Iranian Journal of Mathematical Sciences and Informatics
Vol. 12, No. 2 (2017), pp 1-14
DOI: 10.7508/ijmsi.2017.2.001

On (Semi-)Edge-primality of Graphs

Wai-Chee Shiu[a], Gee-Choon Lau[b,*] , Sin-Min Lee[c]

[a]Department of Mathematics, Hong Kong Baptist University
224 Waterloo Road, Kowloon Tong, Hong Kong, P.R. China.
[b]Faculty of Computer & Mathematical Sciences, University Teknologi MARA
40450 Shah Alam, Selangor, Malaysia.
[c]34803, Hollyhock Street,
Union City, CA 94587, U.S.A.

E-mail: wcshiu@math.hkbu.edu.hk
E-mail: geeclau@yahoo.com
E-mail: sinminlee@gmail.com

Abstract. Let $G = (V, E)$ be a (p, q)-graph. A bijection $f : E \to \{1, 2, 3, \ldots, q\}$ is called an edge-prime labeling if for each edge uv in E, we have $GCD(f^+(u), f^+(v)) = 1$ where $f^+(u) = \sum_{uw \in E} f(uw)$. Moreover, a bijection $f : E \to \{1, 2, 3, \ldots, q\}$ is called a semi-edge-prime labeling if for each edge uv in E, we have $GCD(f^+(u), f^+(v)) = 1$ or $f^+(u) = f^+(v)$. A graph that admits an edge-prime (or a semi-edge-prime) labeling is called an edge-prime (or a semi-edge-prime) graph. In this paper we determine the necessary and/or sufficient condition for the existence of (semi-) edge-primality of bipartite and tripartite graphs.

Keywords: Prime labeling, Edge-prime labeling, Semi-edge-prime labeling, Bipartite graphs, Tripartite graphs.

2000 Mathematics subject classification: 05C25, 05C78.

*Corresponding Author

Received 05 July 2016; Accepted 14 January 2017

1. Introduction

Let $G = (V(G), E(G))$ (or $G = (V, E)$ for short if not ambiguous) be a simple, finite and undirected graph of order $|V| = p$ and size $|E| = q$. All notation not defined in this paper can be found in [1].

The concept of prime labeling was originated by Entringer and it was introduced in a paper by Tout et al. [8]. A graph G with p vertices and q edges is said to have a prime labeling if $f : V \to \{1, 2, \dots, p\}$ is bijective and for every edge $e = uv$ of G, $GCD(f(u), f(v)) = 1$. If there is no ambiguous, we use (a, b) instead of $GCD(a, b)$. Currently, the two most prominent open conjectures involving vertex labelings are the following:

(1) All tree graphs have a prime vertex labeling (Entringer-Tout Conjecture);
(2) All unicyclic graphs have a prime vertex labeling (Seoud and Youssef [7]).

In 2011, Haxell and Pikhurko [4] proved that all large trees are prime. In 1991, Deretsky et al. [2] introduced the notion of dual of prime labeling which is known as vertex prime labeling. A graph with q edges has vertex prime labeling if its edges can be labeled with distinct integers $\{1, 2, \dots, q\}$ such that for each vertex of degree at least two the greatest common divisor of the labels on its incident edges is 1. For convenience, we will use $[a, b]$ to denote the set of integers between a and b inclusively.

A conjecture: "Any 2-regular graph has a vertex prime labeling if and only if it does not have two odd cycles." was proposed.

An excellent survey on graph labeling is maintained by Gallian [5]. In this paper, we introduce a variant of prime labeling of graphs.

Definition 1.1. Let $G = (V, E)$ be a (p, q)-graph. A bijection $f : E \to [1, q]$ is called an *edge-prime labeling* if for each edge uv in E, we have $(f^+(u), f^+(v)) = 1$, where $f^+(u) = \sum_{uv \in E} f(uw)$. A graph that admits an edge-prime labeling is called an *edge-prime graph*.

Note that this is not a generalization of integer-magic spectra [6] and Bary-Centric Labeling [9]. In Section 2, we obtained a necessary and sufficient condition for disjoint union of path to be edge-prime. We also proved that all 2-regular graphs are edge-prime. In Sections 3 and 4, we proved that many bipartite and tripartite graphs are edge-prime (or not edge-prime). In Section 5, we defined semi-edge-prime and show that certain bipartite and tripartite graphs are semi-edge-prime graphs.

2. Edge-Prime Labelings of Some Simplest Graphs

Lemma 2.1. *Suppose e_1, e_2, e_3 are any 3 successive edges of a graph such that the end-vertices of $e_2 = uv$ are of degree 2. If there exist an edge labeling f such that $f(e_1)+f(e_2)$ and $f(e_2)+f(e_3)$ are not both even, and that $|f(e_1)-$*

[DOI: 10.7508/ijmsi.2017.2.001]

$f(e_3)| = 2^m$, $m \geq 0$, then the induced vertex labels of the 2 end-vertices of e_2 are relatively prime.

Proof. Without loss of generality, assume that $f(e_1) > f(e_3)$. The given labeling f guarantees that $(f^+(u), f^+(v)) = (f(e_1) + f(e_2), f(e_2) + f(e_3)) = (f(e_1) - f(e_3), f(e_2) + f(e_3)) = (2^m, f(e_2) + f(e_3))$. If $f(e_2) + f(e_3)$ is odd, we have $(2^m, f(e_2) + f(e_3)) = 1$. Otherwise, we must have $f(e_1) + f(e_2)$ is odd and $m = 0$ so that $(2^m, f(e_2) + f(e_3)) = (1, f(e_2) + f(e_3)) = 1$. Hence, the lemma holds. □

Theorem 2.2. *Let G be the disjoint union of paths. Then G is edge-prime if and only if it has at most one component of P_2.*

