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Labeling Subgraph Embeddings and Cordiality of Graphs

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Abstract. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$, a vertex labeling $f : V(G) \to \mathbb{Z}_2$ induces an edge labeling $f^+ : E(G) \to \mathbb{Z}_2$ defined by $f^+(xy) = f(x) + f(y)$, for each edge $xy \in E(G)$. For each $i \in \mathbb{Z}_2$, let $v_f(i) = |\{u \in V(G) : f(u) = i\}|$ and $e_{f^+}(i) = |\{xy \in E(G) : f^+(xy) = i\}|$. A vertex labeling f of a graph G is said to be friendly if $|v_f(1) - v_f(0)| \leq 1$. The friendly index set of the graph G, denoted by $FI(G)$, is defined as $\{|e_{f^+}(1) - e_{f^+}(0)|$: the vertex labeling f is friendly$\}$. The full friendly index set of the graph G, denoted by $FFI(G)$, is defined as $\{e_{f^+}(1) - e_{f^+}(0)$: the vertex labeling f is friendly$\}$. A graph G is cordial if $-1, 0$ or $1 \in FFI(G)$. In this paper, by introducing labeling subgraph embeddings method, we determine the cordiality of a family of cubic graphs which are double-edge blow-up of $P_2 \times P_n, n \geq 2$. Consequently, we completely determined friendly index and full product cordial index sets of this family of graphs.

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1. Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Let A be an abelian group. A labeling $f : V(G) \to A$ induces an edge labeling $f^+ : E(G) \to A$ defined by $f^+(x,y) = f(x) + f(y)$, for each edge $(x,y) \in E(G)$. For $i \in A$, let $v_f(i) = |\{v \in V(G)\colon\ f(v) = i\}|$ and $e_{f^+}(i) = |\{e \in E(G) : f^+(e) = i\}|$. Let $c(f) = \{|e_{f^+}(i) - e_{f^+}(j)|\colon\ (i,j) \in A \times A\}$. A labeling f of a graph G is said to be A-friendly if $|v_f(i) - v_f(j)| \le 1$ for all $(i,j) \in A \times A$. If $c(f)$ is (i,j)-matrix for some A-friendly labeling f, then f is said to be A-cordial. The notion of A-cordial labelings was first introduced by Hovey [5], who generalized the concept of cordial graphs of Cahit [1].

In this paper, we will exclusively focus on $A = \mathbb{Z}_2$, and drop the reference to the group. A vertex v is called a k-vertex if $f(v) = k$, $k \in \{0,1\}$, an edge e is called a k-edge if $f^+(e) = k$, $k \in \{0,1\}$. A vertex labeling f of a graph G is said to be friendly if $|v_f(1) - v_f(0)| \le 1$.

In [4] the following concept was introduced.

Definition 1.1. The *friendly index set* $FI(G)$ of a graph G is defined as $\{|e_{f^+}(1) - e_{f^+}(0)| :$ the vertex labeling f is friendly$\}$.

The following result was established in [6]:

Theorem 1.2. *For any graph G with q edges, the friendly index set $FI(G) \subseteq \{0,\ 2,\ \ldots\ ,\ q\}$ if q is even, and $FI(G) \subseteq \{1,\ 3,\ \ldots\ ,\ q\}$ if q is odd.*

For more details of known results and open problems on friendly index sets, the reader can see [7, 8, 9].

Shiu and Kwong [12] extended $FI(G)$ to $FFI(G)$.

Definition 1.3. The *full friendly index set* $FFI(G)$ of a graph G is defined as $\{e_{f^+}(1) - e_{f^+}(0) :$ the vertex labeling f is friendly$\}$.

Hence, a graph G is cordial if $-1, 0$ or $1 \in FFI(G)$. Moreover, the cordiality of G can be determined by finding the $FI(G)$ or $FFI(G)$. Shiu and Kwong [12] determined $FFI(P_2 \times P_n)$. Shiu and Lee [13] determined the full friendly index sets of twisted cylinders. Shiu and Wong [15] determined the full friendly index sets of cylinder graphs. Shiu and Ho [10] determined the full friendly index sets of some permutation Petersen graphs, they also determined the full friendly index sets of slender and flat cylinder graphs [11]. Shiu and Ling [14] determined the full friendly index sets of Cartesian products of two cycles. Sinha and Kaur [16] studied the full friendly index sets of some graphs such as K_n, C_n, fans F_n, $F_{2,m}$ and $P_3 \times P_n$. Interested readers may refer to [2] for more results on cordiality of graphs. In general, it is difficult to obtain the full friendly index sets of graphs. The problem on the full friendly index sets and cordiality of general cubic graphs is still beyond our reach at this moment.

