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# Atomic Systems in 2-inner Product Spaces

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ABSTRACT. In this paper, the concept of a family of local atoms in a 2-inner product space is introduced and then this concept is generalized to an atomic system for an operator. Next a characterization of atomic systems is proved. This characterization lead us to obtain a new frame which is a generalization of frames in 2-inner product spaces.

**Keywords:** 2-inner product space, 2-normed space, Family of local atoms, Atomic system, Frame.

# 2000 Mathematics subject classification: 42C15, 46C50.

### 1. Introduction and Preliminaries

Frames in Hilbert spaces were introduced by Duffin and Schaffer [9] in the context of nonharmonic Fourier series in 1952. In 1986, frames were brought to life by Daubechies *et al.* [7]. Now frames play an important role not only in the theoretics but also in many kinds of applications, and have been widely applied in signal processing [13], sampling [10, 11], coding and communications [19], filter bank theory [2], system modeling [8], and so on.

Atomic systems for bounded linear operators on Hilbert spaces have been introduced by L. Găvruţa in [15] as a generalization of families of local atoms

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- [12]. A sequence  $\{f_j\}_{j\in\mathbb{N}}$  in a Hilbert space  $\mathcal{H}$  is called an *atomic system* for a bounded linear operator K on  $\mathcal{H}$  if
- i) the series  $\sum_{j\in\mathbb{N}} c_j f_j$  converges for all  $c=(c_j)\in l^2:=\{\{b_j\}_{j\in\mathbb{N}}: \sum_{j\in\mathbb{N}} |b_j|^2<\infty\};$
- ii) there exists C > 0 such that for every  $f \in \mathcal{H}$  there exists  $a_f = (a_j) \in l^2$  such that  $||a_f||_{l^2} \leq C||f||$  and  $Kf = \sum_{j \in \mathbb{N}} a_j f_j$ .

It is proved that this concept is equivalent to K-frames, where K is a bounded linear operator on separable Hilbert space  $\mathcal{H}$  [15].

A sequence  $\{f_j\}_{j\in\mathbb{N}}$  is said to be a K-frame for  $\mathcal{H}$  if there exist constants A, B>0 such that

$$A\|K^*f\|^2 \le \sum_{j\in\mathbb{N}} |\langle f, f_j \rangle|^2 \le B\|f\|^2, \ \forall f \in \mathcal{H}.$$

We refer to [20] for more results on these concepts. In addition, the authors generalized these concepts and gave some new results in Hilbert modules [5] and Banach spaces [6]. Note that frames in Hilbert spaces are just a particular case of K-frames, when K is the identity operator on these Hilbert spaces.

The concepts of 2-inner product spaces and 2-normed spaces have been studied by many authors [3, 4, 14, 16, 17, 18]. In the sequel, we introduce 2-inner product and 2-normed spaces.

**Definition 1.1.** Suppose that X is a vector space of dimension grater than 1 over the field  $\mathbb{F}$  ( $either \mathbb{R} \ or \mathbb{C}$ ). If there exists a mapping  $\langle ., .|. \rangle : X \times X \times X \to \mathbb{F}$  with the properties

- 1.  $\langle f,f|h\rangle \geq 0$  and  $\langle f,f|h\rangle = 0$  if and only if f and h are linearly dependent:
- 2.  $\langle f, f | h \rangle = \langle h, h | f \rangle$ ;
- 3.  $\langle g, f | h \rangle = \overline{\langle f, g | h \rangle};$
- 4.  $\langle \alpha f, g | h \rangle = \alpha \langle f, g | h \rangle$  for  $\alpha \in \mathbb{F}$ ;
- 5.  $\langle f_1 + f_2, g | h \rangle = \langle f_1, g | h \rangle + \langle f_2, g | h \rangle$ ,

then the pair  $(X, \langle ., .|. \rangle)$  is called a 2-inner product space. The map  $\langle ., .|. \rangle$  is said to be a 2-inner product on X.

Some basic properties of 2-inner product  $\langle ., .|. \rangle$  can be immediately obtained as follows (see [3, 4]).

