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Fractal Dimension of Graphs of Typical Continuous Functions on Manifolds

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ABSTRACT. If M is a compact Riemannian manifold and C(M,R) is the set of all real valued continuous functions defined on M, then we show that for a typical element $f \in C(M,R)$, $\overline{dim}_B(graph(f))$ is as big as possible and for a typical $f \in C(M,R)$, $\underline{dim}_B(graph(f))$ is as small as possible.

Keywords: Manifold, Fractal, Box dimension.

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1. Introduction

A subset A of a topological space X is called to be comeagre, if there is a countable collection $\{W_i\}$ of open and dense subsets of X such that $\bigcap_i W_i \subset A$. Complement of a comeagre subset is called a meagre subset. A meagre subset can be considered as subset of a countable union of nowhere dense subsets and they are negligible in some sense. So, we say that some property holds for typical elements of X, if it holds on a comeagre subset. Study of properties of typical elements in X is a classic and interesting problem. One can find many papers dealing with typical elements when X is supposed to be the space C(W,R) of all continuous functions defined on a compact topological space W, endowed with the metric topology defined by the metric $d(f,g) = sup_{x \in W} |f(x) - g(x)|$. A well known theorem due to Banach [1], states that typical elements of C([0,1],R) are nowhere differentiable, so the image or graph of a typical f in C([0,1],R) is a

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fractal set. Calculating fractal dimensions (including box dimension, Hausdorff dimension, packing dimension, etc) of the image of f or graph(f) is a well known problem and one can find many results in the literature. It is proved in [6] that for a typical $g \in C([0,1],R)$, $dim_H(graph(g)) = 1$. It is proved in [3] that if $W \subset R$ is bounded with only finitely many isolated points and $X = \{f \in C(W,R) : f \text{ is uniformly countinuous } \}$, then for a typical $f \in X$, $\overline{dim}_B(graph(f))$ is as big as possible and $\underline{dim}_B(graph(f))$ is as small as possible. In the previous paper [7] we generalized Banach's theorem to the set C(M,R), where M is a compact Riemannian manifold. Now, we show in the present paper that the main results of [3] about upper and lower box dimensions are also true when W is replaced by a compact Riemannian manifold M.

2. Preliminaries

In what follows, M is a compact Riemannian manifold with the Riemannian metric d, and C(M,R) will denote the collection of all continuous functions defined on M endowed with the metric d defined by $d(f,g) = \max_{x \in M} |f(x) - g(x)|$.

If (X, d_1) and (Y, d_2) are metric spaces then we will consider the usual product metric d on $X \times Y$ defined by $d((x_1, y_1), (x_2, y_2)) = \sqrt{d_1^2(x_1, x_2) + d_2^2(y_1, y_2)}$.

If E is a bounded subset of M then the upper box dimension of E is defined by

$$\overline{dim}_{B}(E) = limsup_{\delta \to 0} \frac{N_{\delta}(E)}{-log\delta}.$$

Where, $N_{\delta}(E)$ is the minimum number of balls of radius δ (or minimum number of sets of diameter at most δ) covering E (The lower box dimension $\underline{dim}_{B}(E)$ is defined in similar way). Another definition for dimension, which is widely used in fractal geometry is Hausdorff dimension (see [4]).

Now, we mention some facts which we need in the proofs of theorems.

Remark 2.1. If E is a bounded subset of R^m then $\overline{dim}_B(E \times I^n) = \overline{dim}_B(E) + n$. The similar result is true if we replace \overline{dim}_B by \underline{dim}_B or dim_H .

Proof. We give the proof for $\overline{dim}_B(E \times I) = \overline{dim}_B(E) + 1$. The general case comes by induction. If $\delta > 0$ then the smallest number of intervals of length δ covering I is equal to $\left[\frac{1}{\delta}\right]$ or $\left[\frac{1}{\delta}\right] + 1$. If U_{δ} (I_{δ}) is a bounded subset of R^m (I) with diameter δ , then the diameter of $U_{\delta} \times I_{\delta}$ is equal to $\sqrt{2}\delta$. So,

$$N_{\sqrt{2}\delta}(E \times I) \le ([\frac{1}{\delta}] + 1)N_{\delta}(E)$$

Then we have

$$\begin{split} \overline{\dim}_B(E\times I) &= limsup_{\delta\to 0} \frac{log(N_{\sqrt{2}\delta}(E\times I))}{-log(\sqrt{2}\delta)} \\ &\leq limsup_{\delta\to 0} \frac{log([\frac{1}{\delta}]+1)N_{\delta}(E))}{-log(\sqrt{2}\delta)} \\ &= 1 + limsup_{\delta\to 0} \frac{N_{\delta}(E)}{-log\delta} = 1 + \overline{dim}_B(E) \end{split}$$

Also we know that $\overline{dim}_B(E \times I^n) \ge \overline{dim}_B(E) + n$ (see [4]). So we get the equality.