Definition 1.4. Let G and H be two graphs such that u and v are two particular vertices of H. An edge xy of G is *blown-up* by H at u and v if xy is replaced by H by identifying x and u, and y and v respectively.

Definition 1.5. Let $P_2 \times P_n$ $(n \geq 2)$ be the ladder graph of order $2n$ and size $3n-2$ with vertex set $V = \{u_i, v_i : 1 \leq i \leq n\}$ and edge set $E = \{u_iu_{i+1}, v_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{u_iv_i : 1 \leq i \leq n\}$. Let $K_4^- = K(1,1,2)$ be the complete tripartite graph with degree 2 vertices u, v. The *double-edge blow-up* graph of $P_2 \times P_n$, denoted $DB(n-2)$, is obtained by blowing up the edges u_1v_1 and u_nv_n by a K_4^- at u and v respectively such that a cubic graph is obtained.

In what follow, we let $m = n-2 \geq 0$ so that $|V(DB(m))| = 2(m+4)$, $|E(DB(m))| = 3m+12$.

EXAMPLE 1.6. The graph $DB(2)$ is illustrated in Figure 1.

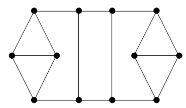

FIGURE 1. Graph $DB(2)$

In this paper, we introduce a labeling subgraph embeddings method to obtain the full friendly index sets of $DB(m)$. Consequently, the cordiality of $DB(m)$ is determined.

2. PRELIMINARIES

We now present some derived results and prove some results which will be used to obtain our main results.

Theorem 2.1. *(Shiu and Wong[15]) Let f be a labeling of a graph G that contains a cycle C as its subgraph. If C contains a 1-edge, then the number of 1-edges in C is a positive even number.*

Theorem 2.2. *In any friendly labeling f of $DB(m)$, if any two vertex labels are exchanged, then $e_{f^+}(1)$ changes by -6, -4, -2, 0, 2, 4, or 6.*

Proof. Since the graph $DB(m)$ is cubic, any vertex u is adjacent to three vertices u_1, u_2, and u_3. In a friendly labeling of $DB(m)$, suppose that the vertices u, u_1, u_2, and u_3 are labeled by x, x_1, x_2, and x_3 ($x, x_1, x_2, x_3 \in \{0,1\}$) respectively. When we change the label of u to $1-x$, the number of 1-edges changes by -3, -1, 1, or 3.

For any two vertices of u and v in $DB(m)$, there are five cases that three of them are listed in Figure 2.

Exchange the labels of u and v, $e_{f^+}(1)$ changes by -6, -4, -2, 0, 2, 4, or 6 in (a), and by -4, -2, 0, 2, or 4 in (b) and (c). In the fourth case, u and v

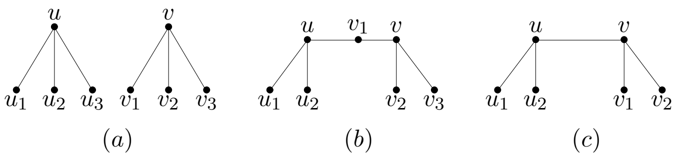

FIGURE 2. Three possible structures of $DB(m)$

are the two degree 3 vertices of a K_4^- subgraph that gives no change to $e_{f^+}(1)$. In the fifth case, u and v are the two degree 2 vertices of a K_4^- subgraph with $e_{f^+}(1)$ changes by 0 or 2. Hence, if any two vertex labels are exchanged, then $e_{f^+}(1)$ changes by -6, -4, -2, 0, 2, 4, or 6. $\square$

Theorem 2.3. *If f_1 and f_2 be two friendly labelings of $DB(m)$ such that f_2 is obtained from f_1 by exchanging two distinct vertex labels under f_1, then $(e_{f_1^+}(1) - e_{f_1^+}(0)) - (e_{f_2^+}(1) - e_{f_2^+}(0)) \equiv 0 \pmod 4$.*

Proof. Since $|V(DB(m))| = 2(m+4)$, any friendly labeling f_1 of $DB(m)$ gives $v_f(1) = v_f(0)$. Hence, exchanging two vertex labels under f_1 gives a new friendly labeling f_2.

