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On a Metric on Translation Invariant Spaces

M. Mortazavizadeh[a], R. Raisi Tousi[a,*] , R. A. Kamyabi Gol[b]

[a]Department of Mathematics, Ferdowsi University of Mashhad, P. O. Box 1159-91775, Mashhad, Islamic Republic of Iran.
[b]Department of Mathematics, Ferdowsi University of Mashhad, P. O. Box 1159-91775, Mashhad, Islamic Republic of Iran, Centre of Excellence in Analysis on Algebraic Structures (CEAAS).

E-mail: mortazavizadeh@mail.um.ac.ir
E-mail: raisi@um.ac.ir
E-mail: kamyabi@um.ac.ir

Abstract. In this paper we define a metric on the collection of all translation invariant spaces on a locally compact abelian group, and we study some topological properties of the metric space.

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1. Introduction

Translation invariant spaces on a locally compact abelian group G are closed subspaces of $L^2(G)$ that are invariant under translations by a closed cocompact subgroup of G. Especially, a shift invariant space is a closed subspace of $L^2(G)$ that is invariant under translations by elements of a uniform lattice in G. Shift invariant spaces are applicable in various areas of mathematical analysis and its applications such as approximation, wavelet, and frame theory. These spaces are studied on $\mathbb{R}^n$ in [3] by Bownik and on locally compact

*Corresponding Author

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abelian groups in [9, 10] by the second and the third authors (see also [6]). Translation invariant spaces in the setting of locally compact abelian groups are studied in [4, 2]. Introducing a metric on the class of translation invariant spaces provides a platform for topological investigations on them and it is interesting on its own right. In this paper, we introduce a translation metric on the collection of all translation invariant subspaces of $L^2(G)$, where G is a locally compact abelian group. We then study some topological properties and convergence in this metric space. This paper is organized as follows. In the rest of this section, we state some preliminaries and notation related to locally compact abelian groups and translation invariant spaces. In Section 2, following an idea from [1], we define a translation metric on $TI(G)$, the collection of all translation invariant subspaces of $L^2(G)$, and we show that $TI(G)$ is a complete, noncompact, and disconnected metric space. In Section 3, we study convergence in the translation metric.

Throughout this paper, we assume that G is a second countable locally compact abelian group. Let $\widehat{G}$ denote the dual group of G equipped with the compact convergence topology and the Fourier transform, $\hat{}: L^1(G) \longrightarrow C_0(\widehat{G})$, $\varphi \longrightarrow \hat{\varphi}$, be defined by $\hat{\varphi}(\xi) = \int_G \varphi(x)\overline{\xi}(x)dx$. The Fourier transform can be extended to a unitary isomorphism from $L^2(G)$ to $L^2(\widehat{G})$ known as the Plancherel transform [8, Theorem 4.25].

If L is a closed cocompact subgroup of G, then the subgroup $L^\perp = \{\xi \in \widehat{G};\ \xi(L) = \{1\}\}$, the annihilator of L in $\widehat{G}$, is a closed cocompact subgroup of $\widehat{G}$. For closed cocompact subgroup L of G, a fundamental domain of L in G is a measurable set S_L in G such that every $x \in G$ can be uniquely written in the form $x = ks$, where $k \in L$ and $s \in S_L$, whose existence is guaranteed by [11, Lemma 1.1]. For more details on locally compact abelian groups, we refer to the usual textbooks about locally compact groups, e.g.[8, 11].

For $\mathcal{A} \subseteq L^2(G)$, the translation invariant space generated by $\mathcal{A}$, is defined by $S(\mathcal{A}) = \{T_k\varphi : k \in L, \varphi \in \mathcal{A}\}$. If $\mathcal{A} = \{\varphi\}$, then $S(\varphi)$ is called a principal translation invariant space. The mapping $\mathcal{T}$, defined from $L^2(G)$ to $L^2(S_{L^\perp}, l^2(L^\perp))$ by

$$\mathcal{T}\varphi(\xi) = (\hat{\varphi}(\xi\eta))_{\eta \in L^\perp}, \tag{1.1}$$

is called the fiberization mapping, where $L^2(S_{L^\perp}, l^2(L^\perp))$ is the space of square integrable vector valued functions on $S_{L^\perp}$ to $l^2(L^\perp)$. For a locally compact abelian group G with its closed cocompact subgroup L, a range function is defined to be