Proof. (Sufficiency) List all the path(s) from the shortest length to the longest length. Label the consecutive edges of each path from 1 to $|E(G)|$ such that every 2 adjacent edge labels must differ by 1. By Lemma 2.1, the induced vertex labels of every 2 adjacent internal vertices are relatively prime. It is also easy to verify that the induced vertex labels of each pendant vertex and its adjacent vertex are relatively prime.
(Necessity) We prove by contrapositive. If G has at least 2 components of P_2, then a P_2 will have its edge labeled by integer > 1. Such a labeling is not edge-prime. □

Corollary 2.3. *A 1-regular graph is edge-prime if and only if it is K_2.*

Theorem 2.4. *All 2-regular graphs are edge-prime.*

Proof. Let $G = \sum_{i=1}^{j} C_{n_i}$ be a 2-regular graph which is the disjoint union of n_i-cycles, $1 \leq i \leq j$. Without loss of generality, assume that $3 \leq n_1 \leq n_2 \leq \cdots \leq n_j$. We shall label C_{n_1} by using the first n_1 integers, and label C_{n_2} by the next n_2 integers and so on. Suppose $a+1 \geq 1$ is the smallest available edge label for a cycle C_n. Let $e_1, e_2, \ldots, e_n$ be successive edges in C_n. Consider the following four cases.

(1) Suppose $n = 4k$ for some $k \geq 1$. Label the 4 successive edges of C_4 by $a+1$, $a+2$, $a+3$, $a+4$ if $k = 1$. Suppose $k \geq 2$. Define $\sigma : \{e_i \mid 1 \leq i \leq 2k\} \to [a+1, a+4k]$ by $\sigma(e_{i+4}) = \sigma(e_i) + 8$ for $1 \leq i \leq 2k-4$ with initial values $\sigma(e_1) = a+1$, $\sigma(e_2) = a+2$, $\sigma(e_3) = a+5$ and $\sigma(e_4) = a+6$. Also define $\sigma : \{e_i \mid 2k+1 \leq i \leq 4k\} \to [a+1, a+4k]$ by $\sigma(e_{i+4}) = \sigma(e_i) - 8$ for $2k+1 \leq i \leq 4k-4$ with initial values $\sigma(e_{2k+1}) = a+4k-1$, $\sigma(e_{2k+2}) = a+4k$, $\sigma(e_{2k+3}) = a+4k-5$ and $\sigma(e_{2k+4}) = a+4k-4$. One may check that $\sigma : E(C_{4k}) \to [a+1, a+4k]$ is a bijection.
(2) Suppose $n = 4k+1$ for some $k \geq 1$. Label the 5 successive edges of C_5 by $a+1$, $a+4$, $a+5$, $a+2$, $a+3$ if $k = 1$. Suppose $k \geq 2$. Define $\sigma : \{e_i \mid 2 \leq i \leq 2k+1\} \to [a+1, a+4k+1]$ by $\sigma(e_{i+4}) = \sigma(e_i) + 8$

[DOI: 10.7508/ijmsi.2017.2.001]

for $2 \le i \le 2k-3$ with initial values $\sigma(e_2) = a+4$, $\sigma(e_3) = a+5$, $\sigma(e_4) = a+8$ and $\sigma(e_5) = a+9$. Also define $\sigma : \{e_i \mid 2k+2 \le i \le 4k+1\} \to [a+1, a+4k+1]$ by $\sigma(e_{i+4}) = \sigma(e_i) - 8$ for $2k+2 \le i \le 4k-3$ with initial values $\sigma(e_{2k+2}) = a+4k-2$, $\sigma(e_{2k+3}) = a+4k-1$, $\sigma(e_{2k+4}) = a+4k-6$ and $\sigma(e_{2k+5}) = a+4k-5$. Finally define $\sigma(e_1) = a+1$. One may check that $\sigma : E(C_{4k+1}) \to [a+1, a+4k+1]$ is a bijection.

(3) Suppose $n = 4k+2$ for some $k \ge 1$. Label the 6 successive edges of C_6 by $a+1$, $a+4$, $a+3$, $a+2$, $a+5$, $a+6$ if $k = 1$. Suppose $k \ge 2$. Define $\sigma : \{e_i \mid 3 \le i \le 2k+2\} \to [a+1, a+4k+2]$ by $\sigma(e_{i+4}) = \sigma(e_i) + 8$ for $3 \le i \le 2k-2$ with initial values $\sigma(e_3) = a+3$, $\sigma(e_4) = a+4$, $\sigma(e_5) = a+7$ and $\sigma(e_6) = a+8$. Also define $\sigma : \{e_i \mid 2k+3 \le i \le 4k+2\} \to [a+1, a+4k+2]$ by $\sigma(e_{i+4}) = \sigma(e_i) - 8$ for $2k+3 \le i \le 4k-2$ with initial values $\sigma(e_{2k+3}) = a+4k+1$, $\sigma(e_{2k+4}) = a+4k+2$, $\sigma(e_{2k+5}) = a+4k-3$ and $\sigma(e_{2k+6}) = a+4k-2$. Finally define $\sigma(e_1) = a+1$ and $\sigma(e_2) = a+2$. One may check that $\sigma : E(C_{4k+2}) \to [a+1, a+4k+2]$ is a bijection.