Since $e_{f^+}(1) - e_{f^+}(0) = 2e_{f^+}(1) - |E|$, we have $(e_{f_1^+}(1) - e_{f_1^+}(0)) - (e_{f_2^+}(1) - e_{f_2^+}(0)) = 2(e_{f_1^+}(1) - e_{f_2^+}(1)) \equiv 0 \pmod 4$, by Theorem 2.2. $\square$

An edge uv is called an (i,j)-edge if it is incident with an i-vertex and an j-vertex ($i,j \in \{0,1\}$). In the following discussions, all numbers are integer. When the context is clear, we shall also drop the subscript f, f^+. Below are three necessary notations.

(1). Labeling graph: *A graph G with a friendly labeling f such that $e(1) - e(0) = a$ is called a labeling graph of G, denoted by $G(a)$. For easy reading, the $P_2 = a_1a_2$ labeling subgraph is denoted by $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$, and the induced $C_4 = b_1b_2b_4b_3b_1$ (or induced $P_4 = b_1b_2b_4b_3$ or $b_2b_1b_3b_4$) labeling subgraph is denoted by $\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$.*

Throughout this paper, unless stated otherwise, every labeling subgraph $\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$ denotes an induced C_4 or an induced P_4 subgraph with vertices as stated above. Let $G(a)$ be a labeling graph having an edge with end-vertex labels b_1 and b_2 respectively, and another edge with end-vertex labels b_3 and b_4 respectively.

(2). P_2-embedding: *A P_2-embedding onto $G(a)$ at (b_1,b_2)-edge and (b_3,b_4)-edge is obtained by replacing uv and $u'v'$ by a length 2 path uxv and $u'x'v'$ respectively and embedding an edge xx' with corresponding end-vertex labels a_1*

and a_2 such that a new labeling graph $G(b)$ with three extra edges is obtained. Such a P_2-embedding is denoted by

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}.$$

(3). C_4-embedding: *A C_4-embedding onto $G(a)$ at (b_1, b_2)-edge and (b_3, b_4)-edge is obtained by replacing uv and $u'v'$ by a length 3 path $uxyv$ and $u'x'y'v'$ respectively and embedding an edge xx' with corresponding end-vertex labels a_1 and a_3, and another edge yy' with corresponding end-vertex labels a_2 and a_4 such that a new labeling graph $G(b)$ with extra 6 edges is obtained. Such a C_4- or P_4-embedding is denoted by*

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} + \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}.$$

Example 2.4. In Figure 3, a P_2-embedding onto $P_2 \times P_4$ with $e(1) - e(0) = -6$ gives a labeling graph of $P_2 \times P_5$ with $e(1) - e(0) = -7$. The embedding is denoted by

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

Similarly, a C_4-embedding onto the same $P_2 \times P_4$ gives a labeling graph of $P_2 \times P_6$ with $e(1) - e(0) = -4$. The embedding is denoted by

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

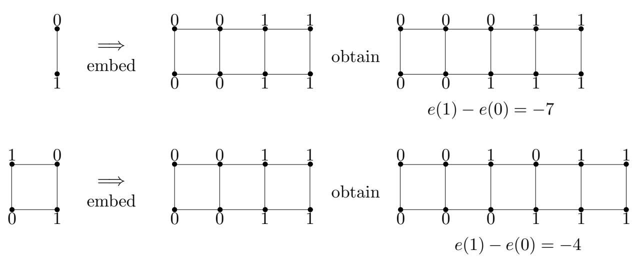

Figure 3. A P_2- and a C_4-embedding onto a $P_2 \times P_4$, respectively

3. Full friendly index sets of $DB(m)$

In the following discussions, all P_2- or C_4-embeddings onto $DB(m)$ only apply to edges u_iu_{i+1} and v_iv_{i+1} for some $i \in \{0, 1, \ldots, n-1\}$.

Lemma 3.1. *For any friendly labeling f of $DB(m)$, we have*

(1) $e(1) \le 10$ *if* $m = 0$*;*
(2) $e(1) \le 3m + 8$ *if* $m > 0$.