- $\langle 0, g | h \rangle = \langle f, 0 | h \rangle = \langle f, g | 0 \rangle = 0;$
- $\langle f, \alpha g | h \rangle = \overline{\alpha} \langle f, g | h \rangle$ ;
- $\langle f, g | \alpha h \rangle = |\alpha|^2 \langle f, g | h \rangle$ ;

for all  $f, g, h \in X$  and  $\alpha \in \mathbb{F}$ .

One of the most important properties of 2-inner product is the Cauchy-Schwarz inequality

$$|\langle f, g | h \rangle|^2 \le \langle f, f | h \rangle \langle g, g | h \rangle, \quad f, g, h \in X.$$

For a given 2-inner product space  $(X, \langle ., .|. \rangle)$  we can define a function  $\|., .\|$  on  $X \times X$  by

$$||f,h|| = \langle f,f|h\rangle^{\frac{1}{2}} \tag{1.1}$$

for all  $f, h \in X$ .

The above mentioned function satisfies the following conditions:

- a. ||f,h|| > 0 and ||f,h|| = 0 if and only if f and h are linearly dependent;
- b. ||f, h|| = ||h, f||;
- c.  $\|\alpha f, h\| = |\alpha| \|f, h\|, \alpha \in \mathbb{F};$
- d.  $||f_1 + f_2, h|| \le ||f_1, h|| + ||f_2, h||$ .

A 2-norm on a vector space X is a function  $\|.,.\|$  defined on  $X \times X$  satisfying the conditions (a) to (d) and  $(X,\|.,.\|)$  is called a linear 2-normed space. Whenever a 2-inner product space  $(X,\langle.,.|.\rangle)$  is given, we consider it as a linear 2-normed space  $(X,\|.,.\|)$  via the 2-norm defined by (1.1).

Let X be a 2-inner product space. A sequence  $\{f_j\}$  is called convergent if there exists  $f \in X$  such that  $\lim_{j\to\infty} \|f_j - f, h\| = 0$ , for all  $h \in X$ . Similarly, we can define a Cauchy sequence in X. Also, X is said to be a 2-Hilbert space if it is complete (see [18]).

Now we are ready to state the concept of a 2-frame which was introduced in [1]. A sequence  $\{f_j\}$  in a 2-Hilbert space  $(X, \langle ., .|. \rangle)$  is called a 2-frame associated to  $h \in X$  if there exist A, B > 0 such that

$$A||f,h||^2 \le \sum_{j} |\langle f, f_j | h \rangle|^2 \le B||f,h||^2, \forall f \in X.$$
 (1.2)

If the right side of (1.2) holds, then  $\{f_j\}$  is called a 2-Bessel sequence.

In this paper, we shall introduce 2-atomic systems as a generalization of families of local 2-atoms. A characterization of 2-atomic systems is given. This leads us to obtain a generalization of 2-frame.

## 2. Main Results

In this section we are going to define the concept of a family of local 2-atoms. Next we will generalize this concept to a 2-atomic system for a linear operator and then a generalization of 2-frames will be studied.

In the sequel we assumed that  $(X, \langle ., .|. \rangle)$  is a 2-Hilbert space,  $h \in X$  and  $\langle h \rangle$  is the subspace generated by h.

**Definition 2.1.** Let  $\{f_j\}$  be a 2-Bessel sequence in a 2-inner product space X,  $h \in X$  and Y be a closed subspace of X. We say that  $\{f_j\}$  is a family of local 2-atoms for Y associated to h if there exists a sequence of bilinear functionals  $\{c_j\}$  on  $X \times \langle h \rangle$  such that

i) 
$$\sum_{j} |c_{j}(f,h)|^{2} \leq C||f,h||^{2}$$
, for some  $C > 0$ ;  
ii)  $f = \sum_{j} c_{j}(f,h)f_{j}$ ,  
for all  $f \in Y$ .

Note that a map  $c_j: X \times \langle h \rangle \to \mathbb{F}$  is called a bilinear functional if the following conditions hold for every  $f, g \in X$  and  $\alpha \in \mathbb{F}$ .