(4) Suppose $n = 4k+3$ for some $k \ge 0$. If $n = 3$, then label the 3 edges of C_3 by $a+1$, $a+2$, $a+3$. If $n = 7$, then label the 7 edges of C_7 by $a+1$, $a+2$, $a+3$, $a+4$, $a+7$, $a+5$, $a+6$. Suppose $k \ge 2$. Define $\sigma : \{e_i \mid 4 \le i \le 2k+3\} \to [a+1, a+4k+3]$ by $\sigma(e_{i+4}) = \sigma(e_i) + 8$ for $4 \le i \le 2k-1$ with initial values $\sigma(e_4) = a+4$, $\sigma(e_5) = a+7$, $\sigma(e_6) = a+8$ and $\sigma(e_7) = a+11$. Also define $\sigma : \{e_i \mid 2k+4 \le i \le 4k+3\} \to [a+1, a+4k+3]$ by $\sigma(e_{i+4}) = \sigma(e_i) - 8$ for $2k+4 \le i \le 4k-1$ with initial values $\sigma(e_{2k+4}) = a+4k+1$, $\sigma(e_{2k+5}) = a+4k+2$, $\sigma(e_{2k+6}) = a+4k-3$ and $\sigma(e_{2k+7}) = a+4k-2$. Finally define $\sigma(e_1) = a+1$, $\sigma(e_2) = a+2$ and $\sigma(e_3) = a+3$. One may check that $\sigma : E(C_{4k+3}) \to [a+1, a+4k+3]$ is a bijection.

By Lemma 2.1, the labeling above is edge-prime. □

Example 2.5. Let $G = C_3 + C_8 + C_9 + C_{10} + C_{11}$. We label the components of G as follows:

(1) Label the 3 successive edges of C_3 by $1, 2, 3$.
(2) Label the 8 successive edges of C_8 by $4, 5, 8, 9, 10, 11, 6, 7$.
(3) Label the 9 successive edges of C_9 by $12, 15, 16, 19, 20, 17, 18, 13, 14$.
(4) Label the 10 successive edges of C_{10} by 21, 22, 23, 24, 27, 28, 29, 30, 25, 26.
(5) Label the 11 successive edges of C_{11} by 31, 32, 33, 34, 37, 38, 41, 39, 40, 35, 36.

It is readily verified that the labeling is edge-prime.

From the proof of Theorem 2.4, we have

Theorem 2.6. *If G is edge-prime, then $G + C_n$ is edge-prime.*

[DOI: 10.7508/ijmsi.2017.2.001]

Proof. Let f be an edge-prime labeling of G and h be an edge-prime labeling of C_n as defined in Theorem 2.4. Define an edge labeling g of $G+C_n$ such that $g(e) = f(e)$ if $e \in E(G)$, and $g(e) = h(e) + |E(G)|$ otherwise. Clearly, g is an edge-prime labeling. □

Corollary 2.7. *If G is edge-prime, then $G + H$ is edge-prime, where H is a 2-regular graph.*

We note that under the edge-prime labeling defined in the proof of Theorem 2.4 by choosing $a = 0$, all the induced vertex labels of C_4 and C_6 are prime. We now give edge-prime labelings of even cycles of order at most 34 such that all the induced vertex labels are primes.

n	Labels of successive edges of C_n
8	1, 2, 5, 8, 3, 4, 7, 6
10	1, 2, 5, 8, 3, 10, 9, 4, 7, 6
12	1, 2, 5, 12, 11, 8, 3, 10, 9, 4, 7, 6
14	1, 2, 5, 14, 3, 8, 11, 12, 7, 4, 9, 10, 13, 6
16	1, 2, 5, 14, 15, 16, 3, 8, 11, 12, 7, 4, 9, 10, 13, 6
18	1, 2, 5, 14, 15, 16, 3, 8, 11, 18, 13, 10, 9, 4, 7, 12, 17, 6
20	1, 2, 5, 14, 15, 16, 3, 8, 11, 18, 13, 10, 19, 20, 9, 4, 7, 12, 17, 6
22	1, 2, 5, 14, 15, 16, 3, 8, 11, 18, 13, 10, 19, 22, 21, 20, 9, 4, 7, 12, 17, 6
24	1, 2, 5, 24, 17, 12, 7, 4, 9, 20, 21, 22, 19, 10, 13, 18, 11, 8, 3, 16, 15, 14, 23, 6
26	1, 2, 11, 18, 5, 8, 21, 16, 25, 4, 19, 24, 13, 10, 9, 14, 23, 20, 3, 26, 15, 22, 7, 6, 17, 12
28	1, 2, 11, 18, 5, 8, 21, 16, 25, 4, 19, 24, 13, 28, 9, 10, 27, 14, 23, 20, 3, 26, 15, 22, 7, 6, 17, 12
30	1, 30, 29, 2, 11, 18, 5, 8, 21, 16, 25, 4, 19, 24, 13, 28, 9, 10, 27, 14, 23, 20, 3, 26, 15, 22, 7, 6, 17, 12
32	1, 30, 29, 2, 11, 18, 5, 8, 21, 16, 25, 4, 19, 24, 13, 28, 31, 10, 9, 32, 27, 14, 23, 20, 3, 26, 15, 22, 7, 6, 17, 12
34	1, 30, 29, 2, 11, 18, 5, 8, 21, 16, 25, 4, 19, 24, 13, 28, 31, 10, 33, 34, 9, 32, 27, 14, 23, 20, 3, 26, 15, 22, 7, 6, 17, 12

Similarly, it is easy to verify that each odd cycle of order up to 11 admits an edge-prime labeling such that all but one induced vertex labels are prime.

Conjecture 2.1. *There exist edge-prime labelings for even cycles such that all induced vertex labels are primes, and for odd cycles such that all but one induced vertex labels are prime.*

3. Edge-Prime Labelings of Some Bipartite and Tripartite Graphs

The following useful lemma can be found in any book of number theory:

[DOI: 10.7508/ijmsi.2017.2.001]

Lemma 3.1. *For any integers a, b, c,*

1. $(a,b) = (a,-b) = (a+bc,b)$*;*

2. if $(a,b) = (a,c) = 1$*, then* $(a,bc) = 1$*.*

Let (X,Y) be the bipartition of $K(2,n)$, where $X = \{x_1, x_2\}$ and $Y = \{y_j \mid 1 \leq j \leq n\}$. Define $\sigma_n : E(K(2,n)) \to [1,2n]$ by $\sigma_n(x_1y_j) = 2j-1$ and $\sigma_n(x_2y_j) = 2n+2-2j$, $1 \leq j \leq n$. Then $\sigma_n^+(y_j) = 2n+1$ for all j, $\sigma_n^+(x_1) = n^2$ and $\sigma_n^+(x_2) = n^2+n$. The labeling σ_n is called the *basic labeling of* $K(2,n)$.

