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Linear Resolutions of Powers of Generalized Mixed Product Ideals

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Abstract. Let L be the generalized mixed product ideal induced by a monomial ideal I. In this paper we compute powers of the generalized mixed product ideals and show that L^k has a linear resolution if and only if I^k has a linear resolution for all k. We also introduce the generalized mixed polymatroidal ideals and prove that powers and monomial localizations of a generalized mixed polymatroidal ideal are again generalized mixed polymatroidal ideal.

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1. Introduction

The class of ideals of mixed products is a special class of squarefree monomial ideals. In 2001 Restuccia and Villarreal [8] introduced mixed product ideals and they classified those among these ideals which are normal. Rinaldo and Ionescu [7] studied the Castelnuovo-Mumford regularity, the depth and dimension of mixed product ideals and characterized when they are Cohen-Macaulay. Rinaldo [9] studied the Betti numbers of their finite free resolutions and Hoa

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and Tam [6] computed the regularity and some other algebraic invariants of mixed products of arbitrary graded ideals.

Mixed product ideals, as introduced by Restuccia and Villarreal are of the form $(I_qJ_r + I_pJ_s)S$, where for integers a and b, the ideal I_a (resp. J_b) is the ideal generated by all squarefree monomials of degree a in the polynomial ring $A = K[x_1, \ldots, x_n]$ (resp. of degree b in the polynomial ring $B = K[y_1, \ldots, y_m]$), where $0 < p < q \leq n$, $0 < r < s \leq m$ and $S = K[x_1, \ldots, x_n, y_1, \ldots, y_m]$. Thus the ideal $L = (I_qJ_r + I_pJ_s)S$ is obtained from the monomial ideal $I = (x^qy^r, x^py^s)$ by replacing x^q by I_q, x^p by I_p, y^r by J_r and y^s by J_s.

The first author together with Herzog and Yassemi introduced the generalized mixed product ideals [4] which generalized the mixed product ideals introduced by Restuccia and Villarreal and also generalized the expansion construction by Bayati and Herzog [1]. For this construction we choose for each i a set of new variables $x_{i1}, x_{i2}, \ldots, x_{im_i}$ and replace each of the factor $x_i^{a_i}$ in each minimal generator $x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}$ of I by a monomial ideal in $T_i = K[x_{i1}, x_{i2}, \ldots, x_{im_i}]$ generated in degree a_i. Indeed in the paper [1] a similar construction, called expansion, is made. There however, each $x_i^{a_i}$ is replaced by $(x_{i1}, \cdots, x_{im_i})^{a_i}$, while in our generalized mixed product ideals each $x_i^{a_i}$ is replaced by an arbitrary monomial ideal of T_i generated in degree a_i. In [4] the minimal graded free resolution of generalized mixed product ideals are computed and it is shown that a generalized mixed product ideal L induced by I has the same regularity as I, provided the ideals which replace the pure powers $x_i^{a_i}$ all have a linear resolution. As a consequence we obtained the result that under the above assumptions, L has a linear resolution if and only if I has a linear resolution. We also prove that the projective dimension of L can be expressed in terms of the multi-graded shifts in the resolution of I and the projective dimension of the ideals which replace the pure powers.

In Section 1 we compute powers of the generalized mixed product ideal L induced by a monomial ideal I and we show that L^k has a linear resolution if and only if I^k has a linear resolution for all k, provided the ideals which replace the pure powers $x_i^{a_i}$ all have a linear resolution, see Theorem 2.5.

In Section 2 we introduce the generalized mixed polymatroidal ideals. The class of generalized mixed polymatroidal ideals is a special class of generalized mixed product ideals where for each i we replace each factor $x_i^{a_i}$ in each minimal generator $x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}$ of I by a polymatroidal ideal in T_i generated in degree a_i. We also show that powers of a generalized mixed polymatroidal ideal is generalized mixed polymatroidal ideal, see Theorem 3.2, and that the monomial localizations of a generalized mixed polymatroidal ideal at monomial prime ideals is again a generalized mixed polymatroidal ideal, see Theorem 3.3.

