0-presented R -module is 1-presented.
- (7) A ring extension $R \subseteq R'$ with characteristic $p > 0$ is called a *purely inseparable extension* [10] if for every element $r' \in R'$, there exists a non-negative integer n such that $r'^{p^n} \in R$.
- (8) For any commutative ring R of prime characteristic $p > 0$, assume that $F_R : R \rightarrow R^{(e)}$ is the e -th iterated Frobenius map in which $R^{(e)} \cong R$. Then, the *perfect closure* [9] of R , denoted by R^∞ , is defined as the limit of the following direct system:

$$R \xrightarrow{F_R} R \xrightarrow{F_R} R \xrightarrow{F_R} \cdots$$

- (9) M is called *(n, d) -injective* [18] if $\text{Ext}_R^{d+1}(N, M) = 0$ for any n -presented right R -module N . It is clear that M is $(0, 0)$ -injective if and only if M is injective.
- (10) Assume that $S \geq R$ is a unitary ring extension. Then, the ring S is called *right R -projective* [6] in case, for any right S -module M_S with an S -module N_S , $N_R \mid M_R$ implies $N_S \mid M_S$, where $N \mid M$ means that N is a direct summand of M .
- (11) The ring extension $S \geq R$ is called a *finite normalizing extension* [8] in case there is a finite subset $\{s_1, \dots, s_n\} \subseteq S$ such that $S = \sum_{i=1}^n s_i R$ and $s_i R = R s_i$ for $i = 1, \dots, n$.
- (12) A finite normalizing extension $S \geq R$ is called an *almost excellent extension* [12] in case ${}_R S$ is flat, S_R is projective, and the ring S is right R -projective.

In this paper, we introduce the dual concepts of *presented dimensions* of R -modules. We also, introduce the *copresented dimension* of any R -module M :

$\text{FEd}(M) = \inf\{m \mid \text{there exists an injective resolution } 0 \rightarrow M \rightarrow E^0 \rightarrow \cdots \rightarrow E^m \rightarrow \cdots \rightarrow E^{m+i} \rightarrow \cdots, \text{ such that } E^{m+i} \text{ are finitely cogenerated for}$

$i = 0, 1, 2, \dots\}$. If $K = \ker(E^m \rightarrow E^{m+1})$, then K has an infinite finite copresentation. It is clear that any copresented dimension is finitely copresented dimension (see [16]). Also, the copresented dimension of ring R is defined to be:

$$\text{FED}(R) = \sup\{\text{FEd}(M) \mid M \text{ is a finitely cogenerated module}\}.$$

Then, some basic properties of the copresented dimensions of modules are studied. For example, it is shown that if $\text{FEd}(M) < \infty$, then $\text{id}(M) \leq n$ if and only if $\text{Ext}_R^{n+1}(N, M) = 0$ for every strongly copresented R -module N . Also, it is proved that $\text{FED}(R \oplus S) = \sup\{\text{FED}(R), \text{FED}(S)\}$, for any two rings R and S . Also, some characterizations of the copresented dimensions of modules on Ring Extensions are determined. For instance, let $S \geq R$ be a finite normalizing extension with S_R projective as an R -module, then for any right R -module M_R , we have $\text{FED}(\text{Hom}_R(S, M))_S \leq \text{FEd}(M_R)$. Finally, we give a sufficient condition under which $\text{FED}(S) \leq \text{FED}(R)$ and or $\text{FED}(R) < \text{FED}(S) + \max\{k, d\}$, where $k = \text{id}(S_R)$ and $d = \sup\{\text{FEd}(M_R) \mid M \in \text{Mod} - S \text{ and } \text{FEd}(M_S) = 0\}$.

2. MAIN RESULTS

We start this section with the following definition which is the dual of the presented dimension of a module.

Definition 2.1. For any R -module M , we define the copresented dimension of M to be $\text{FEd}(M) = \inf\{m \mid \text{there exists an injective resolution } 0 \rightarrow M \rightarrow E^0 \rightarrow \dots \rightarrow E^m \rightarrow \dots \rightarrow E^{m+i} \rightarrow \dots, \text{ so that } E^{m+i} \text{ are finitely cogenerated for } i = 0, 1, 2, \dots\}$. In particular, a module M is called strongly copresented module if $\text{FEd}(M) = 0$.

Proposition 2.2. For any R -module M , $\text{FEd}(M) \leq \text{id}(M) + 1$.