- (i)  $c_j(\alpha f + g, h) = \alpha c_j(f, h) + c_j(g, h);$
- (ii)  $c_j(f, \alpha h) = \alpha c_j(f, h)$ .

In the following proposition, it is proved that every family of local 2-atoms is indeed a 2-frame sequence.

**Proposition 2.2.** Suppose that  $\{f_j\}$  is a family of local 2-atoms for Y, a closed subspace of 2-inner product space X, then  $\{f_j\}$  is a 2-frame for Y associated to h.

*Proof.* It is enough to show that  $\{f_j\}$  has a lower bound. Since  $\{f_j\}$  is a family of local 2-atoms, there exists a sequence of bilinear functionals  $\{c_j\}$  such that  $\sum_j |c_j(f,h)|^2 \leq C||f,h||^2, f \in Y$ , for some C > 0.

$$\begin{split} \|f,h\|^4 &= (\langle f,f|h\rangle)^2 \\ &= (\langle f,\sum_j c_j(f,h)f_j|h\rangle)^2 \\ &= (\sum_j \overline{c_j(f,h)}\langle f,f_j|h\rangle)^2 \\ &\leq \sum_j |c_j(f,h)|^2 \sum_j |\langle f,f_j|h\rangle|^2 \\ &\leq C\|f,h\|^2 \sum_j |\langle f,f_j|h\rangle|^2, \end{split}$$

it means that  $\frac{1}{C}||f,h||^2 \leq \sum_j |\langle f,f_j|h\rangle|^2$ .

Assume that  $(X, \langle ., ., ; . \rangle)$  is a 2-Hilbert space and  $h \in X$ . The algebraic complement of  $\langle h \rangle$  in X is denoted by  $M_h$ , i.e.  $\langle h \rangle \oplus M_h = X$ .

One may see that

$$\langle f, q \rangle_h = \langle f, q | h \rangle, \ f, q \in X.$$

defines a semi-inner product on X (see [1]). This semi-inner product induces the following inner product on the quotient space  $\frac{X}{\langle h \rangle}$  denoted by  $M_h$  as follows:

$$\langle f + \langle h \rangle, g + \langle h \rangle \rangle_h = \langle f, g \rangle_h, \ f, g \in X.$$

So  $M_h$  with respect to  $||f||_h := \sqrt{\langle f, f \rangle_h}, f \in M_h$ , is a normed space. The completion of the inner product space  $M_h$  is denoted by  $X_h$ .

With these notations, one can rewrite (1.2) as follows:

$$A||f||_h^2 \le \sum_j |\langle f, f_j \rangle_h|^2 \le B||f||_h^2, \ \forall f \in X_h.$$

Now we are going to generalize the concept of a family of local 2-atoms.

**Definition 2.3.** Let X be a 2-inner product space and fix  $h \in X$ . Let  $K_h$  be a bounded linear operator on the Hilbert space  $X_h$ . A sequence  $\{f_j\} \subseteq X$  is

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called a 2-atomic system for  $K_h$  associated to h if

- i)  $\{f_i\}$  is a 2-Bessel sequence;
- ii) for any  $f \in X_h$  there exists  $a_f = \{a_j\} \in \ell^2$  such that  $K_h f = \sum_j a_j f_j$ , where  $\|a_f\|_{\ell^2} \leq C\|f, h\|_X$  and C is a positive constant.

Note that the convergence of the series  $\sum_j a_j f_j$  is in the topology of X. Also if  $\{f_j\} \subseteq X_h$  then the convergence of the series  $\sum_j a_j f_j$  is in the topology of X implies its convergence in  $X_h$ .

A characterization of a 2-atomic system corresponding to  $h \in X$  is given as follows which lead us to obtain a generalization of 2-frame.

