# Generalized Approximate Amenability of Direct Sum of Banach Algebras

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ABSTRACT. In the present paper for two $\mathfrak{A}$-module Banach algebras $A$ and $B$, we investigate relations between $\varphi$-$\mathfrak{A}$-module approximate amenability of $A$, $\psi$-$\mathfrak{A}$-module approximate amenability of $B$, and $\varphi \oplus \psi$-$\mathfrak{A}$-module approximate amenability of $A \oplus B$ ($l^1$-direct sum of $A$ and $B$), where $\varphi \in \mathrm{Hom}_{\mathfrak{A}}(A)$ and $\psi \in \mathrm{Hom}_{\mathfrak{A}}(B)$.

**Keywords:** Banach algebra, Module derivation, Module approximate amenability.

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## 1. INTRODUCTION

The notion of approximate amenable Banach algebras was introduced and extensively studied by Ghahramani and Loy in [5]. They showed in [6] that if A and B are approximately amenable Banach algebras and one of A or B has a bounded approximate identity, then $A \oplus B$ is approximately amenable, but in general the direct sum of two approximately amenable Banach algebras need not be approximately amenable (see [7]).

The concept of module amenable Banach algebras was introduced by Amini in [1], and the notion of module approximate amenable Banach algebras was studied by Pourmahmood and Bodaghi in [15]. Recently, some authors have

studied $\varphi$-derivations, and $\varphi$-amenability of Banach algebra $A$, whenever $\varphi$ is a continuous homomorphism on $A$ (see [8, 9, 10, 11, 12]).

The aim of the present paper is to investigate generalized approximate amenability of $A \oplus B$.

The organization of this paper is as follows:

Section 2 is devoted to the notations and definitions which are needed throughout the paper.

In section 3 for $\mathfrak{A}$-module Banach algebras $A$ and $B$ where each has a bounded approximate identity we show that $A$ is $\varphi$-$\mathfrak{A}^{\#}$-module approximately amenable and $B$ is $\psi$-$\mathfrak{A}^{\#}$-module approximately amenable if and only if $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}^{\#}$-module approximately amenable.

In section 4 we show that if $\mathfrak{A}$ has a bounded approximately identity and $\frac{A}{J_{A,\mathfrak{A}}}$ and $\frac{B}{J_{B,\mathfrak{A}}}$ are unital, then $A$ is $\varphi$-$\mathfrak{A}$-module approximately amenable and $B$ is $\psi$-$\mathfrak{A}$-module approximately amenable if and only if $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable.

## 2. Preliminaries

Let $\mathfrak{A}$ and $A$ be Banach algebras such that $A$ is a Banach $\mathfrak{A}$-bimodule with compatible actions given by

$$\alpha.(ab) = (\alpha.a)b, \quad (ab).\alpha = a(b.\alpha) \qquad (a, b \in A, \alpha \in \mathfrak{A}).$$

Let $X$ be a Banach $A$-bimodule and a Banach $\mathfrak{A}$-bimodule with compatible left actions defined by

$$\alpha.(a.x) = (\alpha.a).x, \ a.(\alpha.x) = (a.\alpha).x, \ (\alpha.x).a = \alpha.(x.a)$$

$$(a \in A, \alpha \in \mathfrak{A}, x \in X), \tag{2.1}$$

and similar for the right or two-sided actions. Then we say that $X$ is a Banach $A$-$\mathfrak{A}$-module. A Banach $A$-$\mathfrak{A}$-module $X$ is called commutative $A$-$\mathfrak{A}$-module, if $\alpha.x = x.\alpha$ ($\alpha \in \mathfrak{A}, x \in X$). Note that in general, $A$ dose not satisfy the compatibility condition $a.(\alpha.b) = (a.\alpha).b$ ($a, b \in A, \alpha \in \mathfrak{A}$).

If $X$ is a commutative Banach $A$-$\mathfrak{A}$-module , then so is $X^*$, where the actions of $A$ and $\mathfrak{A}$ on $X^*$ are defined as follows

$$\langle \alpha.f, x \rangle = \langle f, x.\alpha \rangle, \ \langle a.f, x \rangle = \langle f, x.a \rangle \ (a \in A, \alpha \in \mathfrak{A}, x \in X, f \in X^*),$$

and similar for the right actions.