Proof. It is a direct consequence of Definition 2.1. □

EXAMPLE 2.3. Let $R = \mathbb{Z}$. Since $\text{id}(\mathbb{Z}_{p^\infty}) = 0$, we have $\text{FEd}(\mathbb{Z}_{p^\infty}) \leq 1$. On the other hand, $\mathbb{Z}_{p^\infty}$ is finitely cogenerated by [1, p.124]. So by Definition 2.1, $\text{FEd}(\mathbb{Z}_{p^\infty}) = 0$.

Now, we study the behavior of the copresented dimension on the exact sequences. Before this we need the following lemma.

Lemma 2.4. Let $0 \rightarrow A \xrightarrow{f'} B \xrightarrow{f} C \rightarrow 0$ be a short exact sequence of R -modules. Then:

- (1) If $0 \rightarrow A \rightarrow A^0 \rightarrow A^1 \rightarrow \dots$ and $0 \rightarrow C \rightarrow C^0 \rightarrow C^1 \rightarrow \dots$ are injective resolutions of A and C , respectively. Then the exact sequence

$$0 \longrightarrow B \longrightarrow A^0 \oplus C^0 \longrightarrow A^1 \oplus C^1 \longrightarrow \dots$$

is an injective resolution of B .

- (2) If $0 \rightarrow B \rightarrow B^0 \rightarrow B^1 \rightarrow \dots$ and $0 \rightarrow C \rightarrow C^0 \rightarrow C^1 \rightarrow \dots$ are injective resolutions of B and C , respectively. Then the exact sequence

$$0 \rightarrow A \rightarrow B^0 \rightarrow D^0 \rightarrow D^1 \rightarrow \dots$$

is an injective resolution of A , where $D^i = C^i \oplus B^{i+1}$ for any $i \geq 0$.

- (3) If $0 \rightarrow B \rightarrow B^0 \rightarrow B^1 \rightarrow \dots$ and $0 \rightarrow A \rightarrow A^0 \rightarrow A^1 \rightarrow \dots$ are injective resolutions of B and A , respectively. Then the exact sequence

$$0 \rightarrow C \rightarrow F^0 \rightarrow E^0 \rightarrow E^1 \rightarrow \dots$$

is an injective resolution of C , where $F^0 = B^0 \oplus A^1$ and $E^i = A^0 \oplus B^{i+1} \oplus A^{i+2}$ for any $i \geq 0$.

Proof. (1) The proof is similar to that of [3, Theorem 2.4].

(2) Let $0 \rightarrow B \rightarrow B^0 \rightarrow B^1 \rightarrow \dots$ be an injective resolution of B . Then, the exact sequences

$0 \rightarrow K \rightarrow B^1 \rightarrow B^2 \rightarrow \dots$ and $0 \rightarrow B \rightarrow B^0 \rightarrow K \rightarrow 0$ exist, where $K = \frac{B^0}{B}$. Now, we consider the following commutative diagram:

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \downarrow & & \downarrow & \\
 0 & \rightarrow & A & \rightarrow & B & \rightarrow & C \rightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 0 & \rightarrow & A & \rightarrow & B^0 & \rightarrow & D \rightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & K & = & K \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

By (1), there is an exact sequence

$$0 \rightarrow D \rightarrow D^0 \rightarrow D^1 \rightarrow D^2 \rightarrow \dots$$

of injective R -modules D^i such that $D^i = C^i \oplus B^{i+1}$ for any $i \geq 0$.

Combining this sequence with the exact sequence $0 \rightarrow A \rightarrow B^0 \rightarrow D \rightarrow 0$, we get the exact sequence

$$0 \rightarrow A \rightarrow B^0 \rightarrow D^0 \rightarrow D^1 \rightarrow \dots,$$

where B^0 and D^i are injective for any $i \geq 0$.

(3) Let $0 \rightarrow A \rightarrow A^0 \rightarrow A^1 \rightarrow \dots$ be an injective resolution of A . Then, the exact sequences

$0 \rightarrow K \rightarrow A^1 \rightarrow A^2 \rightarrow \dots$ and $0 \rightarrow A \rightarrow A^0 \rightarrow K \rightarrow 0$ exist, where $K = \frac{A^0}{A}$. Now, we consider the following commutative diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \longrightarrow & A^0 & \longrightarrow & F & \longrightarrow & C \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & K & = & K & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

By (1), there is an exact sequence

$$0 \longrightarrow F \longrightarrow F^0 \longrightarrow F^1 \longrightarrow F^2 \longrightarrow \dots$$

of injective R -modules F^i such that $F^i = B^i \oplus A^{i+1}$ for any $i \geq 0$.