Proof. Given any friendly labeling f of $DB(m)$, it is clear that each K_4^- subgraph contains at least one 0-edge. Hence, $e(0) \ge 2$. Consequently, $e(1) \le 10$ if $m = 0$. Suppose $m \ge 1$. By Theorem 2.1, each K_4^- subgraph contains at least one 0-edge. In order to get $min\{e(0)\}$, each K_4^- must contain exactly one 0-edge. Assuming $m = 1$. It is easy to verify that $e(0) \ge 4$. Assuming $m \ge 2$ and $e(0) = 3$. Without loss of generality, we may assume $f(u_i) = f(v_j) = x$ for odd i and even j, whereas $f(u_i) = f(v_j) = 1 - x$ for even i and odd j. As f is friendly, the four degree 3 vertices of both K_4^- must be assigned with two x and two $1 - x$, $x \in \{0, 1\}$. Consequently, $e(0) = 4$, which is a contradiction. Hence, $e(0) \ge 4$ and $e(1) \le 3m + 8$ if $m > 0$. □

Lemma 3.2. *For any friendly labeling f of $DB(m)$, we have*

(1) $e(1) \ge 2$ *if* $m \ge 0$ *is even;*
(2) $e(1) \ge 3$ *if* $m > 0$ *is odd.*

Proof. Let x_1, x_2 (respectively, x_3, x_4) be the 2 common neighbors of u_1, v_1 (respectively, u_n, v_n). By Theorem 2.1, any largest induced cycle of $DB(m)$, say C, contains at least two 1-edges. When m is even, $e(1) = 2$ can be attained by labeling x_1, x_2, u_i and v_i ($1 \le i \le (m+2)/2$) by 1 and the remaining vertices by 0. Assume m is odd. Since $v(1) = v(0) = m + 4$ and C has $2m + 6 \ge 6$ vertices, we have $|v(1) - v(0)| = 0$ or 2. Note that the two K_4^- subgraphs may contain 0, 1 or 2 1-edges in C. In each possibility, we can verify that $e(1) \ge 3$. The equality can be attained by labeling $x_1, x_2, u_i, u_{(m+3)/2}$ and v_i ($1 \le i \le (m+1)/2$) by 1 and the remaining vertices by 0. The lemma holds. □

Lemma 3.3. $FFI(DB(0)) = \{-4 + 4i : -1 \le i \le 3\}$.

Proof. By Theorems 1.2, 2.3 and Lemmas 3.1, 3.2, $FFI(DB(0)) \subseteq \{-4 + 4i : -1 \le i \le 3\}$. The labeling graphs in Figure 4 show that the equality holds. □

Lemma 3.4. *For even $m \ge 2$, $FFI(DB(m)) = \{-3m - 8 + 4i : 0 \le i \le \frac{3(m+2)}{2}\}$.*

Proof. By Theorems 1.2, 2.3 and Lemmas 3.1, 3.2, $FFI(DB(m)) \subseteq \{-3m - 8 + 4i : 0 \le i \le \frac{3(m+2)}{2}\}$. We prove the equality holds by induction on m.
Suppose $m = 2$. We show that there exist labeling graphs of $DB(2)(-14 + 4i)$, $0 \le i \le 6$, by doing the following seven C_4-embeddings:

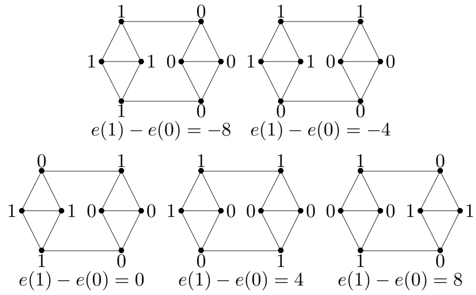

FIGURE 4. Labeling graphs of $DB(0)$ with $e(1)-e(0) \in \{-4+4i : -1 \leq i \leq 3\}$

Case (1). In $DB(0)(-8)$, embed $\begin{bmatrix}1 & 0\\1 & 0\end{bmatrix} + \begin{bmatrix}1 & 0\\1 & 0\end{bmatrix}$ to decrease $e(1)-e(0)$ by 6. Hence, we obtain $DB(2)(-14)$, and in this labeling graph, there exists the labeling subgraph $\begin{bmatrix}1 & 0\\1 & 0\end{bmatrix}$.

Case (2). In $DB(0)(-8)$, embed $\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix} + \begin{bmatrix}1 & 0\\1 & 0\end{bmatrix}$ to decrease $e(1)-e(0)$ by 2. Hence, we obtain $DB(2)(-10)$, and in this labeling graph, there exists the labeling subgraph $\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}$.