$$J : S_{L^\perp} \longrightarrow \{closed\ subspaces\ of\ l^2(L^\perp)\}.$$

The mapping J is called measurable if the mapping $\xi \longmapsto \langle P_{J_V}(\xi)f, g\rangle$ is measurable for each $f, g \in l^2(L^\perp)$, where $P_{J_V}(\xi)$ is the orthogonal projection onto $J(\xi)$. By [4, Theorem 3.8] a closed subspace $V \subseteq L^2(G)$ is translation invariant if and only if $V = \{\varphi \in L^2(G) : \mathcal{T}\varphi(\xi) \in J(\xi)\ for\ a.e.\ x \in S_{L^\perp}\}$, where J is a measurable range function and $\mathcal{T}$ is the mapping as in (1.1). Identifying range functions which are equivalent almost everywhere, the correspondence between translation invariant spaces and measurable range functions is one to one and onto. For a translation invariant space V with range function J, let $P_{J_V}(\xi)$ be the projection of $l^2(L^\perp)$ onto $J(\xi)$ for $\xi \in S_{L^\perp}$. The spectral function of V is defined to be the mapping $\sigma_V : \widehat{G} \longrightarrow [0,1]$ given by

$$\sigma_V(\xi\eta) = \|P_{J_V}(\xi)(e_\eta)\|^2,\ \xi \in S_{L^\perp}, \eta \in L^\perp,$$

where (e_η) denotes the standard basis for $l^2(L^\perp)$. The local trace function associated to V is defined as

$$\tau_{V,f}(\xi) = \langle f, P_{J_V}(\xi) f\rangle,\ f \in l^2(L^\perp), \xi \in \widehat{G}.$$

Note that there is a close relation between the local trace function associated to V and the spectral function of V (see the proof of Corollary 3.2). For more information on spectral function and local trace function we refer to [5, 7].

2. Translation Metric

In this section we introduce and investigate topological properties of a translation metric θ, a metric on the collection of all translation invariant subspaces of $L^2(G)$. Let $TI(G)$ denote the collection of all translation invariant subspaces of $L^2(G)$. For each V and W in $TI(G)$ define

$$\theta(V,W) = \inf\{\alpha > 0 : m(\{\xi \in S_{L^\perp} : \|P_{J_V}(\xi) - P_{J_W}(\xi)\| > \alpha\}) = 0\}, \quad (2.1)$$

where J_V and J_W are the measurable range functions associated with V and W, $P_{J_V}(\xi)$ and $P_{J_W}(\xi)$, $\xi \in S_{L^\perp}$, are the orthogonal projections onto $J_V(\xi)$ and $J_W(\xi)$ respectively, $\|.\|$ denotes the operator norm, and m is the Haar measure of $\widehat{G}$. In the forthcoming proposition, we show that θ is a metric on $TI(G)$, which is called translation metric. Note that if V and W are translation invariant spaces, then $\theta(V,W) \le \epsilon$ if and only if, $\|P_{J_V}(\xi) - P_{J_W}(\xi)\| \le \epsilon$, for a.e. $\xi \in S_{L^\perp}$.

Proposition 2.1. *With the notation as above, θ is a metric on $TI(G)$.*

Proof. Positivity of θ follows from the definition. For V and W in $TI(G)$, if $\theta(V,W) = 0$, one can find a sequence (α_n) of positive numbers converging to 0 and a set E of measure zero such that $\|P_{J_V(\xi)} - P_{J_W(\xi)}\| \le \alpha_n$, for all $n \in \mathbb{N}$ and for $\xi \in S_{L^\perp}$. It follows that $\|P_{J_V(\xi)} - P_{J_W(\xi)}\| = 0$ for a.e. $\xi \in S_{L^\perp}$, so the projections onto $J_W(\xi)$ and $J_V(\xi)$ are the same a.e. and hence $V = W$, in the sence of the usual convention that two translation invariant spaces are equal if the corresponding range functions are equal a.e. On the other hand, $V = W$ implies that $J_V(\xi) = J_W(\xi)$ for a.e. $\xi \in S_{L^\perp}$, which in turn implies that $\|P_{J_V(\xi)} - P_{J_W(\xi)}\| > 0$ only on a set of measure 0. Hence $\theta(V,W) = 0$. For the triangle inequality, if U, V, and W are translation invariant spaces and $\epsilon > 0$, one can get $M_1, M_2 > 0$ such that $M_1 < \theta(V,U) + \frac{\epsilon}{2}$,

$M_2 < \theta(U,W) + \frac{\epsilon}{2}$, $m(\{\xi \in S_{L^\perp} : \|P_{J_V(\xi)} - P_{J_U(\xi)}\| > M_1\}) = 0$, and $m(\{\xi \in S_{L^\perp} : \|P_{J_U(\xi)} - P_{J_W(\xi)}\| > M_2\}) = 0$. Applying the triangle inequality for the norm gives

$$\|P_{J_V(\xi)} - P_{J_W(\xi)}\| \leq M_1 + M_2 \quad a.e\ \xi \in S_{L^\perp}.$$

It follows that $\theta(V,W) \leq \theta(V,U)+\theta(U,W)$. Finally, it's obvious that $\theta(V,U) = \theta(U,V)$. □

In the sequel, we show that $TI(G)$ is a complete, noncompact, and disconnected metric space. The following lemma is used in the proof of the next theorem.