It is clear that $F = A^0 \oplus C$. So, the exact sequence $0 \rightarrow C \rightarrow F \rightarrow A^0 \rightarrow 0$ exists. Let $K = \frac{F^0}{F}$, then we obtain the following commutative diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & C & \longrightarrow & F & \longrightarrow & A^0 \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 0 & \longrightarrow & C & \longrightarrow & F^0 & \longrightarrow & E \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & K & = & K \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

Therefore by (1), the sequence

$$0 \longrightarrow E \longrightarrow E^0 \longrightarrow E^1 \longrightarrow E^2 \longrightarrow \dots$$

is an injective resolution of E , where $E^i = A^0 \oplus F^{i+1} = A^0 \oplus B^{i+1} \oplus A^{i+2}$ for any $i \geq 0$. Combining this sequence with the exact sequence $0 \rightarrow C \rightarrow F^0 \rightarrow E \rightarrow 0$, we get the exact sequence

$$0 \longrightarrow C \longrightarrow F^0 \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \dots,$$

where F^0 and E^i are injective for any $i \geq 0$. □

Theorem 2.5. Let $0 \rightarrow A \xrightarrow{g} B \xrightarrow{f} C \rightarrow 0$ be an exact sequence of R -modules. Then $\text{FEd}(B) \leq \max\{\text{FEd}(A), \text{FEd}(C)\}$, $\text{FEd}(C) \leq \max\{\text{FEd}(B), \text{FEd}(A) + 1\}$, $\text{FEd}(A) \leq \max\{\text{FEd}(B), \text{FEd}(C) - 1\}$.

Proof. Assume that $\mathbf{E}'$ is an injective resolution of A and $\mathbf{E}''$ is an injective resolution of C . Thus by Lemma 2.5(1), there exists an injective resolution $\mathbf{E}$ of B such that

$$0 \rightarrow \mathbf{E}'^A \rightarrow \mathbf{E}^B = \mathbf{E}'^A \oplus \mathbf{E}''^C \rightarrow \mathbf{E}''^C \rightarrow 0$$

is an exact sequence of complexes. Hence for every $m \geq \max\{\text{FEd}(A), \text{FEd}(C)\}$, E^m is finitely cogenerated. So, we deduce that $\text{FEd}(B) \leq \max\{\text{FEd}(A), \text{FEd}(C)\}$.

Assume that $\mathbf{E}''$ is an injective resolution of C and $\mathbf{E}$ is an injective resolution of B . Thus by Lemma 2.5(2), the exact sequence

$$0 \longrightarrow A \longrightarrow E^0 \longrightarrow D^0 \longrightarrow D^1 \longrightarrow \cdots \longrightarrow D^d \longrightarrow \cdots$$

is an injective resolution of A . So for every $d \geq \max\{\text{FEd}(B), \text{FEd}(C) - 1\}$, D^d is finitely cogenerated. Thus, we have that $\text{FEd}(A) \leq \max\{\text{FEd}(B), \text{FEd}(C) - 1\}$. Also, it is prove that $\text{FEd}(C) \leq \max\{\text{FEd}(B), \text{FEd}(A) + 1\}$. $\square$

The proof of the following Corollary is similar to the proof of [19, Corollary 2.7].

Corollary 2.6. *If $\text{FEd}(M_1), \text{FEd}(M_2), \dots, \text{FEd}(M_d)$ are finite, then:*

$$\text{FEd}(\oplus M_i) = \max\{\text{FEd}(M_i) \mid i = 1, \dots, d\}.$$

Proof. For the case $m = 2$, the exact sequences

$$0 \rightarrow M_1 \rightarrow M_1 \oplus M_2 \rightarrow M_2 \rightarrow 0$$

and

$$0 \rightarrow M_2 \rightarrow M_2 \oplus M_1 \rightarrow M_1 \rightarrow 0$$

exist. Thus by Theorem 2.5, we deduce that

$$\text{FEd}(M_2) \leq \max\{\text{FEd}(M_1 \oplus M_2), \text{FEd}(M_1) - 1\},$$

$$\text{FEd}(M_1) \leq \max\{\text{FEd}(M_1 \oplus M_2), \text{FEd}(M_2) - 1\}$$

and

$$\text{FEd}(M_1 \oplus M_2) \leq \max\{\text{FEd}(M_1), \text{FEd}(M_2)\}.$$

Assume that $\text{FEd}(M_1) < \text{FEd}(M_2)$. Then $\text{FEd}(M_1) \leq \text{FEd}(M_2) - 1$, and we have:

$$\text{FEd}(M_2) \leq \max\{\text{FEd}(M_1 \oplus M_2), \text{FEd}(M_2) - 2\} = \text{FEd}(M_1 \oplus M_2).$$