Remark 2.2. If M is a compact metric space and $f: M \to R$ is a locally lipschitz function, then f is globally lipschitz.

Proof. Since f is locally lischitz and M is compact, then there is a finite collection of open cover of balls B_i , $1 \le i \le m$, and constants L_i such that

$$d(f(x), f(y)) \le L_i d(x, y), \quad x, y \in B_i$$

Since M is compact then the function $F: M \times M \to R$, defined by F(x,y) = d(f(x), f(y)) has a maximum which we denote it by N. Let δ be the lebesgue's number related to the covering B_i of M, and put $L = max\{\frac{N}{\delta}, L_i : i\}$. Then for given $x, y \in M$, either there is a B_i such that $x, y \in B_i$ or $d(x, y) \geq \delta$. In the first case we have $d(f(x), f(y)) \leq Ld(x, y)$. In the second case we have

$$d(f(x), f(y)) \le N \le \frac{N}{\delta} d(x, y) \le Ld(x, y)$$

If M and N are compact differentiable manifolds and $f: M \to N$ is continuously differentiable, then f is a lipschitz function. So, we get the following remark easily.

Remark 2.3. If M and N are compact Riemannian manifolds and $\phi: M \to N$ is a map such that ϕ and its inverse are continuously differentiable, then the map $\psi: M \times R \to N \times R$ defined by $\psi(x,y) = (\phi(x),y)$ is bilipschitz.

Remark 2.4. If M is a compact Riemannian manifold, $f: M \to R$ is continuously differentiable, $g: M \to R$ is continuous and k = f + g, then $\overline{dim}_B(graph(k)) = \overline{dim}_B(graph(g))$. The same result is true for \underline{dim}_B .

Proof. Consider the map $\psi: graph(g) \to graph(k)$, defined by $\psi(x, g(x)) = (x, k(x))$. We show that ψ and ψ^{-1} are Lipschitz functions. We have

$$d(\psi(x,g(x)),\psi(y,g(y))) = d((x,k(x)),(y,k(y))) = \sqrt{d^2(x,y) + (k(x) - k(y))^2}$$

Since f is continuously differentiable, it is locally Lischitz and by Remark 2.2, it must be Lischitz. Then, there exist a positive number N such that $|f(x) - f(y)| \le Nd(x, y), x, y \in M$. Thus

$$\begin{split} (k(x) - k(y))^2 &= (f(x) - f(y) + g(x) - g(y))^2 \le (Nd(x, y) + |g(x) - g(y)|)^2 \\ &= N^2 d^2(x, y) + 2Nd(x, y)|g(x) - g(y)| + |g(x) - g(y)|^2 \\ &\le N^2 d^2(x, y) + N^2 d^2(x, y) + |g(x) - g(y)|^2 + |g(x) - g(y)|^2 \\ &= 2N^2 d^2(x, y) + 2|g(x) - g(y)|^2 \end{split}$$

Then

$$\begin{split} &d(\psi(x,g(x)),\psi(y,g(y))) \leq \sqrt{d^2(x,y) + 2N^2d^2(x,y) + 2|g(x) - g(y)|^2} \\ \leq &\sqrt{2(N^2+1)}\sqrt{d^2(x,y) + (g(x) - g(y))^2} = &\sqrt{2(N^2+1)}d((x,g(x)),(y,g(y))). \end{split}$$

Therefore, ψ is Lipschitz. In a similar way we can show that ψ^{-1} is Lipschitz.

Remark 2.5. (generalized StoneWeierstrass Theorem). Suppose X is a compact Hausdorff space and A is a subalgebra of C(X, R) which contains a nonzero constant function. Then A is dense in C(X, R) if and only if it separates points.

3. Results

Lemma 3.1. If $f: M \to R$ is continuously differentiable and $\epsilon > 0$, then there exists $g \in C(M,R)$ such that $d(f,g) < \epsilon$ and $\overline{dim}_B(graph(g)) = n+1$, n = dimM.