Lemma 2.2. *Let (J_n) be a sequence of measurable range functions, and let $(P_n(\xi))$ be the corresponding sequence of orthogonal projections onto J_n's. Suppose that $(P_n(\xi))$ converges to the orthogonal projection $P(\xi)$ in the operator norm for $\xi \in S_{L^\perp}$. If $J(\xi)$ is the range of $P(\xi)$, then J is a measurable range function.*

Proof. Let $f \in l^2(L^\perp)$. Setting $F_n(\xi) = P_n(\xi)f$ and $F(\xi) = P(\xi)f$, we have

$$\|F_n(\xi) - F(\xi)\| \leq \|P_n(\xi) - P(\xi)\|\|f\|. \tag{2.2}$$

It now follows that $F(\xi) = \lim F_n(\xi)$. Thus F is the limit of a sequence (F_n) of vector valued measurable functions and hence is measurable. That is J is measurable. □

Theorem 2.3. *The space $TI(G)$ is complete in the translation metric.*

Proof. Suppose (V_n) is Cauchy in $TI(G)$. Then $(P_{J_{V_n}}(\xi))$ is Cauchy in the Banach space $BL(l^2(L^\perp))$, the space of all bounded linear operators on $l^2(L^\perp)$. Hence it converges to an orthogonal projection $P(\xi)$ for a.e. $\xi \in S_{L^\perp}$. Let $J(\xi)$ be the closed subspace of $l^2(L^\perp)$ associated with the orthogonal projection $P(\xi)$. Consider the translation invariant space $V := \{\varphi \in L^2(G) : \mathcal{T}\varphi(\xi) \in J(\xi)\ a.e.\ \xi \in S_{L^\perp}\}$, we have $J_V(\xi) = J(\xi)$ for a.e. $\xi \in S_{L^\perp}$, and hence $P_{J_V}(\xi) = P(\xi)$ for a.e. $\xi \in S_{L^\perp}$. Consequently, (V_n) converges to V in the translation metric. □

As a consequence of Theorem 2.3 we have the following corollary. Let $PTI(G)$ denote the collection of all principal translation invariant subspaces of $L^2(G)$.

Corollary 2.4. *The space $PTI(G)$ is complete in the translation metric.*

Proof. Suppose that (V_n) is a Cauchy sequence in $PTI(G)$. By Theorem 2.3, (V_n) converges to some $V \in TI(G)$. We need only to show that V has a single generator. For $0 < \epsilon < 1$, choose $p \in \mathbb{N}$ such that $\theta(V_n, V) < \epsilon$ for all $n \geq p$. This implies that $\|P_{J_{V_n}}(\xi) - P_{J_V}(\xi)\| < \epsilon$ for a.e. ξ whenever $n \geq p$. Hence $\dim J_V(\xi) = \dim J_{V_n}(\xi) = 1$ for a.e. ξ ([13, Theorem 4.35]). This proves that V can be generated by a single function, and hence $V \in PTI(G)$. □

Let $FTI(G)$ be the collection of all translation invariant spaces generated by a fixed number of elements of $L^2(G)$. With the same proof as Corollary 2.4, one can see that $FTI(G)$ is complete in the translation metric. Indeed, we have the following corollary.

Corollary 2.5. *The collection $FTI(G)$ is complete in the translation metric.*

Now we show that $TI(G)$ is not a compact metric space.

Proposition 2.6. *The space $TI(G)$ is not compact in the translation metric topology.*

Proof. Using [12, Theorem 45.1], it is enough to show that $TI(G)$ is not totally bounded in the translation metric. First choose a countable basis $\{\varphi_1, \varphi_2, \dots\}$ for $L^2(G)$. Set $V_m = S(\mathcal{A}_m)$, where $\mathcal{A}_m = \{\varphi_1, \varphi_2, \dots, \varphi_m\}$. Then $V_m \subset V_{m+1}$ for any m, and hence $\|P_{J_{V_m}}(\xi) - P_{J_{V_{m+1}}}(\xi)\| = 1$ for all $\xi \in S_{L^\perp}$ ([13, Theoram 4.30]). That is $\theta(V_m, V_{m+1}) = 1$ for all m. Hence for $\epsilon = \frac{1}{2}$, no finite collection of ϵ-balls can contain all V_m's. □

In the next theorem we show that the metric space $TI(G)$ is disconnected.