Also, similarly $\text{FEd}(M_1) \leq \text{FEd}(M_1 \oplus M_2)$. So, we conclude that $\text{FEd}(M_1 \oplus M_2) = \max\{\text{FEd}(M_1), \text{FEd}(M_2)\}$. $\square$

Proposition 2.7. *Let n be a non-negative integer. Then the following statements are equivalent:*

- (1) $\text{id}(M) \leq n$ for every strongly copresented R -module M ;
- (2) $\text{Ext}_R^{n+1}(N, M) = 0$ for every strongly copresented R -module N .

Proof. (1) $\Rightarrow$ (2) This is obvious.

(2) $\Rightarrow$ (1) We use the induction on n . Let $n = 0$. Since $\text{Ext}_R^1(N, M) = 0$ for any strongly copresented R -module N , by using the exact sequence $0 \rightarrow M \rightarrow E^0 \rightarrow L^0 \rightarrow 0$ where E^0 is finitely cogenerated and L^0 is strongly copresented, we deduce that $\text{Ext}_R^1(L^0, M) = 0$. Therefore by [7, Theorem 7.31], the exact sequence above is split. So, M is injective and hence $\text{id}(M) \leq 0$. Assume that

$n > 0$. By [7, Corollary 6.42], we have that $\text{Ext}_R^{n+1}(N, M) \cong \text{Ext}_R^n(N, L^0) = 0$. Thus by induction hypothesis, $\text{id}(L^0) \leq n - 1$. Therefore from the exact sequence above, we deduce that $\text{id}(M) \leq n$. $\square$

Proposition 2.8. *Let $\text{FEd}(M) \leq 1$. Then the following statements are equivalent:*

- (1) $\text{id}(M) \leq n$;
- (2) $\text{Ext}_R^{n+1}(N, M) = 0$ for every strongly copresented R -module N .

Proof. Since $\text{FEd}(M) \leq 1$, the exact sequence $0 \rightarrow M \rightarrow E^0 \rightarrow L^0 \rightarrow 0$ exists, where E^0 is injective and L^0 is strongly copresented. Thus, $\text{Ext}_R^{n+1}(N, M) = 0$ for any strongly copresented R -module N if and only if $\text{Ext}_R^n(N, L^0) = 0$ if and only if $\text{id}(L^0) \leq n - 1$ (by Proposition 2.7) if and only if $\text{id}(M) \leq n$. $\square$

Theorem 2.9. *Let $\text{FEd}(M) < \infty$. Then the following statements are equivalent:*

- (1) $\text{id}(M) \leq n$;
- (2) $\text{Ext}_R^{n+1}(N, M) = 0$ for every strongly copresented R -module N .

Proof. (1) $\Rightarrow$ (2) It is clear.

(2) $\Rightarrow$ (1) If $\text{FEd}(M) = m$, then the exact sequence

$$0 \rightarrow M \rightarrow E^0 \rightarrow E^1 \rightarrow \dots \rightarrow E^{m-1} \xrightarrow{d^{m-1}} E^m \xrightarrow{d^m} \dots \rightarrow E^{m+j} \rightarrow \dots$$

exists, where E^i is finitely cogenerated for any $i \geq m$. By Proposition 2.2, $n + 1 \geq m$. Let $\text{Ext}_R^{n+1}(N, M) = 0$ for every strongly copresented R -module N . Thus by [7, Corollary 6.42], we have

$$\text{Ext}_R^{n+1}(N, M) \cong \text{Ext}_R^{n-m+1}(N, \text{coker } d^{m-1}) = 0.$$