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2. Powers Of Generalized Mixed Product Ideals

Let $S = K[x_1, \ldots, x_n]$ be the polynomial ring over a field K in the variables $x_1, \ldots, x_n$, and let $I \subset S$ be a monomial ideal with $I \neq S$ whose minimal set of generators is $G(I) = \{\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_m}\}$. Here $\mathbf{x}^{\mathbf{a}} = x_1^{\mathbf{a}(1)} x_2^{\mathbf{a}(2)} \cdots x_n^{\mathbf{a}(n)}$ for $\mathbf{a} = (\mathbf{a}(1), \ldots, \mathbf{a}(n)) \in \mathbb{N}^n$. Next we consider the polynomial ring T over K in the variables

$$x_{11}, \ldots, x_{1m_1}, x_{21}, \ldots, x_{2m_2}, \ldots, x_{n1}, \ldots, x_{nm_n}.$$

In [4] we introduced the generalized mixed product ideals. For $i = 1, \ldots, n$ and $j = 1, \ldots, m$ let $L_{i,\mathbf{a}_j(i)}$ be a monomial ideal in the variables $x_{i1}, x_{i2}, \ldots, x_{im_i}$ such that

$$L_{i,\mathbf{a}_j(i)} \subset L_{i,\mathbf{a}_k(i)} \quad \text{whenever} \quad \mathbf{a}_j(i) \geq \mathbf{a}_k(i). \tag{2.1}$$

Given these ideals we define for $j = 1, \ldots, m$ the monomial ideals

$$L_j = \prod_{i=1}^{n} L_{i,\mathbf{a}_j(i)} \subset T, \tag{2.2}$$

and set $L = \sum_{j=1}^{m} L_j$. The ideal L is called a *generalized mixed product ideal* induced by I.

We want to study linear resolutions of powers of the generalized mixed product ideals. For this purpose we first compute powers of such ideals.

Proposition 2.1. *Let L be the generalized mixed product ideal induced by the monomial ideal I as described in (2.2). Then*

$$L^k = \sum_{k_1, \ldots, k_m \geq 0, \sum_{j=1}^{m} k_j = k} \prod_{i=1}^{n} \prod_{j=1}^{m} L_{i,\mathbf{a}_j(i)}^{k_j}$$

for all $k \geq 1$.

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Proof. Let L be the generalized mixed product ideal induced by a monomial ideal I with $G(I) = \{\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_m}\}$. Then for all $k \geq 1$ we have

$$\begin{aligned} L^k &= (\sum_{j=1}^{m} L_j)^k \\ &= (\sum_{j=1}^{m} \prod_{i=1}^{n} L_{i,\mathbf{a}_j(i)})^k \\ &= \sum_{k_1,\ldots,k_m \geq 0, \sum_{j=1}^{m} k_j = k} (L_{1,\mathbf{a}_1(1)}^{k_1} L_{2,\mathbf{a}_1(2)}^{k_1} \cdots L_{n,\mathbf{a}_1(n)}^{k_1}) \cdots \\ &\quad (L_{1,\mathbf{a}_m(1)}^{k_m} L_{2,\mathbf{a}_m(2)}^{k_m} \cdots L_{n,\mathbf{a}_m(n)}^{k_m}) \\ &= \sum_{k_1,\ldots,k_m \geq 0, \sum_{j=1}^{m} k_j = k} (L_{1,\mathbf{a}_1(1)}^{k_1} L_{1,\mathbf{a}_2(1)}^{k_2} \cdots L_{1,\mathbf{a}_m(1)}^{k_m}) \cdots \\ &\quad (L_{n,\mathbf{a}_1(n)}^{k_1} L_{n,\mathbf{a}_2(n)}^{k_2} \cdots L_{n,\mathbf{a}_m(n)}^{k_m}) \\ &= \sum_{k_1,\ldots,k_m \geq 0, \sum_{j=1}^{m} k_j = k} \prod_{i=1}^{n} \prod_{j=1}^{m} L_{i,\mathbf{a}_j(i)}^{k_j}. \end{aligned}$$

□

Now we prove that the monomial ideal $L(I)^2$ is induced by I^2

Lemma 2.2. *Let $L(I) = \sum_{k=1}^{r} \prod_{i=1}^{n} L_{i,\mathbf{a}_k(i)}$ and $L(J) = \sum_{l=1}^{s} \prod_{i=1}^{n} L_{i,\mathbf{b}_l(i)}$ be generalized mixed product ideals, respectively, induced(2.2) by the monomial ideals I and J with $G(I) = \{\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_r}\}$ and $G(J) = \{\mathbf{x}^{\mathbf{b}_1}, \ldots, \mathbf{x}^{\mathbf{b}_s}\}$. We assume that*

$$L_{i,\mathbf{a}_k(i)} L_{i,\mathbf{b}_l(i)} \subset L_{i,\mathbf{a}_p(i)} L_{i,\mathbf{b}_q(i)} \text{ whenever } \mathbf{a}_k + \mathbf{b}_l \geq \mathbf{a}_p + \mathbf{b}_q. \tag{2.3}$$