Case (3). In $DB(0)(-4)$, embed $\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix} + \begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}$ to decrease $e(1)-e(0)$ by 2. Hence, we obtain $DB(2)(-6)$, and in this labeling graph, there exists the labeling subgraph $\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}$.

Case (4). In $DB(0)(0)$, embed $\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix} + \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$ to decrease $e(1)-e(0)$ by 2. Hence, we obtain $DB(2)(-2)$, and in this labeling graph, there exists the labeling subgraph $\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}$.

Case (5). In $DB(0)(4)$, embed $\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix} + \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$ to decrease $e(1)-e(0)$ by 2. Hence, we obtain $DB(2)(2)$, and in this labeling graph, there exists the labeling subgraph $\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}$.

Case (6). In $DB(0)(8)$, embed $\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix} + \begin{bmatrix}1 & 0\\1 & 0\end{bmatrix}$ to decrease $e(1)-e(0)$ by 2. Hence, we obtain $DB(2)(6)$, and in this labeling graph, there exists the labeling subgraph $\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}$.

Case (7). In $DB(0)(8)$, embed $\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} + \begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}$ to increase $e(1)-e(0)$ by 2. Hence, we obtain $DB(2)(10)$, and in this labeling graph, there exists the labeling subgraph $\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$.

The labeling graphs of $DB(2)$ that we have obtained are illustrated in Figure 5.

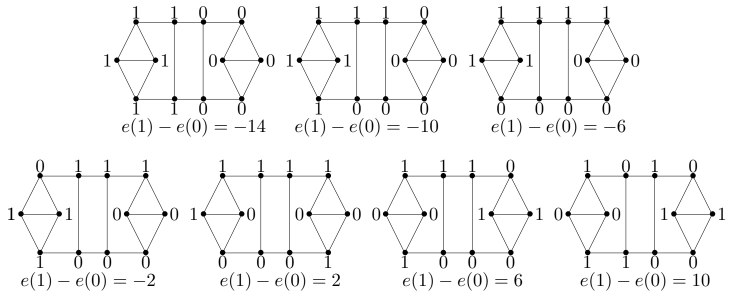

FIGURE 5. Labeling graphs of $DB(2)$ with $e(1)-e(0) \in \{-14+4i : 0 \le i \le 6\}$

Hence, $FFI(DB(2)) = \{-14+4i : 0 \le i \le 6\}$.

Note that in the labeling graphs in Figure 5 (except $DB(2)(-14)$ and $DB(2)(10)$), there exist the labeling subgraph $\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}$. In $DB(2)(-14)$, there exists the labeling subgraph $\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}$. In $DB(2)(10)$, there exists the labeling subgraph $\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$.

Since the embedding $\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix} + \begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}$ decreases $e(1)-e(0)$ by 2, we do this embedding onto $DB(2)(-14+4i)$, $1 \le i \le 5$, to obtain $DB(4)(-16+4i)$, $1 \le i \le 5$), and in these labeling graphs, there exists the labeling subgraph $\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}$.

Similarly, we do the embeddings $\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix} + \begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}$ and $\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix} + \begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}$ onto $DB(2)(-14)$ to decrease $e(1)-e(0)$ by 2 and 6 respectively. Thus, we obtain $DB(4)(-16)$ and $DB(4)(-20)$ with the labeling subgraphs $\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}$ and $\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}$, respectively.

Next, we do the embeddings $\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}+\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$ and $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}+\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$ onto $DB(2)(10)$ so that $e(1)-e(0)$ is decreased by 2 and is increased by 6, respectively. Thus, we obtain $DB(4)(8)$ and $DB(4)(16)$ with the labeling subgraphs $\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}$ and $\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$, respectively.

In $DB(2)(6)$, we do the embedding $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}+\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}$ to increase $e(1)-e(0)$ by 6. Hence, we obtain $DB(4)(12)$, and there exists the labeling subgraph $\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}$.

Hence, $FFI(DB(4))=\{-20+4i : 0\le i\le 9\}$.

Now, assume that for even $k\ge 6$, $FFI(DB(k))=\{-3k-8+4i : 0\le i\le \frac{3(k+2)}{2}\}$ such that the labeling graph $DB(k)(-3k-8+4i)$, $(1\le i\le \frac{3k+4}{2})$, has a labeling subgraph $\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}$, and that in $DB(k)(-3k-8)$ and $DB(k)(3k+4)$, there exist the labeling subgraphs $\begin{bmatrix}1 & 0\\1 & 0\end{bmatrix}$ and $\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$, respectively.