Proof. Let N be a compact Riemannian manifold. Consider a function $g_1 \in C(I, \mathbb{R}^+)$ such that $\overline{dim}_B(graph(g_1)) = 2$ and put

$$q_2: I^n = I \times I^{n-1} \to R^+, \quad q_2(t_1, t_2) = q_1(t_1).$$

Then

$$graph(g_2) = \{((t_1, t_2), g_1(t_1)), (t_1, t_2) \in I \times I^{n-1}\} \simeq \{((t_1, g_1(t_1)), t_2), (t_1, t_2) \in I \times I^{n-1}\} = graph(g_1) \times I^{n-1}.$$

So, by Remark 2.1

$$\overline{dim}_B(graph(q_2)) = 2 + n - 1 = n + 1.$$

Consider a chart (U, ϕ) on N such that $I^n \subset \phi(U)$ and put $W = \phi^{-1}(I^n)$. Now, put $g_3 = g_2o\phi : W \to R$. By Remark 2.3, the function $\psi : W \times R \to I^n \times R$, defined by $\psi(x,y) = (\phi(x),y)$ is bilipschitz. Since $\psi(graph(g_3)) = graph(g_2)$, then $\overline{dim}_B(graph(g_3)) = n+1$. Extend the function g_3 to a continuous function $g_4 : N \to R$. Since $graph(g_3) \subset graph(g_4)$ then $\overline{dim}_B(graph(g_4)) = n+1$. Now put N = graph(f). We know that N is a submanifold of $M \times R$, which with the induced metric is a riemannian manifold. Given $\delta > 0$, the function $g_5 = \delta g_4 : N \to R$ is a positive function such that $\overline{dim}(graph(g_5)) = \overline{dim}(graph(g_4)) = \overline{dim}(grap$

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n+1. By compactness condition we can choose δ small enough such that for all $y = (x, f(x)) \in N$, $g_5(y) < \epsilon$.

Now, consider the function $g_6: M \to R$, defined by $g_6(x) = g_5(x, f(x))$ and put $\psi: M \times R \to N \times R$, $\psi(x,y) = ((x,f(x)),y)$. We have

$$\psi: graph(g_6) = graph(g_5)$$

By Remark 2.3, ψ is bilipshitze, so

$$\overline{dim}_B(graph(g_6)) = \overline{dim}_B(graph(g_5)) = n + 1$$

Put $g: M \to R$, $g(x) = f(x) + g_6(x)$. Since f is differentiable, then by Remark $2.4, \overline{dim}_B(graph(g)) = \overline{dim}_B(graph(g_6)) = n+1.$ Also, we have d(f,g) = $\max_{x \in M} |g(x) - f(x)| = \max_{x \in M} |g_6(x)| = \max_{x \in M} g_5(x, f(x)) < \epsilon.$

Theorem 3.2. Let M be a compact Riemannian manifold, dim(M) = n, and C(M,R) be the set of all continuous functions defined on M. Then for typical members f in C(M,R), $\underline{dim}_{B}(graph(f)) = n$.

Proof. Put

$$A = \{ f \in C(M, R) : dim_B(graph(f)) = n \}.$$

Let $f \in A$ and consider a positive number $\epsilon > 0$ and $g \in C(M, R)$ such that $d(f,g) < \epsilon$. If a collection of balls of radius δ in $M \times R$ covers graph(f) and $\epsilon < \delta$, then the same number of balls with radius 2δ covers graph(g). Since each ball of radius 2δ can be covered by 4^{n+1} balls of radius δ , then

$$N_{\delta}(graph(g)) \le 4^{n+1}N_{\delta}(graph(f))$$

If $\delta < 1$ then

$$\frac{logN_{\delta}(graph(g))}{-log(\delta)} \leq (n+1)\frac{lo4}{-log\delta} + \frac{logN_{\delta}(graph(f))}{-log\delta}$$

Since $\underline{dim}_B(graph(f)) = n$ and $\lim_{\delta \to 0} \frac{\log 4}{-\log \delta} = 0$, then for each $k \in N$ there exists $\delta = \delta(f, k) > 0$ such that

$$\frac{\log N_{\delta}(graph(g))}{-log(\delta)} \leq (n+1)\frac{\log 4}{-log\delta} + \frac{\log N_{\delta}(graph(f))}{-log\delta} < n + \frac{1}{k}$$

Put

$$U_{f,k} = \{ g \in C(M,R) : d(f,g) < \delta(f,k) \}$$

and

$$W_k = \bigcup_{(f \in A)} U_{f,k}$$

 $W_k = \bigcup_{(f \in A)} U_{f,k}$ $W_{f,k}$ is an open set in C(M,R) such that for each $g \in W_k,$

$$\underline{dim}_B(graph(g) < n + \frac{1}{k}.$$

Clearly $A \subset \bigcap_k W_k$. If $g \in \bigcap_k W_k$ then $\underline{dim}_B(graph(g)) \leq n$, and since for all $g \in C(M,R)$, $n \leq \underline{dim}_{B}(graph(g))$ then $\underline{dim}_{B}(graph(g)) = n$. Thus $\bigcap_k W_k = A$. Now, we show that W_k is dense for all k, then the proof will be complete. Given $g \in C(M,R)$ and $\epsilon > 0$. By Remark 2.5, collection of differentiable functions is dense, so there exists a differentiable function $f: M \to R$ such that $d(f,g) < \epsilon$. But for a differentiable function f, $\underline{dim}_B(graph(f)) = \overline{dim}_B(graph(f)) = n$. So $f \in A \subset W_k$.