Suppose that $G(IJ) = \{\mathbf{x}^{\mathbf{c}_1}, \ldots, \mathbf{x}^{\mathbf{c}_t}\}$. Then given $\mathbf{c}_j$, there exist $\mathbf{a}_k$ and $\mathbf{b}_l$ such that $\mathbf{c}_j = \mathbf{a}_k + \mathbf{b}_l$. We set

$$L_{i,\mathbf{c}_j(i)} = L_{i,\mathbf{a}_k(i)} L_{i,\mathbf{b}_l(i)},$$

Furthermore, let

$$L(IJ) = \sum_{j=1}^{t} \prod_{i=1}^{n} L_{i,\mathbf{c}_j(i)}.$$

Then $L(IJ)$ is a generalized mixed product ideal, and $L(IJ) = L(I)L(J)$.

Proof. Condition (2.3) implies that $L_{i,\mathbf{a}_k(i)} L_{i,\mathbf{b}_l(i)} = L_{i,\mathbf{a}_p(i)} L_{i,\mathbf{b}_q(i)}$ if $\mathbf{a}_k + \mathbf{b}_l = \mathbf{a}_p + \mathbf{b}_q$. Thus the definition of $L_{i,\mathbf{c}_j(i)}$ is independent of the presentation of $\mathbf{c}_j$ as a sum of $\mathbf{a}_k$ and $\mathbf{b}_l$, and hence well defined. It also implies that

$$L_{i,\mathbf{c}_j(i)} \subset L_{i,\mathbf{c}_k(i)} \quad \text{whenever} \quad \mathbf{c}_j(i) \geq \mathbf{c}_k(i).$$

Thus shows that $L(IJ)$ is indeed a generalized mixed product ideal of IJ.

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Now we want to show that $L(I)L(J) = L(IJ)$. We have

$$L(I)L(J) = \sum_{k=1}^{r}\sum_{l=1}^{s}\prod_{i=1}^{n}(L_{i,\mathbf{a}_k(i)}L_{i,\mathbf{b}_l(i)}),$$

and

$$L(IJ) = \sum_{j=1}^{t}\prod_{i=1}^{n}L_{i,\mathbf{c}_j(i)}.$$

If $\mathbf{c}_j = \mathbf{a}_k + \mathbf{b}_l$, then $\prod_{i=1}^{n} L_{i,\mathbf{c}_j(i)} = \prod_{i=1}^{n}(L_{i,\mathbf{a}_k(i)}L_{i,\mathbf{b}_l(i)})$. This shows that $L(IJ) \subset L(I)L(J)$.

Conversely, take a summand $\prod_{i=1}^{n}(L_{i,\mathbf{a}_k(i)}L_{i,\mathbf{b}_l(i)})$ of $L(I)L(J)$. Since IJ is generated by the elements $\mathbf{x}^{\mathbf{a}_k+\mathbf{b}_l}$, $k = 1, \ldots, r$ and $l = 1, \ldots, s$, and since $\mathbf{x}^{\mathbf{c}_1}, \ldots, \mathbf{x}^{\mathbf{c}_t}$ is a minimal set of generators of IJ, there exists $\mathbf{c}_j$ such that $\mathbf{a}_k + \mathbf{b}_l \geq \mathbf{c}_j$. Then condition (2.3) guarantees that $\prod_{i=1}^{n}(L_{i,\mathbf{a}_k(i)}L_{i,\mathbf{b}_l(i)}) \subset \prod_{i=1}^{n} L_{i,\mathbf{c}_j(i)}$. This shows that $L(I)L(J) \subset L(IJ)$. □

As a consequence of this lemma one easily obtains the following result.

Theorem 2.3. *Let L be the generalized mixed product ideal induced by the monomial ideal I satisfying condition (3). Then for all $k \geq 1$, L^k is a generalized mixed product ideal induced by the monomial ideal I^k.*

Proof. Let $G(I) = \{\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_m}\}$, and let $L(I) = \sum_{j=1}^{m}\prod_{i=1}^{n} L_{i,\mathbf{a}_j(i)}$.

We prove the assertion by induction on k. If $k = 1$, then the assertion is trivial, and for $k = 2$ Lemma 2.2 implies that $L(I)^2$ is a generalized mixed product ideal and $L(I^2) = L(I)^2$.