Hence, we do the embedding $\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}+\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}$ onto $DB(k)(-3k-8+4i)$, $1\le i\le \frac{3k+4}{2}$, to obtain $DB(k+2)(-3k-10+4i)$, $1\le i\le \frac{3k+4}{2}$, and in these labeling graphs, there exists the labeling subgraph $\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}$.

Similarly, we do the embeddings $\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}+\begin{bmatrix}1 & 0\\1 & 0\end{bmatrix}$ and $\begin{bmatrix}1 & 0\\1 & 0\end{bmatrix}+\begin{bmatrix}1 & 0\\1 & 0\end{bmatrix}$ onto $DB(k)(-3k-8)$ to obtain $DB(k+2)(-3k-10)$ and $DB(k+2)(-3k-14)$ respectively, and there exist the labeling subgraphs $\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}$ and $\begin{bmatrix}1 & 0\\1 & 0\end{bmatrix}$, respectively.

Next, we do the embeddings $\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}+\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$ and $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}+\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$ onto $DB(k)(3k+4)$ such that $e(1)-e(0)$ is decreased by 2 and is increased by 6 respectively. Thus, we obtain $DB(k+2)(3k+2)$ and $DB(k+2)(3k+10)$ having the labeling subgraphs $\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}$ and $\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$, respectively.

Finally, in $DB(k)(3k)$, we do the embedding $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}+\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}$ to increase $e(1)-e(0)$ by 6. Hence, we obtain $DB(k+2)(3k+6)$, and there exists the labeling subgraph $\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}$.

By mathematical induction, we have $FFI(DB(k+2))=\{-3(k+2)-8+4i : 0\le i\le \frac{3(k+4)}{2}\}$. The proof is complete. □

By Lemmas 3.3 and 3.4, we have

Theorem 3.5. *For even m, $FFI(DB(m)) =$*

(1) $\{-4+4i : -1 \leq i \leq 3\}$ *if $m = 0$;*
(2) $\{-3m-8+4i : 0 \leq i \leq \frac{3(m+2)}{2}\}$ *if $m \geq 2$.*

We now consider odd $m \geq 1$.

Theorem 3.6. *For odd $m \geq 1$, $FFI(DB(m)) = \{-3m-6+4i : 0 \leq i \leq \frac{3m+5}{2}\}$.*

Proof. By Theorems 1.2, 2.3 and Lemmas 3.1, 3.2, we have $FFI(DB(m)) \subseteq \{-3m-6+4i : 0 \leq i \leq \frac{3m+5}{2}\}$. We prove the equality holds by induction on m. The labeling graphs in Figures 6 and 7 show that the equality holds for $m = 1, 3$.

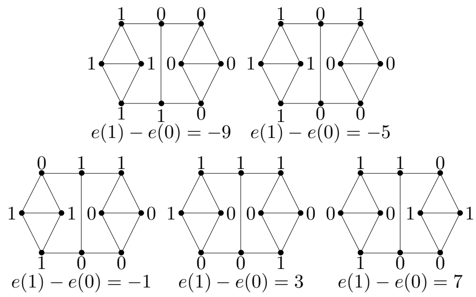

FIGURE 6. Labeling graphs of $DB(1)$ with $e(1)-e(0) \in \{-9+4i : 0 \leq i \leq 4\}$

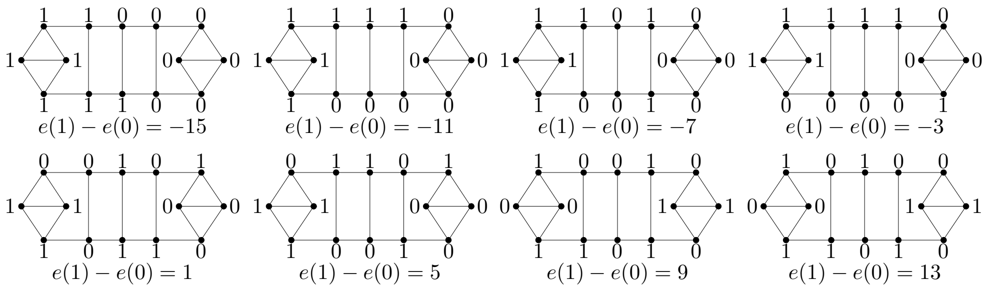

FIGURE 7. Labeling graphs of $DB(3)$ with $e(1)-e(0) \in \{-15+4i : 0 \leq i \leq 7\}$

Note that in the labeling graphs in Figure 7 (except the labeling graphs $DB(3)(-15)$ and $DB(3)(13)$), there exist the labeling subgraph $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$.