Since $\text{coker } d^{m-1}$ is strongly copresented, Proposition 2.8 implies that

$$\text{id}(\text{coker } d^{m-1}) \leq n - m$$

and so, we deduce that $\text{id}(M) \leq n$. $\square$

Corollary 2.10. *Let $D(R) < \infty$. Then:*

$$D(R) = \sup\{\text{pd}(N) \mid N \text{ is strongly copresented}\}.$$

Proof. Assume that $D(R) \leq m$. Thus, $\text{pd}(N') \leq m$ for any R -module N' . So, for any strongly copresented R -module N , $\text{pd}(N) \leq m$. Conversely, let $\text{pd}(N) \leq m$ for every strongly copresented R -module N . Thus $\text{Ext}_R^{m+1}(N, M) = 0$ for every strongly presented R -module M . Since $D(R) < \infty$, $\text{FEd}(M) < \infty$ by Proposition 2.2. Therefore by Theorem 2.9, $\text{id}(M) \leq m$ and hence by [19, corollary 3.7], $D(R) \leq m$. $\square$

Definition 2.11. For any ring R , we define the copresented dimension of R to be $\text{FED}(R) = \sup\{\text{FEd}(M) \mid M \text{ is a finitely cogenerated module}\}$.

EXAMPLE 2.12. Let $R = k[x^3, x^3y, xy^3, y^3]$, where k is a field with characteristic $p = 3$. By Definition 2.11 and Proposition 2.2, $\text{FED}(R^\infty) \leq D(R^\infty) + 1$, where R^∞ is perfect closure of R . On the other hand, $k[x, y]$ is purely inseparable over R . Also, by [9, Proposition 3.3], $(k[x, y])^\infty$ is coherent. Therefore by [10, Remark 1.4], R^∞ is coherent. Since R is reduced, [2, Proposition 5.5] implies that $\text{FED}(R^\infty) \leq \dim(R) + 1$ and so, $\text{FED}(R^\infty) \leq 3$.

Proposition 2.13. *The following statements are equivalent:*

- (1) $\text{FED}(R) = 0$;
- (2) Every finitely cogenerated module has an infinite finite copresented;
- (3) Every finitely cogenerated module is finitely copresented;
- (4) R is co-noetherian.

Proof. The implication (1) $\implies$ (2) $\implies$ (3) follow immediately from Definition 2.11.

(3) $\implies$ (4) $\implies$ (1) are trivial. $\square$

Corollary 2.14. *If $\text{FED}(R) \leq 0$, then R is n -cocoherent.*

Proof. Since every n -copresented module M is finitely cogenerated, Proposition 2.13 implies that M is $(n + 1)$ -copresented. $\square$

Next, we study the copresented dimension of the direct sum of rings. But before this we need the following lemma.

Lemma 2.15. *Let $f : R \rightarrow S$ be a ring epimorphism. If M_S is a right S -module (hence a right R -module) and N_R is a right R -module, then the following statements hold:*

- (1) $M \otimes_R S \cong M_S$.
- (2) If f is flat and N_R is a finitely cogenerated right R -module, then $N \otimes_R S$ is a finitely cogenerated right S -module.
- (3) If f is flat, then M_S is a finitely cogenerated right S -module if and only if M_R is a finitely cogenerated right R -module.
- (4) If f is projective, then M_S is an injective right S -module if and only if M_R is an injective right R -module.

Proof. (1) This is clear.

(2) For any family of submodules $\{N_i \otimes_R 1_S | i \in I\}$ in $N \otimes_R S$, if $\bigcap (N_i \otimes_R 1_S) = 0$, then we need to show that $\bigcap_{i \in F} (N_i \otimes_R 1_S) = 0$ for some finite subset F of I . Since f is flat, we have that $\bigcap_{i \in I} N_i \otimes_R 1_S = 0$. So, $\bigcap_{i \in I} N_i = 0$ and hence by hypothesis $\bigcap_{i \in F} N_i = 0$ for some finite subset F of I . Therefore, $\bigcap_{i \in F} (N_i \otimes_R 1_S) = \bigcap_{i \in F} N_i \otimes_R 1_S = 0$.