Lemma 3.2. *Keep the notation defined above. Suppose* $a \in \mathbb{Z}$*. Let* $f : E(K(2,n)) \to [a+1, a+2n]$*, where* $f = \sigma_n + a$*. If* $(n, 2a+1) = 1$*, then* $(f^+(x_i), f^+(y_j)) = 1$ *for* $1 \leq j \leq n$ *and* $i = 1, 2$*.*

Proof. Clearly $f^+(x_1) = n(n+a)$, $f^+(x_2) = n^2+n+na$ and $f^+(y_j) = 2(n+a)+1$.
By Lemma 3.1 and the hypothesis we have $(n+a, 2(n+a)+1) = 1$ and $(n, 2n+2a+1) = (n, 2a+1) = 1$. By Lemma 3.1 again we have $(f^+(x_1), f^+(y_j)) = (n(n+a), 2(n+a)+1) = 1$ for all j.
Similarly, $(f^+(x_2), f^+(y_j)) = (n^2+n+na, 2n+2a+1) = (-n^2-na, 2n+2a+1) = (n(n+a), 2(n+a)+1) = 1$ for all j. □

Theorem 3.3. *The disjoint union of m complete bipartite graph $K(2,n)$'s, $mK(2,n)$, is edge-prime for $m, n \geq 1$.*

Proof. Let $G_i \cong K(2,n)$, $1 \leq i \leq m$. By using the basic labeling of $K(2,n)$ we define $f_i : E(G_i) \to [2(i-1)n+1, 2in]$, where $f_i = \sigma_n + 2(i-1)n$, $1 \leq i \leq m$. Let the combining labeling for the whole graph $mK(2,n)$ be f. Since $(4(i-1)n+1, n) = 1$, by Lemma 3.2 we obtain that f is an edge-prime labeling. □

Theorem 3.4. *For* $n \geq 1$, $\sum\limits_{k=1}^{n} K(2,k)$ *is edge-prime.*

Proof. Label $K(2,k)$ by $\sigma_k + k(k-1)$, $1 \leq k \leq n$. We can see that the labeling is a bijection from $E(\sum\limits_{k=1}^{n} K(2,k)) \to [1, n(n+1)]$. Since $(k, 2k(k-1)+1) = 1$, by Lemma 3.2 we have the theorem. □

Conjecture 3.1. $\sum\limits_{i=1}^{m} K(2,n_i)$ *is edge-prime, where* $m \geq 2$*.*

For $1 \leq i \leq m$, let $G_i \cong K(2,n_i)$ with bipartition (X_i, Y_i), where $X_i = \{x_{i-1}, x_i\}$, $Y_i = \{y_{i,1}, \dots, y_{i,n_i}\}$ and $x_0 = x_m$. Let $B(n_1, \dots, n_m) = \bigcup\limits_{i=1}^{m} G_i$. If $n_1 = \cdots = n_m = n$, then we denote the sequence $n_1, n_2, \dots, n_m$ by $n^{[m]}$ for short. Note that $B(1^{[m]}) = C_{2m}$.

Theorem 3.5. *Suppose* $(m-1, 2n+1) = 1$ *where* $m \geq 2$ *and* $n \geq 1$*. The bipartite graph* $B(n^{[m]})$ *is edge-prime.*

Proof. Keep the notation defined above. Label G_i by $\sigma_n + 2(i-1)n$, where σ_n is the basic labeling of $K(2,n)$. Let the combining labeling be f. Then $f^+(x_i) = (n^2+n+2(i-1)n^2) + (n^2+2in^2) = 4in^2+n$ for $1 \leq i \leq m-1$; $f^+(x_0) = (n^2+n+2(m-1)n^2)+(n^2) = 2mn^2+n$; and $f^+(y_{i,j}) = 4in-2n+1$, for all j.
Since $(n, 4in \pm 2n+1) = 1$ and $(4in+1, 4in \pm 2n+1) = (4in+1, \pm 2n) = 1$, $(f^+(x_i), f^+(y_{i,j})) = (4in^2+n, 4in-2n+1) = 1$ and $(f^+(x_i), f^+(y_{i+1,j})) = (4in^2+n, 4in+2n+1) = 1$ for $1 \leq i \leq m-1$.
Finally, from the hypothesis, $(2mn+1, 2n+1) = (1-m, 2n+1) = 1$ and $(2mn+1, 4mn-2n+1) = (2mn+1, -2n-1) = (2mn+1, 2n+1) = 1$, $(f^+(x_0), f^+(y_{1,j})) = (2mn^2+n, 2n+1) = 1$ and $(f^+(x_0), f^+(y_{m,j})) = (2mn^2+n, 4mn-2n+1) = 1$. □

Conjecture 3.2. *$B(n^{[m]})$ is edge-prime, where $m \geq 2$, $n \geq 2$.*

The generalized theta graph $\theta(s_1, \ldots, s_k)$ consists of a pair of end vertices joined by $k \geq 3$ internally disjoint paths of lengths $s_1, \ldots, s_k \geq 1$.

Theorem 3.6. *For $n \geq 3$, the generalized theta graph $\theta(3^{[n]})$ is edge-prime.*

Proof. Let $G = \theta(3^{[n]})$ with $V(G) = \{u, x, v_i, w_i \mid 1 \leq i \leq n\}$ and $E(G) = \{uv_i, v_iw_i, w_ix \mid 1 \leq i \leq n\}$. Define a labeling f as follows:

(1) $f(uv_i) = i$ for $1 \leq i \leq n$;
(2) $f(v_iw_i) = 2n+1-i$ for $1 \leq i \leq n$;
(3) $f(w_ix) = 2n+i$ for $1 \leq i \leq n$.