Now let $k > 2$. Then $L(I^k) = L(I^{k-1}I)$ and we have as in the case $k = 2$ that $L(I^{k-1}I) = L(I^{k-1})L(I)$. By induction hypothesis and Lemma 2.2, $L(I)^{k-1}$ is a generalized mixed product ideal and $L(I)^{k-1} = L(I^{k-1})$. Therefore $L(I)^k$ is a generalized mixed product ideal and $L(I)^k = L(I^k)$, as desired. □

Example 2.4. Consider the mixed product ideals introduced by Restuccia and Villarreal [8]. A mixed product ideal is a monomial ideal of the form $L = I_qJ_r + I_sJ_t$ where the ideals I_q and I_s are monomial ideals generated in degree q and s, respectively, in the variables $x_1, \ldots, x_n$ and J_r and J_t are monomial ideals generated in degree r and t in the variables $y_1, \ldots, y_m$. Thus L is induced by the ideal $I = (x^qy^r, x^sy^t)$. Now by using Proposition 2.1 we see that

$$L^k = \sum_{k_1,k_2\geq 0, k_1+k_2=k}(I_q^{k_1}I_s^{k_2})(J_r^{k_1}J_t^{k_2})$$

for all k. Then Theorem 2.3 implies that L^k induced by $(x^qy^r, x^sy^t)^k$.

In [4] a double complex is constructed whose total complex provided a multigraded free resolution of the generalized mixed product ideal. Also we show

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that a generalized mixed product ideal L induced by I has the same regularity as I, provided for $i = 1, \ldots, n$ and $j = 1, \ldots, m$ the ideals $L_{i,\mathbf{a}_j(i)}$ have a linear resolution. As a consequence we obtained that L has a linear resolution if and only if I has a linear resolution.

The main result of this section is the following

Theorem 2.5. *Let L be the generalized mixed product ideal induced by a monomial ideal I satisfying condition (3). Also assume that for $i = 1, \ldots, n$ and $j = 1, \ldots, m$ the ideals $L_{i,\mathbf{a}_j(i)}$ have an $\mathbf{a}_j(i)$-linear resolution. Then L^k has a linear resolution if and only if I^k has a linear resolution for all $k \geq 1$.*

Proof. Combining [4, Corollary 2.3] and Theorem 2.3 and we obtain the assertion. □

3. Generalized Mixed Polymatroidal Ideals

The class of polymatroidal ideals is one of the rare classes of monomial ideals with the property that all powers of an ideal in this class have a linear resolution. This is due to the fact that the powers of a polymatroidal ideal are again polymatroidal [3, Theorem 12.6.3] and polymatroidal ideals have linear resolutions [3, Theorem 12.6.2]. A monomial ideal is called polymatroidal, if its monomial generators correspond to the bases of a discrete polymatroid, see [3]. Let $J \subset S = K[x_1, \ldots, x_n]$ be a monomial ideal generated in single degree. We denote $G(J)$ the unique minimal set of monomial generators of J. Then J is said to be polymatroidal, if for any two elements $u, v \in G(J)$ such that $\deg_{x_i} u > \deg_{x_i} v$ there exists an index j with $\deg_{x_j} u < \deg_{x_j} v$ such that $x_j(u/x_i) \in J$.

Now we define monomial ideals which are a special class of the generalized mixed product ideals.

let $I \subset S = K[x_1, \ldots, x_n]$ be a monomial ideal with minimal set of generators $G(I) = \{\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_m}\}$. For $i = 1, \ldots, n$ and $j = 1, \ldots, m$ let $L_{i,\mathbf{a}_j(i)}$ be a polymatroidal ideal in the variables $x_{i1}, x_{i2}, \ldots, x_{im_i}$ such that

$$L_{i,\mathbf{a}_j(i)} \subset L_{i,\mathbf{a}_k(i)} \quad \text{whenever} \quad \mathbf{a}_j(i) \geq \mathbf{a}_k(i). \tag{3.1}$$

Given these ideals we define for $j = 1, \ldots, m$ the polymatroidal ideals

$$L_j = \prod_{i=1}^{n} L_{i,\mathbf{a}_j(i)} \subset T, \tag{3.2}$$

and set $L = \sum_{j=1}^{m} L_j$. We call L a *generalized mixed polymatroidal ideal* induced by I.