(3) ($\implies$): Let $\psi : M \rightarrow \prod_{i \in I} R$ is a monomorphism, then we claim that $\pi : M \rightarrow \prod_{i \in F} R$ is a monomorphism for some finite subset F of I . We have the following commutative diagram:

$$\begin{array}{ccc} M & \xrightarrow{\psi} & \prod_{i \in I} R \\ \downarrow \cong & & \downarrow g \\ M & \xrightarrow{h} & \prod_{i \in I} S, \end{array}$$

where since g is epimorphism and ψ is monomorphism, h is monomorphism. So by hypothesis, $\alpha : M \rightarrow \prod_{i \in F} S$ is a monomorphism for some finite subset F of I . Therefore the following commutative diagram:

$$\begin{array}{ccc} M & \xrightarrow{\gamma} & \prod_{i \in F} R \\ \downarrow \cong & & \downarrow \beta \\ M & \xrightarrow{\alpha} & \prod_{i \in F} S, \end{array}$$

where β is epimorphism and α is monomorphism, implies that γ is monomorphism.

($\Leftarrow$) : This follows from (1) and (2)

(4) By [5, Lemma 3.3], M_S is an (n, d) -injective right S -module if and only if M_R is an (n, d) -injective right R -module. If $n = 0, d = 0$, Then (4) is hold. $\square$

Theorem 2.16. Assume that R and S are two rings. Then:

$$\text{FED}(R \oplus S) = \sup\{\text{FED}(R), \text{FED}(S)\}.$$

Proof. We first show that $\text{FED}(R \oplus S) \leq \sup\{\text{FED}(R), \text{FED}(S)\}$. Consider $\text{FED}(R) = n, \text{FED}(S) = m$ and $n \geq m$. Also, let M be a finitely cogenerated right $(R \oplus S)$ -module. Then M has a unique decomposition $M = A \oplus B$, where A, B are right modules of rings R and S , respectively. By [15, Lemma 1.1], A and B are finitely cogenerated right $(R \oplus S)$ -module. So by Lemma 2.15, A is finitely cogenerated right R -module and B is finitely cogenerated right S -module. Therefore $\text{FED}(A) \leq n$ and $\text{FED}(B) \leq m$, and hence there is an exact sequences

$$\begin{aligned} 0 \rightarrow A \rightarrow E_a^0 \rightarrow E_a^1 \rightarrow \cdots \rightarrow E_a^{n-1} \rightarrow E_a^n \rightarrow \cdots, \\ 0 \rightarrow B \rightarrow E_b^0 \rightarrow E_b^1 \rightarrow \cdots \rightarrow E_b^{m-1} \rightarrow E_b^m \rightarrow \cdots \end{aligned}$$

of injective right R -modules E_a^i and injective right S -modules E_b^i such that E_a^i, E_b^i are finitely cogenerated for any $i \geq n$ and $i \geq m$, respectively. So, we deduce that the exact sequence

$$0 \rightarrow A \oplus B \rightarrow E_a^0 \oplus E_b^0 \rightarrow E_a^1 \oplus E_b^1 \rightarrow \cdots \rightarrow E_a^{n-1} \oplus E_b^{m-1} \rightarrow E_a^n \oplus E_b^m \rightarrow \cdots$$

exists, where by Lemma 2.15, every $E_a^i \oplus E_b^i$ is injective right $(R \oplus S)$ -module and also, every $E_a^i \oplus E_b^i$ is finitely cogenerated for any $i \geq n$. Therefore, we have $\text{FED}(R \oplus S) \leq \sup\{\text{FED}(R), \text{FED}(S)\}$.

Conversely, Assume that $\text{FED}(R \oplus S) = d$. If M is a finitely cogenerated right R -module. Then by Lemma 2.15, M is a finitely cogenerated right $(R \oplus S)$ -module and hence $\text{FED}(M_{(R \oplus S)}) \leq d$. Thus, the exact sequence $0 \rightarrow M \rightarrow E^0 \rightarrow E^1 \rightarrow \cdots \rightarrow E^{d-1} \rightarrow E^d \rightarrow \cdots$ of injective right $(R \oplus S)$ -modules

E^i exists, where every E^i is finitely cogenerated for any $i \geq d$. Let $E^i = C^i \oplus D^i$, where C^i is a R -module and D^i is a S -module. On the other hand, M is a right R -module, so we have the exact sequence $0 \rightarrow M \rightarrow C^0 \rightarrow C^1 \rightarrow \dots \rightarrow C^{d-1} \rightarrow C^d \rightarrow \dots$ of R -modules. But, every C^i is injective right $(R \oplus S)$ -module and also every C^i is finitely cogenerated right $(R \oplus S)$ -module for $i \geq d$. So by [15, Lemma 1.1] and Lemma 2.15, C^i is an injective right R -module and it is finitely cogenerated R -module for $i \geq d$. Therefore $\text{FEd}(M) \leq d$ and hence $\text{FEd}(R) \leq d$. Similarly, $\text{FEd}(S) \leq d$ and implies that $\sup\{\text{FEd}(R), \text{FEd}(S)\} \leq \text{FEd}(R \oplus S)$. $\square$