Clearly, $f^+(u) = n(n+1)/2, f^+(v_i) = 2n+1, f^+(w_i) = 4n+1$ and $f^+(x) = n(5n+1)/2$. It can be verified that $(f^+(v_i), f^+(w_i)) = (f^+(u), f^+(v_i)) = (f^+(w_i), f^+(x)) = 1$. Hence, f is an edge-prime labeling. □

Theorem 3.7. *For $n \geq 3$, the generalized theta graph $\theta(4^{[n]})$ is edge-prime.*

Proof. Let $G = \theta(4^{[n]})$ with $V(G) = \{u, y, v_i, w_i, x_i \mid 1 \leq i \leq n\}$ and $E(G) = \{uv_i, v_iw_i, w_ix_i, x_iy \mid 1 \leq i \leq n\}$. Define a labeling f similarly to that of Theorem 3.6:

(1) $f(uv_i) = i$ for $1 \leq i \leq n$;
(2) $f(v_iw_i) = 2n+1-i$ for $1 \leq i \leq n$;
(3) $f(w_ix_i) = 2n+i$ for $1 \leq i \leq n$;
(4) $f(x_iy) = 4n+1-i$ for $1 \leq i \leq n$.

Clearly, $f^+(u) = n(n+1)/2, f^+(v_i) = 2n+1, f^+(w_i) = 4n+1, f^+(x_i) = 6n+1$ and $f^+(y) = n(7n+1)/2$. It can be verified that $(f^+(v_i), f^+(w_i)) = (f^+(w_i), f^+(x_i)) = (f^+(u), f^+(v_i)) = (f^+(x_i), f^+(y)) = 1$. Hence, f is an edge-prime labeling. □

Theorem 3.8. *The generalized theta graph $\theta(n,n,n)$ is edge-prime for $n \geq 2$.*

[DOI: 10.7508/ijmsi.2017.2.001]

Proof. For $n = 2, 3, 4$, the results follow from Theorems 3.3, 3.6 and 3.7. We may assume $n \geq 5$. Let $V(\theta(n,n,n)) = \{x, y, u_i, v_i, w_i \mid 1 \leq i \leq n-1\}$ and

$$E(\theta(n,n,n)) = \{xu_1, xv_1, xw_1, u_{n-1}y, v_{n-1}y, w_{n-1}y\} \cup \{u_iu_{i+1}, v_iv_{i+1}, w_iw_{i+1} \mid 1 \leq i \leq n-2\}.$$

Define a labeling f as follows:

(a) $f(xu_1) = 1$, $f(xv_1) = 2$, $f(xw_1) = 3$;
(b) $f(u_{i-1}u_i) = 3i$, $f(v_{i-1}v_i) = 3i-1$, $f(w_{i-1}w_i) = 3i-2$ for even $i \geq 2$;
(c) $f(u_{i-1}u_i) = 3i-2$, $f(v_{i-1}v_i) = 3i-1$, $f(w_{i-1}w_i) = 3i$ for odd $i \geq 3$.
(d) $f(u_{n-1}y) = 3n$, $f(v_{n-1}y) = 3n-1$, $f(w_{n-1}y) = 3n-2$ if n is even; $f(u_{n-1}y) = 3n-2$, $f(v_{n-1}y) = 3n-1$, $f(w_{n-1}y) = 3n$ if n is odd.

Observe that $f^+(x) = 6, f^+(u_i) = f^+(v_i) = f^+(w_i) = 6i+1$ for $1 \leq i \leq n-1$, $f^+(y) = 9n-3$. Clearly, $(f^+(x), f^+(u_1)) = 1$. For $1 \leq i \leq n-2$, $(f^+(u_i), f^+(u_{i+1})) = (6i+1, 6i+7) = (6i+1, 6) = (1, 6) = 1$. Moreover, $(f^+(u_{n-1}), f^+(y)) = (6n-5, 9n-3) = (6n-5, 3n+2) = (3n-7, 3n+2) = (3n-7, 9) = 1$ since $3n-7$ is not a multiple of 3. Hence, f is an edge-prime labeling. □

Conjecture 3.3. *All generalized theta graphs are edge-prime.*

4. Edge-Prime Labelings of Some Trees

Definition 4.1. For $n \geq 1$, the star $St(n)$ is called the graph of diameter 2 with n edges attach to the apex vertex c.

Definition 4.2. The *n-galaxy* $St(a_1, a_2, \ldots, a_n)$ is called the disjoint union of $n \geq 2$ stars $St(a_i)$, $i = 1, 2, \ldots, n$.

Theorem 4.3. *The star $St(n)$ is edge-prime if and only if $n \leq 2$.*

Proof. The sufficiency is obvious. Suppose $n \geq 3$. Let c be the apex vertex and let f be an edge-prime labeling of $St(n)$. Clearly, $f^+(c) = n(n+1)/2$. If n is odd, then $(n, n(n+1)/2) = n$; and if n is even, then $(n/2, n(n+1)/2) = n/2$. Hence, $St(n)$ is not edge-prime. □

Theorem 4.4. *The galaxy $St(1,n)$ is edge-magic if and only if $n \leq 2$.*

Proof. The sufficiency is obvious. Suppose $n \geq 3$ and $St(1,n)$ is edge-prime. Then we must label the component K_2 by 1 and all other edges by 2 to $n+1$. The apex vertex of $St(n)$ component has label $n(n+3)/2$. If n is odd, then $(n, n(n+3)/2) = n$. If n is even, then $(n/2, n(n+3)/2) = n/2$. Hence, $St(1,n)$ is not edge-prime. □