**Theorem 2.4.** Let  $K_h$  be a bounded linear operator on  $X_h$ . Then for a sequence  $\{f_j\} \subseteq X_h$  the following statements are equivalent:

- (i)  $\{f_i\}$  is a 2-atomic system for  $K_h$ ;
- (ii) there exist A, B > 0 such that

$$A\|K_h^*f\|_h^2 \leq \sum_j |\langle f, f_j|h\rangle|^2 \leq B\|f\|_h^2, \forall f \in X_h;$$

(iii)  $\{f_j\} \subseteq X_h$  is a 2-Bessel sequence and there exists a 2-Bessel sequence  $\{g_j\}$  such that

$$K_h f = \sum_j \langle f, g_j | h \rangle f_j, f \in X_h;$$

(iv)  $\{f_j\} \subseteq X_h$  is a 2-Bessel sequence and there exists a 2-Bessel sequence  $\{g_j\}$  such that

$$K_h^* f = \sum_j \langle f, f_j | h \rangle g_j, f \in X_h;$$

(v)  $\{Q_h f_j\}$  is a 2-atomic system for the bounded linear operator  $Q_h K_h$ , where  $Q_h$  is an injective operator on  $X_h$ .

*Proof.*  $i \to ii$ ) For every  $f \in X_h$  we have

$$\begin{split} \|K_h^*f\|^2 &= \|K_h^*f, h\|^2 \\ &= \sup\{|\langle K_h^*f, g|h\rangle|^2 : g \in X_h, \|g, h\| = 1\} \\ &= \sup\{|\langle f, K_h g|h\rangle|^2 : g \in X_h, \|g, h\| = 1\}. \end{split}$$

By definition of a 2-atomic system for  $K_h$ , there exists C>0 such that  $K_hg=\sum_j b_j f_j$  with  $\|b_g\|_{\ell^2}=\|\{b_j\}\|_{\ell^2}\leq C\|g,h\|$  and so

$$\begin{split} \|K_h^*f\|^2 &= \sup\{|\langle f, \sum_j b_j f_j | h \rangle|^2 : g \in X_h, \|g, h\| = 1\} \\ &= \sup\{|\sum_j \overline{b_j} \langle f, f_j | h \rangle|^2 : g \in X_h, \|g, h\| = 1\} \\ &\leq \sup\{\sum_j |b_j|^2 \sum_j |\langle f, f_j | h \rangle|^2 : g \in X_h, \|g, h\| = 1\} \\ &\leq C^2 \|g, h\|^2 \sum_j |\langle f, f_j | h \rangle|^2 \\ &= C^2 \sum_j |\langle f, f_j | h \rangle|^2. \end{split}$$

It means that  $\frac{1}{C^2} ||K_h^* f||^2 \le \sum_j |\langle f, f_j | h \rangle|^2$ .

 $ii \to iii$ ) Similar to Theorem 3 of [15], there exists a 2-Bessel sequence  $\{g_j\} \in X_h$  such that

$$K_h f = \sum_j \langle f, g_j \rangle_h f_j = \sum_j \langle f, g_j | h \rangle f_j.$$

 $iii \rightarrow iv$ ) For  $f, g \in X_h$  we have

$$\langle K_h f, g \rangle_h = \langle \sum_j \langle f, g_j | h \rangle f_j, g \rangle_h$$

$$= \sum_j \langle f, g_j | h \rangle \langle f_j, g | h \rangle$$

$$= \sum_j \langle f, g_j \rangle_h \langle f_j, g \rangle_h$$

$$= \langle f, \sum_j \langle g, f_j | h \rangle g_j \rangle_h,$$

that is  $K_h^* f = \sum_j \langle f, f_j | h \rangle g_j$ .

 $iv \rightarrow iii$ ) It is similar to  $iii \rightarrow iv$  so we omit it.

 $i \to v$ ) Since  $\{f_j\}$  is a 2-atomic system for  $K_h$ , for any  $f \in X_h$  there exists  $a_f = \{a_j\} \in \ell^2$  such that  $K_h f = \sum_j a_j f_j$  so  $Q_h K_h f = \sum_j a_j Q_h f_j$ , i.e.  $\{Q_h f_j\}$  is a 2-atomic system for  $Q_h K_h$ .