Moreover, in $DB(3)(-15)$, there exist a C_4-embedding onto two induced P_4 labeling subgraphs $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. In $DB(3)(-11)$, there exists a C_4-embedding onto two induced P_4 labeling subgraphs $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. In $DB(3)(9)$, there exists the labeling subgraph $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. In $DB(3)(13)$, there exist two labeling subgraphs $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ (which is an induced P_4 subgraph) and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Now, assume that for odd $m = k \geq 5$, $FFI(DB(k)) = \{-3k - 6 + 4i : 0 \leq i \leq \frac{3k+5}{2}\}$ such that the labeling graphs $DB(k)(-3k - 6 + 4i), 1 \leq i \leq \frac{3k+3}{2}$, has a labeling subgraph $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$. Moreover, in $DB(k)(-3k - 6)$, there exist two labeling subgraphs $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. In $DB(k)(-3k - 2)$, there exist two labeling subgraphs $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. In $DB(k)(3k)$, there exists the labeling subgraph $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. In $DB(k)(3k+4)$, there exist two labeling subgraphs $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

We consider the following 6 cases.

Case (1). In $DB(k)(-3k - 6 + 4i)$, $1 \leq i \leq \frac{3k+3}{2}$, we do the embeddings $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ to decrease $e(1) - e(0)$ by 2. Hence, we obtain $DB(k+2)(-3k - 8 + 4i)$, $1 \leq i \leq \frac{3k+3}{2}$, and in these labeling graphs, there exists the labeling subgraph $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

Case (2). In $DB(k)(-3k-2)$, we do the embedding $\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ to decrease $e(1)-e(0)$ by 6. We obtain $DB(k+2)(-3k-8)$ and in the labeling graph, there exist the labeling subgraph $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Case (3). In $DB(k)(-3k-6)$, we do the embedding $\begin{bmatrix}1\\1\end{bmatrix}+\begin{bmatrix}1&1\\1&1\end{bmatrix}$ and $\begin{bmatrix}0\\0\end{bmatrix}+\begin{bmatrix}0&0\\0&0\end{bmatrix}$ to decrease $e(1)-e(0)$ by 6. We obtain $DB(k+2)(-3k-12)$, and in the labeling graph, there exist the labeling subgraphs $\begin{bmatrix}1&1\\1&1\end{bmatrix}$ and $\begin{bmatrix}0&0\\0&0\end{bmatrix}$.

Case (4). In $DB(k)(3k)$, we do the embedding $\begin{bmatrix}0&1\\0&1\end{bmatrix}+\begin{bmatrix}0&0\\1&0\end{bmatrix}$ to increase $e(1)-e(0)$ by 2. We obtain $DB(k+2)(3k+2)$, and in the labeling graph, there exists the labeling subgraph $\begin{bmatrix}0&0\\1&0\end{bmatrix}$.

Case (5). In $DB(k)(3k+4)$, we do the embedding $\begin{bmatrix}0&1\\0&1\end{bmatrix}+\begin{bmatrix}0&0\\1&0\end{bmatrix}$ to increase $e(1)-e(0)$ by 2. We obtain $DB(k+2)(3k+6)$, and in the labeling graph, there exists the labeling subgraph $\begin{bmatrix}0&0\\1&0\end{bmatrix}$.

Case (6). In $DB(k)(3k+4)$, we do the embedding $\begin{bmatrix}1&0\\0&1\end{bmatrix}+\begin{bmatrix}0&1\\1&0\end{bmatrix}$ to increase $e(1)-e(0)$ by 6. We obtain $DB(k+2)(3k+10)$, and in the labeling graph, there exists the labeling subgraph $\begin{bmatrix}0&1\\1&0\end{bmatrix}$.

By mathematical induction, we have $FFI(DB(m))=\{-3m-6+4i : 0\leq i\leq \frac{3m+5}{2}\}$. The proof is complete. □

Hence, the cordiality and friendly index sets of of $DB(m)$ are determined as follows.