Theorem 2.7. *The space $TI(G)$ is disconnected in the translation metric.*

Proof. It is enough to show that $TI(G)$ has an open and closed proper subset. That $PTI(G)$ is closed follows from the Corollary 2.4. Now we show that it is open. Let $V \in PTI(G)$; put $r = \frac{1}{2}$. We show that $B_r(V) \subseteq PTI(G)$, where $B_r(V)$ is an open ball with center V and radius r. Let $W \in B_r(V)$; then $\theta(V, W) < \frac{1}{2}$, and hence $\dim J_V(\xi) = \dim J_W(\xi)$ for a.e. $\xi \in S_{L^\perp}$ ([13, Theorem 4.35]). Hence $W \in PTI(G)$. Since V is arbitrary, then $PTI(G)$ is an open subspace of $TI(G)$. That is $TI(G)$ is disconnected. □

3. Convergence

In this section we establish a few results about convergence in the translation metric. Indeed we study the relation between convergence of a sequences of translation invariant spaces in the translation metric and uniform convergence of corresponding local trace functions.

Proposition 3.1. *Let (V_n) be a sequence of translation invariant subspaces converging to a translation invariant subspace V in the translation metric. For any $f \in l^2(L^\perp)$, the local trace function $\tau_{V_n,f}$ converges uniformly to $\tau_{V,f}$ in $S_{L^\perp}$, except possibly on a set of measure zero.*

Proof. We have for almost every $\xi \in S_{L^\perp}$,

$$\begin{aligned} |\tau_{V_n,f}(\xi) - \tau_{V,f}(\xi)| &= |\langle f, P_{J_{V_n}}(\xi) f\rangle - \langle f, P_{J_V}(\xi) f\rangle| \\ &= |\langle f, (P_{J_{V_n}}(\xi) - P_{J_V}(\xi)) f\rangle| \\ &\le \|f\|^2 (\|P_{J_{V_n}}(\xi) - P_{J_V}(\xi)\|) \\ &\le \|f\|^2 \theta(V_n, V). \end{aligned}$$

The uniform convergence of local trace functions follows easily. □

Corollary 3.2. *Let (V_n) be a sequence of translation invariant subspaces converging to a translation invariant subspace V in the translation metric. Then the corresponding sequence of spectral functions of V_n converges uniformly to the spectral function of V on $\widehat{G}$ a.e.*

Proof. Let (e_η) denote the standard orthonormal basis of $l^2(L^\perp)$. Then

$$\begin{aligned}\tau_{V,e_\eta}(\xi) &= \langle e_\eta, P_{J_V}(\xi)e_\eta\rangle \\ &= \langle P_{J_V}(\xi)e_\eta, P_{J_V}(\xi)e_\eta\rangle \\ &= \|P_{J_V}(\xi)e_\eta\|^2 \\ &= \sigma_V(\xi+\eta).\end{aligned}$$

This implies by Proposition 3.1 that sequence of spectral functions of V_n converges uniformly to the spectral function of V. □

Proposition 3.3. *Let $V, V_m \in TI(G)$ for any $m \in \mathbb{N}$. Assume that the local trace function $\tau_{V_m,f}$ converges uniformly to $\tau_{V,f}$ on $S_{L^\perp}$, except possibly on a set of measure zero and for all $f \in l^2(L^\perp)$ with $\|f\| = 1$. Then (V_m) converges to V in the translation metric.*

Proof. The result follows from the following identification.

$$\begin{aligned}\sup_{\|f\|=1} |\tau_{V_m,f}(\xi) - \tau_{V,f}(\xi)| &= \sup_{\|f\|=1} |\langle f, (P_{J_{V_m}}(\xi) - P_{J_V}(\xi))f\rangle| \\ &= \|P_{J_{V_m}}(\xi) - P_{J_V}(\xi)\|.\end{aligned}$$

□

Remark 3.4. Recently, translation invariant spaces have been generalized to the setting when the subgroup L is not necessarily discrete or cocompact in [2]. The authors in [2] have utilized the Zak transform instead of the fiberization map $\mathcal{T}$ defined in (1.1), and they have given a characterization of translation invariant spaces in terms of range functions. Our results can be also in this setting phrased in terms of the Zak transform.

Example 3.5. We give an example of a sequence of translation invariant spaces converging in the translation metric. Define $\phi \in L^2(\mathbb{R})$ by $\hat{\phi}(\xi) = \mathbf{1}_{(0,1)}(\xi)$, and suppose (ϕ_n) is the sequence defined by $\hat{\phi}_n(\xi) = \frac{n+1}{n}\mathbf{1}_{(0,1)}(\xi)$. Let $V = S(\phi)$ and $V_n = S(\phi_n)$. A direct calculation shows that $J_V(\xi) = span\{e_0\}$ and $J_{V_n}(\xi) = span\{\frac{n+1}{n}e_0\}$. So the projections onto $J_{V_n}(\xi)$'s and $J_V(\xi)$ are the same and we can conclude that (V_n) converges to V in the translation metric.

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