Proposition 2.17. *Let $S \geq R$ be a finite normalizing extension with S_R projective as an R -module. Then for any right R -module M_R , $\text{FEd}(\text{Hom}_R(S, M))_S \leq \text{FEd}(M_R)$.*

Proof. Assume that $\text{FEd}(M_R) = n$. Then there exists an exact sequence of injective R -modules

$$0 \rightarrow M \rightarrow E^0 \rightarrow E^1 \rightarrow \dots \rightarrow E^{n-1} \rightarrow E^n \rightarrow \dots,$$

where each E^i is finitely cogenerated for any $i \geq n$. Since S is projective, there is an exact sequence

$$0 \rightarrow \text{Hom}_R(S, M) \rightarrow \text{Hom}_R(S, E^0) \rightarrow \dots \rightarrow \text{Hom}_R(S, E^n) \rightarrow \dots$$

of injective S -modules $\text{Hom}_R(S, E^i)$, where by [13, Proposition 8.3], $\text{Hom}_R(S, E^i)$ is finitely cogenerated for any $i \geq n$. Thus $\text{FEd}(\text{Hom}_R(S, M))_S \leq n$ and hence, we have $\text{FEd}(\text{Hom}_R(S, M))_S \leq \text{FEd}(M_R)$. $\square$

Proposition 2.18. *Let $S \geq R$ be a finite normalizing extension, S_R be Projective, and S be R -projective. Then for each right S -module M_S , $\text{FEd}(M_S) \leq \text{FEd}(\text{Hom}_R(S, M))$.*

Proof. By [12, Lemma 1.1], M_S is isomorphic to a direct summand of $\text{Hom}_R(S, M)$. So, from Corollary 2.6, we deduce that $\text{FEd}(M_S) \leq \text{FEd}(\text{Hom}_R(S, M))$. $\square$

Proposition 2.19. *Let $S \geq R$ be an almost excellent extension. Then for each right S -module M_S , $\text{FEd}(M_R) \leq \text{FEd}(M_S)$.*

Proof. Assume that $\text{FEd}(M_S) = n$. So, there exists an exact sequence of injective S -modules

$$0 \rightarrow M \rightarrow E^0 \rightarrow E^1 \rightarrow \dots \rightarrow E^{n-1} \rightarrow E^n \rightarrow \dots,$$

where each E^i is finitely cogenerated for any $i \geq n$. Thus by [18, Proposition 5.1], every E^i is an injective R -module and also, it is a finitely cogenerated R -module for $i \geq n$ by [14, Theorem 5]. Therefore, it follows that $\text{FEd}(M_R) \leq \text{FEd}(M_S)$. $\square$

Corollary 2.20. *Let $S \geq R$ be an almost excellent extension. Then for each right S -module M_S , $\text{FEd}(M_R) = \text{FEd}(M_S) = \text{FEd}(\text{Hom}_R(S, M))$.*

Theorem 2.21. *Asume that $S \geq R$ is a finite normalizing extension and S_R is Projective. Then:*

- (1) *If S is R -projective and $\text{FED}(S) < \infty$, then $\text{FED}(S) \leq \text{FED}(R)$.*
- (2) *If $\text{FED}(R) < \infty$, then $\text{FED}(R) < \text{FED}(S) + \max\{k, d\}$, where $k = \text{id}(S_R)$ and $d = \sup\{\text{FE}d(M_R) \mid M \in \text{Mod} - S \text{ and } \text{FE}d(M_S) = 0\}$.*

Proof. (1) Asume that $\text{FED}(S) = n$ and $\text{FE}d(M_S) = n$ for a finitely cogenerated S -module M . Since S_R is projective, by hypothesis and [12, Lemma 1.1], M_S is isomorphic to a direct summand of $\text{Hom}_R(S, M)$ and hence we have:

$$0 \rightarrow K \rightarrow \text{Hom}_R(S, M) \rightarrow M_S \rightarrow 0.$$