 $v \to i$ ) Since  $\{Q_h f_j\}$  is a 2-atomic system for  $Q_h K_h$ , for any  $f \in X_h$  there exists  $\{b_j\} \in \ell^2$  such that  $Q_h K_h f = \sum_j b_j Q_h f_j$  so  $Q_h (K_h f - \sum_j b_j f_j) = 0$ . Due to injectivity of  $Q_h$ ,  $K_h f = \sum_j b_j f_j$ .

As a result of Theorem 2.4 the following definition is given.

**Definition 2.5.** Let  $K_h$  be a bounded linear operator on  $X_h$ . A sequence  $\{f_j\}$  in X is called 2-K-frame if there exist A, B > such that

$$A\|K_h^*f\|_h^2 \le \sum_j |\langle f, f_j | h \rangle|^2 \le B\|f\|_h^2, \forall f \in X_h.$$

Trivially a 2-frame, which was defined in [1], is a special case of 2-K-frames with  $K_h = I$ .

A consequence of Theorem 2.4 is given as follows.

**Theorem 2.6.** Let  $P_{Y_h}$  be the orthogonal projection on  $Y_h$  as a closed subspace of  $X_h$ . Then for a sequence  $\{f_j\} \subseteq X_h$  the following statements are equivalent:

- (i)  $\{f_i\}$  is a family of local 2-atoms for  $Y_h$ ;
- (ii)  $\{f_i\}$  is a 2-atomic system for  $P_{Y_h}$ ;
- (iii)  $\{f_j\}$  is a 2- $P_{Y_h}$ -frame;
- (iv)  $\{f_j\}$  is a 2-Bessel sequence and there exists a 2-Bessel sequence  $\{g_j\}$  such that

$$P_{Y_h}f = \sum_{j} \langle f, g_j | h \rangle f_j = \sum_{j} \langle f, f_j | h \rangle g_j, f \in X_h;$$

(v)  $\{Q_h f_j\}$  is a 2-atomic system for bounded linear operator  $Q_h P_{Y_h}$ , where  $Q_h$  is an injective operator on  $X_h$ .

*Proof.*  $i \rightarrow ii$  is obvious.

 $ii \longleftrightarrow iii, iii \longleftrightarrow iv, iv \longleftrightarrow ii \text{ and } v \longleftrightarrow i \text{ hold from Theorem 2.4.}$   $iv \to i)$  Since  $P_{Y_h}f = \sum_j \langle f, g_j | h \rangle f_j$ , it is enough to put  $c_j(f,h) = \langle f, g_j | h \rangle$  because it is linear and

$$\sum_{j} |c_{j}(f,h)|^{2} = \sum_{j} |\langle f, g_{j} | h \rangle|^{2} \le D \|f, h\|^{2},$$

where D is the upper 2-frame bound of  $\{g_i\}$ .

Example 2.7. Let  $n \in \mathbb{N}$  be odd and consider  $X = \mathbb{R}^n$  with the following standard two inner product

$$\langle x, y | z \rangle = \det \left( \begin{array}{cc} \langle x, y \rangle & \langle x, z \rangle \\ \langle z, y \rangle & \langle z, z \rangle \end{array} \right)$$

where  $\langle .,. \rangle$  is the inner product of  $\mathbb{R}^n$ . Let  $\{e_1,...,e_n\}$  be the standard basis of  $\mathbb{R}^n$  and  $h=e_n$ . Trivially in this case  $X_h=\mathbb{R}^{n-1}$  and one can see that its induced inner product is the standard inner product of  $\mathbb{R}^{n-1}$ . Now define the operator  $K_h$  on  $X_h$  by

$$K_h(e_{2i}) = e_i, i = 1, 2, ..., \frac{n-1}{2}$$
 and otherwise  $K_h e_i = e_i, i \le n-1$ .

Then one can see that  $e_1, e_1, e_3, e_2, ..., e_{\frac{n-1}{2}}, e_{n-1}$  is a 2- $K_h$ -frame.

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