Theorem 4.5. *For $m \geq n$ and $m + n \equiv 1 \pmod 4$, the galaxy $St(n,m)$ is edge-prime only if $m \geq n+1 \geq 3$ is odd, and all the edges of $St(n)$ receive odd labels.*

[DOI: 10.7508/ijmsi.2017.2.001]

Proof. Let $m+n=4k+1$. Hence $m \geq 2k+1 > n$ and there are $2k+1$ odd integers to label the edges. Note that a component of $St(n,m)$ must receive even number of odd edge labels. It follows that all the edges of this component must receive odd integer labels. Since $m \geq 2k+1$, this component must be $St(n)$. Hence, n is even. It follows that $m \geq n+1 \geq 3$ is odd. □

Theorem 4.6. *For $m \geq n$ and $m+n \equiv 2 \pmod 4$, the galaxy $St(n,m)$ is edge-prime only if $m \geq n+2 \geq 4$ is even, and all the edges of $St(n)$ receive odd labels.*

Proof. Let $m+n=4k+2$. Hence $m \geq 2k+1 \geq n$ and there are $2k+1$ odd integers to label the edges. Similar to the proof of Theorem 4.5, all edges of $St(n)$ receive odd labels and n is even. Hence, m is even. It follows that $m \geq n+2 \geq 4$. □

Corollary 4.7. *The galaxy $St(4,6)$ is not edge-magic.*

Proof. It follows by using Theorem 4.6 and checking each case directly. □

Corollary 4.8. *If the galaxy $St(n^{[2]})$ is edge-magic, then n is even.*

Theorem 4.9. *For $m,n \geq 2$ and $m+n \equiv 0,3 \pmod 4$, the galaxy $St(n,m)$ is edge-prime if $(m+n)(m+n+1)/2$ is the sum of two primes p and q such that p is the sum of m distinct integers in $[1,m+n]$.*

Proof. Suppose $p=\sum\limits_{i=1}^{m} x_i$, where $x_1,\dots,x_m$ are distinct integers in $[1,m+n]$. We label the edges of $St(m)$ by $x_1,\dots,x_m$ consecutively and those of $St(n)$ by the remaining labels. It is clear that we have an edge-prime labeling of $St(n,m)$. □

Example 4.10. We illustrate the case $m+n \equiv 3 \pmod 4$ with the example $(n,m)=(5,6)$. We see that $(5+6)(5+6+1)/2=66$. As 66 can be expressed as the sum of $\{5,61\}$, $\{7,59\}$, $\{13,53\}$, $\{19,47\}$, $\{23,43\}$ and $\{29,37\}$, it is clear that we cannot use $\{5,61\}$, $\{7,59\}$ and $\{13,53\}$ to construct an edge-prime labeling. However, for the remaining three pairs we have $(1,2,3,4,9)$, $(5,6,7,8,10,11)$ for $\{19,47\}$; $(1,2,3,6,11)$, $(4,5,7,8,9,10)$ for $\{23,43\}$; and $(1,3,4,10,11)$, $(2,5,6,7,8,9)$ for $\{29,37\}$.

It is easy to verify that for $m+n \leq 16$, the necessary condition in Theorems 4.5 and 4.6 are sufficient except $m=6, n=4$.

Conjecture 4.1. *The galaxy $St(n,m)$ is edge-prime if and only if*
(1) $m+n \equiv 0,3 \pmod 4$;
(2) $m+n \equiv 1 \pmod 4$ *and* $m \geq n+1 \geq 3$ *is odd;*
(3) $m+n \equiv 2 \pmod 4$ *and* $m \geq n+2 \geq 4$ *is even except* $m=6, n=4$.

Theorem 4.11. *For any $k \geq 1$, $St(2^{[k]})$ is edge-prime.*

[DOI: 10.7508/ijmsi.2017.2.001]

Proof. This is a special case of Theorem 3.3. □

Theorem 4.12. *If $St(3^{[k]})$ is edge-prime, then $k \equiv 0, 3 \pmod 4$.*

Proof. Observe that if the induced label of the apex vertex of a component of $St(3^{[k]})$ is even, then the labeling is not edge-prime. Thus, the induced label of the apex vertex of each component of $St(3^{[k]})$ must be odd. Hence, the corresponding component has 1 or 3 odd edge labels. Suppose there are a components containing 1 odd edge label. Since there are $\lceil 3k/2 \rceil$ odd integers to label the edges, $\lceil 3k/2 \rceil = a + 3(k - a) = 3k - 2a$.
When k is even, we have $3k - 2a = 3k/2$ which implies that $k \equiv 0 \pmod 4$. When k is odd, we have $3k - 2a = (3k + 1)/2$ which implies that $k \equiv 3 \pmod 4$. □

Conjecture 4.2. *$St(3^{[k]})$ is edge-prime if $k \equiv 0, 3 \pmod 4$.*

Theorem 4.13. *If G is edge-prime, then $G + St(2^{[k]})$ is edge-prime for all $k \geq 1$.*

Proof. Let $m = |E(G)|$. We extend the edge-labeling of G to $G + St(2^{[k]})$ by labeling the edges of $St(2^{[k]})$ by $\{m + 1, m + 2\}$, $\{m + 3, m + 4\}, \ldots$, $\{m + 2k - 1, m + 2k\}$ consecutively. It is clear that the extended labeling is edge-prime. □

For $3 \leq j \leq 8$, it is easy to verify that $St(2, j)$, $St(3, 4)$, $St(3^{[3]})$, $St(3^{[4]})$ and $St(2) + K_4$ are edge-prime.