By [14, Lemma 4], $\text{Hom}_R(S, M)$ is finitely cogenerated S -module, since M_R is a finitely cogenerated R -module. So, $\text{FE}d(\text{Hom}_R(S, M)_S) \leq n$. On the other hand, by Theorem 2.5,

$$\text{FE}d(K) \leq \max\{n, n-1\},$$

$$n = \text{FE}d(M_S) \leq \max\{\text{FE}d(\text{Hom}_R(S, M)_S), \text{FE}d(K_S) - 1\} \leq \text{FED}(S) = n.$$

Therefore $\text{FE}d(\text{Hom}_R(S, M)_S) = n$. Thus, Proposition 2.17 implies that

$$\text{FE}d(\text{Hom}_R(S, M)_S) \leq \text{FE}d(M_R)$$

and hence $\text{FED}(S) \leq \text{FED}(R)$.

(2) Asume that $\text{FED}(R) = n$ and $\text{FE}d(M_R) = n$ for a finitely cogenerated R -module M . Since S_R is projective, by [12, Lemma 1.1], M_R is isomorphic to a direct summand of $\text{Hom}_R(S, M)$ which induces the following short exact sequence of R -modules:

$$0 \rightarrow K \rightarrow \text{Hom}_R(S_R, M) \rightarrow M_R \rightarrow 0.$$

It is clear that $\text{Hom}_R(S_R, M)$ is a finitely cogenerated R -module. Thus Theorem 2.5 implies that

$$n = \text{FE}d(M_R) \leq \max\{\text{FE}d(\text{Hom}_R(S_R, M)), \text{FE}d(K_R) - 1\} \leq \text{FED}(R) = n,$$

and hence $\text{FE}d(\text{Hom}_R(S_R, M)) = n$.

If $\text{FE}d(\text{Hom}_R(S, M))_S = m \leq \text{FED}(S)$, then there is an injective resolution

$$0 \longrightarrow \text{Hom}_R(S, M) \xrightarrow{f_0} E^0 \xrightarrow{f_1} E^1 \longrightarrow \dots \longrightarrow E^{m-1} \xrightarrow{f_m} E^m \xrightarrow{f_{m+1}} \dots$$

of $\text{Hom}_R(S, M)$, where every E^i is a finitely cogenerated S -module for any $i \geq m$. Let $D^i = \text{coker}(f_i)$ for every $i \geq 0$. Thus, the following short exact sequences

$$0 \longrightarrow \text{Hom}_R(S, M) \longrightarrow E^0 \rightarrow D^0 \longrightarrow 0,$$

...

$$0 \longrightarrow D^{m-2} \longrightarrow E^{m-1} \longrightarrow D^{m-1} \longrightarrow 0,$$

$$0 \longrightarrow D^{m-1} \longrightarrow E^m \longrightarrow D^m \longrightarrow 0$$

exists, where $\text{FEd}(D^{m-1}) = 0$. But by hypothesis and Proposition 2.2, we have:

$$\text{FEd}(D^i)_R \leq \text{id}(D^i)_R + 1 \leq \text{id}(S_R) + 1 = k + 1, \quad \text{FEd}(D^{m-1})_R \leq d.$$

Therefore by Theorem 2.5, we deduce that:

$$\text{FEd}(D^{m-2})_R \leq \max\{\text{FEd}(E^{m-1})_R, \text{FEd}(D^{m-1})_R + 1\} < \max\{k + 1, d + 1\} = 1 + \max\{k, d\},$$

$$\text{FEd}(D^{m-3})_R \leq \max\{\text{FEd}(E^{m-2})_R, \text{FEd}(D^{m-2})_R + 1\} < 2 + \max\{k, d\},$$

$$\vdots$$

$$\text{FEd}(D^0)_R \leq \max\{\text{FEd}(E^1)_R, \text{FEd}(D^1)_R + 1\} < m - 1 + \max\{k, d\},$$

$$n = \text{FEd}(\text{Hom}_R(S, M))_R \leq \max\{\text{FEd}(E^0)_R, \text{FEd}(D^0)_R + 1\} < m + \max\{k, d\}.$$

Thus $\text{FED}(R) < m + \max\{k, d\} \leq \text{FED}(S) + \max\{k, d\}$ and so, the proof is complete. $\square$

Corollary 2.22. *Let $S \geq R$ be an almost excellent extension. Then $\text{FED}(R) < \text{FED}(S) + \text{id}(S)_R$.*

Proof. By Proposition 2.19 and Theorem 2.21, this is clear. $\square$

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