Corollary 3.7. *For odd $m\geq 1$, $DB(m)$ is cordial. For even $m\geq 0$, we have*

(1) *$DB(m)$ is cordial if $m\equiv 0 \pmod 4$, and*
(2) *$DB(m)$ is not cordial if $m\equiv 2 \pmod 4$.*

Proof. By Theorem 3.6, there exists $|e(1)-e(0)|=1$ for odd $m\geq 1$. So $DB(m)$ is cordial. By Theorem 3.5, there exists $|e(1)-e(0)|=0$ for $m\equiv 0 \pmod 4$, but for $m\equiv 2 \pmod 4$, the minimum value of $|e(1)-e(0)|=2$. Hence, $DB(m)$ is cordial for $m\equiv 0 \pmod 4$, but not for $m\equiv 2 \pmod 4$. □

Corollary 3.8. *For odd m, $FI(DB(m))=\{2i+1 : 0\leq i\leq \frac{3m+5}{2}\}$. For even $m\geq 0$, we have*

(1) *$FI(DB(m))=\{4i : 0\leq i\leq \frac{3m+8}{4}\}$ if $m\equiv 0 \pmod 4$, and*
(2) *$FI(DB(m))=\{4i+2 : 0\leq i\leq \frac{3m+6}{4}\}$ if $m\equiv 2 \pmod 4$.*

Proof. By Theorem 3.6, for odd $m\geq 1$, $FFI(DB(m))=\{-3m-6,-3m-2,-3m+2,\ldots,3m+4\}$. Hence, $FI(DB(m))=\{1,3,\ldots,3m+6\}=\{2i+1 :$

$0 \leq i \leq (3m+5)/2\}$. By Theorem 3.5, $FFI(DB(0)) = \{-8, -4, 0, 4, 8\}$, and for even $m \geq 2$, $FFI(DB(m)) = \{-3m-8, -3m-4, -3m, \ldots, 3m+4\}$. Hence, for $m \equiv 0 \pmod 4$, $FI(DB(m)) = \{0, 4, 8, \ldots, 3m+8\} = \{4i : 0 \leq i \leq (3m+8)/4\}$; and for $m \equiv 2 \pmod 4$, $FI(DB(m)) = \{2, 6, 10, \ldots, 3m+8\} = \{4i+2 : 0 \leq i \leq (3m+6)/4\}$. □

Shiu and Wong [15] introduced the full product-cordial index set of G.

Definition 3.9. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$, a vertex labeling $f : V(G) \to \mathbb{Z}_2$ induces an edge labeling $f^* : E(G) \to \mathbb{Z}_2$ defined by $f^*(xy) = f(x)f(y)$, for each edge $xy \in E(G)$. For each $i \in \mathbb{Z}_2$, let $v_f(i) = |\{u \in V(G) : f(u) = i\}|$ and $e_{f^*}(i) = |\{xy \in E(G) : f^*(xy) = i\}|$. The *full product-cordial index set* of G, denoted $FPCI(G)$, is defined as $\{e_{f^*}(1) - e_{f^*}(0) :$ the vertex labeling f is friendly$\}$.

They obtained the following result.

Lemma 3.10. *(Shiu and Wong*[15]*) Let f be a friendly labeling of G, f^* be a product labeling of G. If G is an r-regular graph of even order, then $e_{f^*}(1) - e_{f^*}(0) = -\frac{1}{2}(|E| + e_{f^+}(1) - e_{f^+}(0))$.*

Since $DB(m)$ is 3-regular graph, $|V| = 2(m+4)$, $|E| = 3m+12$. By Lemma 3.10 and Theorems 3.5, 3.6, we have

Corollary 3.11. *For even m, $FPCI(DB(m)) =$*

(1) $\{-2i : 1 \leq i \leq 5\}$ *if $m = 0$,*
(2) $\{-2i : 1 \leq i \leq \frac{3m+8}{2}\}$ *if $m \geq 2$.*

Corollary 3.12. *For odd $m \geq 1$, $FPCI(DB(m)) = \{-2i-1 : 1 \leq i \leq \frac{3m+7}{2}\}$.*

4. Conclusion

In this paper, we determined the cordiality of a family of cubic graphs by the labeling subgraph embeddings method. The results in [3] show that this method may be used to determine the cordiality of all families of graphs that can be constructed by repeated subgraph embeddings and that the full friendly indices form an arithmetic sequence.

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