Corollary 4.14. *For any $k \geq 1$, $3 \leq j \leq 8$, the graphs $St(2^{[k]}, j)$, $St(2^{[k]}, 3, 4)$, $St(2^{[k]}, 3^{[3]})$, $St(2^{[k]}, 3^{[4]})$ and $St(2^{[k]}) + K_4$ are edge-prime.*

Let Y_n be a tree with

$$V(Y_n) = \{u_1, u_2, v_i \,|\, 1 \leq i \leq n\} \text{ and}$$
$$E(Y_n) = \{u_1v_1, u_2v_1, v_iv_{i+1} \,|\, 1 \leq i \leq n - 1\},$$

where $n \geq 3$.

Theorem 4.15. *The tree Y_n, $n \geq 3$ is edge-prime.*

Proof. Define $f(u_1v_1) = 1$, $f(u_2v_1) = 4$, $f(v_1v_2) = 2$, $f(v_2v_3) = 3$, $f(v_iv_{i+1}) = i + 2$ for $3 \leq i \leq n - 1$. Clearly, f is an edge-prime labeling. □

For $n \geq 2$, let X_n be the tree with $V(X_n) = \{u_1, u_2, u_3, u_4, v_i \,|\, 1 \leq i \leq n\}$ and $E(X_n) = \{u_1v_1, u_2v_1, u_3v_n, u_4v_n, v_iv_{i+1} \,|\, 1 \leq i \leq n - 1\}$.

Theorem 4.16. *The tree X_n is edge-prime, $n \geq 2$.*

Proof. Let $e_1, \ldots, e_{n-1}$ be the successive edges of the path $v_1v_2 \cdots v_n$. Define $f(u_1v_1) = 1$, $f(u_2v_1) = 3$, $f(u_3v_n) = 2$, $f(u_4v_n) = n + 3$ and $f(e_i) = i + 3$, $1 \leq i \leq n - 1$. It follows that $f^+(u_1) = 1$, $f^+(u_2) = 3$, $f^+(u_3) = 2$, $f^+(u_4) =$

[DOI: 10.7508/ijmsi.2017.2.001]

$n+3$, $f^+(v_1)=8$, $f^+(v_n)=2n+7$ and $f^+(v_i)=2i+5$ for $2\le i\le n-1$. Clearly $(f^+(v_1),f^+(u_1))=(f^+(v_1),f^+(u_2))=1$, $(f^+(v_n),f^+(u_3))=(2n+7,2)=1$, $(f^+(v_n),f^+(u_4))=(2n+7,n+3)=(1,n+3)=1$. Moreover, $(f^+(v_1),f^+(v_2))=(8,9)=1$, $(f^+(v_{n-1}),f^+(v_n))=(2n+3,2n+7)=(2n+3,4)=1$ and $(f^+(v_i),f^+(v_{i+1}))=(2i+5,2i+7)=(2i+5,2)=1$ for $2\le i\le n-2$. So f is an edge-prime labeling. □

Let $DS(m,n)$ be the double star with $V(DS(m,n))=\{x,y,u_i,v_j\,|\,1\le i\le m, 1\le j\le n\}$ and $E(DS(m,n))=\{xy,xu_i,yv_j\,|\,1\le i\le m,1\le j\le n\}$.

Theorem 4.17. *For even $n=2^m\ge 2$, $DS(n-1,n)$ is edge-prime if $n+1$ is prime.*

Proof. Label edge xy by $n+1$, label edge(s) xu_1 to xu_{n-1} by odd integers in $[1,2n]\setminus\{n+1\}$, and label edges yv_1 to yv_n by even integers in $[1,2n]$. We have $f^+(x)=n^2$ and $f^+(y)=(n+1)^2$. It can be verified that $(f^+(x),f^+(y))=1$. From the given conditions, we also have $(f^+(x),f^+(u_i))=(f^+(y),f^+(v_j))=1$. The theorem holds. □

Theorem 4.18. *For odd $n=2^m-1\ge 1$, $DS(n,n)$ is edge-prime if n^2+n+1 is prime.*

Proof. Label edge xy by 1, label edge(s) xu_1 to xu_n by odd integers in $[3,2n+1]$, and label edges yv_1 to yv_n by even integers in $[1,2n+1]$. We have $f^+(x)=(n+1)^2$ and $f^+(y)=n^2+n+1$. It can be verified that $(f^+(x),f^+(y))=1$. From the given conditions, we also have $(f^+(x),f^+(u_i))=(f^+(y),f^+(v_j))=1$. The theorem holds. □

Remark 4.19. All star $St(n), n\ge 3$ are non-edge-prime trees of diameter 2 while the trees X_n and Y_n are edge-prime trees of diameter at least 3. Moreover, there are sufficient conditions for trees of diameter 3 (the double star $DS(m,n)$) to admit an edge-prime labeling. We propose the following conjecture.

Conjecture 4.3. *All trees of diameter at least 3 are edge-prime.*

5. Semi-Edge-Prime Labeling

Definition 5.1. Let G be a (p,q)-graph. A bijection $f:E\to[1,q]$ is called a *semi-edge-prime labeling* if for each edge uv in E, we have $(f^+(u),f^+(v))=1$ or $f^+(u)=f^+(v)$. A graph that admits a semi-edge-prime labeling is called a *semi-edge-prime graph.*

We now give some semi-edge-prime graphs.

Theorem 5.2. *For any even $n\ge 2$, the double star $DS(n,n)$ is semi-edge-prime if $n+1$ is prime.*

[DOI: 10.7508/ijmsi.2017.2.001]

Proof. Keep all notation defined in the previous section. Label edge xy by $n+1$, edges xu_1 to xu_n by odd integers in $[1, 2n+1] \setminus \{n+1\}$ and edges yv_1 to yv_n by even integers in $[1, 2n+1]$, respectively. We have $f^+(x) = f^+(y) = (n+1)^2$. Since $n+1$ is prime, it is clear that $((n+1)^2, f^+(u_i)) = ((n+1)^2, f^+(v_i)) = 1$. Since $f^+(x) = f^+(y)$, $DS(n, n)$ is semi-edge-prime. □

Note that, if $n + 1 > 3$ is not prime, the above labeling is not edge-prime nor semi-edge-prime.
Let $C(n, n)$ be a bipartite graph with $V(C(n, n)) = \{x, y, z, w, u_i, v_i \mid 1 \le i \le n\}$ and $E(C(n, n)) = \{xz, yw, xu_i, yu_i, zv_i, wv_i \mid 1 \le i \le n\}$.