Lemma 3.3. If $g \in C(M,R)$ and $\epsilon > 0$, then there exists $k \in C(M,R)$ such that $d(g,k) < \epsilon$ and $\overline{dim}_B(graph(k)) = n+1$.

Proof. By Remark 2.5, for a given $\delta > 0$ there exists a differentiable function $f \in C(M,R)$ such that $d(f,g) < \delta$. Consider a function $f_1 \in C(M,R)$ such that $\overline{dim}_B(graph(f_1)) = n+1$. Since M is compact, for a given number $\delta_2 > 0$ there is a positive number δ_1 such that $|\delta_1 f_1(x)| < \delta_2$ for all $x \in M$. Now, put $k = f + \delta_1 f_1$. By Remark 2.4, we have

$$\overline{dim}_B(graph(k)) = \overline{dim}_B(graph(\delta_1 f_1)) = \overline{dim}_B(graph(f_1)) = n + 1.$$

If we choose δ and δ_2 smaller than $\frac{\epsilon}{2}$, then

$$d(g,k) \le d(g,f) + d(f,k) \le \delta + \delta_1 ||f_1|| \le \delta + \delta_2 < \epsilon.$$

Theorem 3.4. Let M be a compact Riemannian manifold, dim(M) = n, and C(M,R) be the set of all continuous functions defined on M. Then for typical members f in C(M,R), $\overline{dim}_B(graph(f)) = n+1$.

Proof. Clearly for all $f \in C(M,R)$, $\overline{dim}_B(graph(f)) \leq n+1$. Put

$$A = \{ f \in C(M, R) : \overline{dim}_B(graph(f)) = n + 1 \}.$$

Consider $f \in A$, a positive number $\epsilon > 0$ and $g \in C(M,R)$ such that $d(f,g) < \epsilon$. If a collection of balls of radius δ in $M \times R$ covers graph(g) and $\epsilon < \delta$, then the same number of balls with radius 2δ covers graph(f). Since each ball of radius 2δ can be covered by 4^{n+1} balls of radius δ , then

$$N_{\delta}(graph(f)) < 4^{n+1}N_{\delta}(graph(g))$$

So, if $\delta < 1$ then

$$\frac{logN_{\delta}(graph(f))}{-log(\delta)} < (n+1)\frac{lo4}{-log\delta} + \frac{logN_{\delta}(graph(g))}{-log\delta}$$

Since $\overline{dim}_B(graph(f)) = n+1$, then for each $k \in N$ there is $\delta(k) = \delta(f,k) > 0$ such that

$$n+1-\frac{1}{k}<\frac{logN_{\delta(k)}(graph(f))}{-log(\delta(k))}-(n+1)\frac{log4}{-log\delta(k)}<\frac{logN_{\delta(k)}(graph(g))}{-log\delta(k)}$$

Put

$$U_{f,k} = \{ g \in C(M,R) : d(f,g) < \delta(f,k) \}$$

and

$$W_k = \bigcup_{(f \in A)} U_{f,k}$$

 W_k is an open set in C(M,R) such that for each $g \in W_k$,

$$\overline{dim}_B(graph(g) > n+1-\frac{1}{k}$$

Clearly

$$\bigcap_{k} W_{k} = A$$

Now it remains to show that W_k is dense for all k. Let $h \in C(M,R)$ and $\epsilon > 0$ we show that there exists $g \in W_k$ such that $d(h,g) < \epsilon$. Since by Remark 2.5, the collection of all differentiable functions is dense in C(M,R) then there exists a differentiable function $g_1 \in C(M,R)$ such that $d(h,g_1) < \frac{\epsilon}{2}$. Consider a function $f \in A \subset W_k$. Since f is continuous and M is compact then there exists $\delta > 0$ such that $|\delta f(x)| < \frac{\epsilon}{2}$ for all $x \in M$. Now, put $g = g_1 + \delta f$. Since g_1 is differentiable then $\overline{dim}_B(graph(g)) = \overline{dim}_B(graph\delta f) = \overline{dim}_B(graph(f)) = n+1$. So, $g \in A \subset W_k$ and we have

$$d(h,g) \le d(h,g_1) + d(g_1,g) \le \frac{\epsilon}{2} + \max_{x \in M} |\delta f| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

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