Example 3.1. *Let $L = \sum_{i=1}^{s} I_{k_i} J_{r_i}$ be a generalized mixed product ideal, where I_{k_i} (resp. J_{r_i}) is the ideal of the polynomial ring $S = K[x_1, \ldots, x_n, y_1, \ldots, y_m]$*

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generated by all squarefree monomial ideals of degree k_i (resp. r_i) in the variables $\{x_1, \ldots, x_n\}$ (resp. $\{y_1, \ldots, y_m\}$), and where $0 \leq k_1 \leq k_2 \leq \cdots \leq k_s \leq n$, $0 \leq r_s \leq r_{s-1} \leq \cdots \leq r_1 \leq m$. We set $I_0 = J_0 = S$ and $I_{k_i} = (0)$ (resp. $J_{r_i} = (0)$ if $k_i > n$ (resp. $r_i > m$). In our terminology L is a generalized mixed polymatroidal ideal induced by the ideal $(x^{k_1}y^{r_1}, \ldots, x^{k_s}y^{r_s})$.

In this section we want to study monomial localizations of powers of the generalized mixed polymatroidal ideals.

It is natural to ask whether powers of generalized mixed polymatroidal ideals are again generalized mixed polymatroidal ideal. There is a positive answer to this question in the following case.

Theorem 3.2. *Let L be the generalized mixed polymatroidal ideal induced by a monomial ideal I satisfying condition (3). Then L^k is a generalized mixed polymatroidal ideal for all k.*

Proof. Suppose that L be the generalized mixed polymatroidal ideal induced by I with $G(I) = \{\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_m}\}$. Since powers of polymatroidal ideals are again polymatriodal, see [3, Theorem 12.6.3] then $L^{k_j}_{i,\mathbf{a}_j(i)}$ are polymatroidal ideals for $j = 1, \ldots, m$. So by Proposition 2.1 and [3, Theorem 12.6.3] we have

$$L^k = \sum_{k_1,\ldots,k_m \geq 0, \sum_{j=1}^m k_j = k} \underbrace{(L^{k_1}_{1,\mathbf{a}_1(1)} L^{k_2}_{1,\mathbf{a}_2(1)} \cdots L^{k_m}_{1,\mathbf{a}_m(1)})}_{polymatroidal} \cdots$$
$$\underbrace{(L^{k_1}_{n,\mathbf{a}_1(n)} L^{k_2}_{n,\mathbf{a}_2(n)} \cdots L^{k_m}_{n,\mathbf{a}_m(n)})}_{polymatroidal}$$

and hence L^k is a generalized mixed polymatroidal ideal for all k. □

Let $P = (x_{i_1}, \ldots, x_{i_s})$ be a monomial prime ideal of S. The monomial localization of I with respect to P, denoted by $I(P)$, is the ideal in the polynomial ring $S(P) = K[x_{i_1}, \ldots, x_{i_s}]$ which is obtained from I by applying the K-algebra homomorphism $S \to S(P)$ with $x_j \to 1$ for all $x_j \notin \{x_{i_1}, \ldots, x_{i_s}\}$.

Now we compute monomial localizations of the generalized mixed polymatroidal ideal L with respect to prime ideals $P \in V^*(L)$, where $V^*(L)$ denotes the set of graded, respectively monomial prime ideals containing L.

Theorem 3.3. *Let L be the generalized mixed polymatroidal ideal induced by a monomial ideal I. Then $L(P)$ is a generalized mixed polymatroidal ideal for all $P \in V^*(L)$ induced by $I(P)$.*

Proof. For a monomial ideal I with $G(I) = \{\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_m}\}$ let L be the generalized mixed polymatroidal ideal induced by I. Then monomial localization of L is

$$L(P) = (\sum_{j=1}^m L_j)(P) = (\sum_{j=1}^m \prod_{i=1}^n L_{i,\mathbf{a}_j(i)})(P) = \sum_{j=1}^m \prod_{i=1}^n (L_{i,\mathbf{a}_j(i)}(P))$$

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for all $P \in V^*(L)$. We know from [5, Corrolary 2.2] that $L_{i,\mathbf{a}_j(i)}(P)$ is again polymatroidal ideal for all $P \in V^*(L)$. Hence [3, Theorem 12.6.3] implies that

$$L_j(P) = (\prod_{i=1}^{n} L_{i,\mathbf{a}_j(i)})(P) = \prod_{i=1}^{n}(L_{i,\mathbf{a}_j(i)}(P))$$

is a polymatroidal ideal and then $L(P)$ is a generalized mixed polymatroidal ideal for all $P \in V^*(L)$. □

By using Theorems 3.2 and 3.3, we have the following corollary.

Corollary 3.4. *Let L be the generalized mixed polymatroidal ideal induced by a monomial ideal I satisfying condition (3). Then $L^k(P)$ is a generalized mixed polymatroidal ideal for all k and $P \in V^*(L)$.*

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