Theorem 5.3. *For even $n \ge 2$, the bipartite graph $C(n, n)$ is semi-edge-prime.*

Proof. Label the edges of $C(n, n)$ as follows:

(1) Label edges xz and yw by $n + 1$ and $3n + 2$, respectively.
(2) Label edges xu_1 to xu_n by odd integers in $[1, 2n + 1] \setminus \{n + 1\}$ in natural order.
(3) Label edges zv_1 to zv_n by even integers in $[1, 2n + 1]$ in natural order.
(4) Label edges yu_1 to yu_n by even integers in $[2n + 2, 4n + 2] \setminus \{3n + 2\}$ in reversed natural order.
(5) Label edges wv_1 to wv_n by odd integers in $[2n + 2, 4n + 2]$ in reversed natural order.

It is easy to verify that $f^+(x) = f^+(z) = (n + 1)^2$, $f^+(y) = f^+(w) = (n + 1) \times (3n + 2)$, and $f^+(u_i) = f^+(v_i) = 4n + 3$. By Lemma 3.1, $(f^+(x), f^+(u_i)) = (f^+(y), f^+(u_i)) = 1$. Hence, $C(n, n)$ is semi-edge-prime. □

Let $W_n = C_n \vee K_1$ be the wheel graph of order $n+1$ and $F_n = P_n \vee K_1$ be the fan graph of order $n + 1$.

Theorem 5.4. *The wheel graph W_n is semi-edge-prime.*

Proof. Let $V(W_n) = \{u, v_1, v_2, \dots, v_n\}$ and $E(W_n) = \{uv_i, v_iv_{i+1} \mid 1 \le i \le n\}$ $(v_{n+1} = v_1)$. Suppose n is even. Define an edge labeling f by

(1) $f(v_iv_{i+1}) = i + 1$ for odd i;
(2) $f(v_iv_{i+1}) = n + i$ for even i;
(3) $f(uv_i) = 2n - 2i + 1$ for $1 \le i \le n$.

Observe that $f^+(u) = n^2, f^+(v_1) = 4n + 1, f^+(v_i) = 3n + 1$ for $2 \le i \le n$. Clearly, $(3n + 1, 4n + 1) = 1$. By Lemma 3.1, $(n^2, 4n + 1) = (n^2, 3n + 1) = 1$.
Suppose n is odd. Define an edge labeling f by

(1) $f(v_iv_{i+1}) = i + 1$ for odd i;
(2) $f(v_iv_{i+1}) = n + i + 1$ for even i;
(3) $f(uv_i) = 2n - 2i + 1$ for $1 \le i \le n$.

Observe that $f^+(u) = n^2, f^+(v_i) = 3n - 2$. By Lemma 3.1, $(n^2, 3n - 2) = 1$. Hence, W_n is semi-edge-prime. □

[DOI: 10.7508/ijmsi.2017.2.001]

Theorem 5.5. *The fan graph F_n is semi-edge-prime.*

Proof. From the wheel graph W_n and its semi-edge-prime labeling, we delete the edge with the highest edge label to get a fan graph F_n. Observe that all vertex labels remain unchanged except that:

(1) for even n, we have $f^+(v_n) = n + 1, f^+(v_1) = 2n + 1$.
(2) for odd n, we have $f^+(v_{n-1}) = f^+(v_n) = n + 2$.

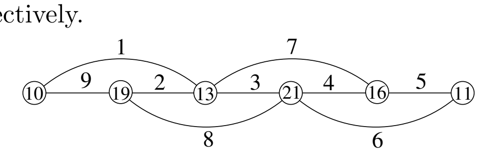
In both cases above, we can show that each pair of adjacent vertices have either identical or relatively prime labels. Hence, F_n is semi-edge-prime. □

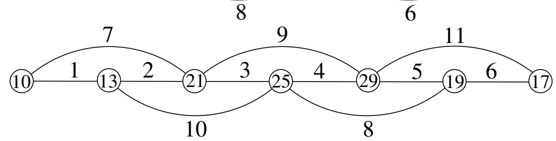
Let $P(k, n)$ be the graph obtained from a path $P_n = u_1 u_2 \cdots u_n$ by joining every two vertices of distant k by an edge. Clearly, $E(P(k, n)) = \{u_i u_{i+1}, u_i u_{i+k} | 1 \le i \le n, i + k \le n\}$.

Theorem 5.6. *The graph $P(2, n)$ is semi-edge-prime if $n \ge 6$.*

Proof. Define an edge labeling f by $f(u_i u_{i+1}) = i$ and $f(u_i u_{i+2}) = 2n-2-i$ for $1 \le i \le n$. It is easy to verify that $f^+(u_1) = 2n-2$, $f^+(u_2) = 2n-1 = f^+(u_n)$, $f^+(u_{n-1}) = 3n-2$, and $f^+(u_i) = 4n-3$ for $3 \le i \le n-2$. It is straight forward to show that every 2 adjacent vertex labels that are distinct are relatively prime. Hence, $P(2, n)$ is semi-edge-prime. □

Note that the above labelings give edge-prime labelings for $P(2, 4)$ and $P(2, 5)$, respectively, and the following labelings give edge-prime labeling for $P(2, 6)$ and $P(2, 7)$, respectively.





Conjecture 5.1. *For $n \ge 8$, $P(2, n)$ is edge-prime.*

Acknowledgments

The authors thank the referees for their valuable comments.

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[DOI: 10.7508/ijmsi.2017.2.001]