Let $A$ and $B$ be Banach $\mathfrak{A}$-bimodules. Then a $\mathfrak{A}$-module morphism from $A$ to $B$ is a norm continuous map $h : A \longrightarrow B$ with $h(a \pm b) = h(a) \pm h(b)$ which is multiplicative, that is

$$h(\alpha.a) = \alpha.h(a), \ \ h(a.\alpha) = h(a).\alpha, \ \ h(ab) = h(a)h(b) \ \ (a \in A, b \in B, \alpha \in \mathfrak{A}).$$

We denote by $\text{Hom}_{\mathfrak{A}}(A, B)$, the space of all such morphism and denote $\text{Hom}_{\mathfrak{A}}(A, A)$ by $\text{Hom}_{\mathfrak{A}}(A)$. In the case that $\mathfrak{A} = \mathbb{C}$, we denote $\text{Hom}_{\mathbb{C}}(A, B)$ by $\text{Hom}(A, B)$ and denote $\text{Hom}_{\mathbb{C}}(A, A)$ by $\text{Hom}(A)$.

Let $X$ be a Banach $A$-bimodule and let $\varphi \in \text{Hom}_{\mathfrak{A}}(A)$. A bounded map $D : A \longrightarrow X$ is called a $\varphi$-$\mathfrak{A}$-module derivation if

$$D(a \pm b) = D(a) \pm D(b), \ D(ab) = D(a).\varphi(b) + \varphi(a).D(b) \ (a, b \in A), \quad (2.2)$$

and

$$D(\alpha.a) = \alpha.D(a), \ D(a.\alpha) = D(a).\alpha \ (a \in A, \alpha \in \mathfrak{A}). \quad\quad (2.3)$$

Although $D$ in general is not linear, but still its boundedness implies its norm continuity.

Let $X$ be a commutative Banach $A$-$\mathfrak{A}$-module. For every $x \in X$ define $ad_x^{\varphi}$ by $ad_x^{\varphi}(a) = \varphi(a).x - x.\varphi(a) \ (a \in A)$. It is easily seen that $ad_x^{\varphi}$ is a $\varphi$-$\mathfrak{A}$-module derivation. A $\varphi$-$\mathfrak{A}$-module derivation $D$ is called $\varphi$-inner if there is $x \in X$ such that $D(a) = ad_x^{\varphi}(a) \ (a \in A)$ and is called approximately $\varphi$-inner if there exists a net $(x_\alpha)_\alpha \subseteq X$ such that $D(a) = \lim_\alpha ad_{x_\alpha}^{\varphi}(a) \ (a \in A)$. A Banach algebra $A$ is called $\varphi$-$\mathfrak{A}$-module amenable if for any commutative Banach $A$-$\mathfrak{A}$-module $X$, each $\varphi$-$\mathfrak{A}$-module derivation $D : A \longrightarrow X^*$ is $\varphi$-inner, and $A$ is called $\varphi$-$\mathfrak{A}$-module approximately amenable if each $\varphi$-$\mathfrak{A}$-module derivation $D : A \longrightarrow X^*$ is approximately $\varphi$-inner (see [1, 15]).

In the case that $\mathfrak{A} = \mathbb{C}$, $\varphi$-$\mathfrak{A}$-module derivations (resp. $\varphi$-$\mathfrak{A}$-module amenable Banach algebras, $\varphi$-$\mathfrak{A}$-module approximately amenable Banach algebras) are called $\varphi$-derivation (resp. $\varphi$-amenable, $\varphi$-approximately amenable) (see [9, 10]).

## 3.  $\varphi \oplus \psi$-Module Approximate Amenability of the Direct Sum of Banach Algebras

We commence this section with the following remark from [1]:

*Remark* 3.1. Assume that $A$ has a bounded approximate identity $(e_\alpha)_\alpha$, and let $M_{\mathfrak{A}}(A)$ denotes the algebra of $\mathfrak{A}$-multipliers of $A$, that is $M_{\mathfrak{A}}(A) = \{(T_1, T_2) : T_1, T_2 \in L_{\mathfrak{A}}(A) : T_1(ab) = T_1(a)b, T_2(ab) = aT_2(b)(a, b \in A)\}$, where $L_{\mathfrak{A}}(A)$ is the space of all $\mathfrak{A}$-module morphisms on $A$. Then $M_{\mathfrak{A}}(A)$ is an $A$-$\mathfrak{A}$-module and $A$ embeds in $M_{\mathfrak{A}}(A)$ via $a \longmapsto (L_a, R_a)$, where $L_a(b) = ab, R_a(b) = ba \ \ (a, b \in A)$. For any element $T = (T_1, T_2)$ of $M_{\mathfrak{A}}(A)$ it is easy to see that $\| T_1 \| = \| T_2 \|$ and if we put $\| T \|$ equal to this common value, then $M_{\mathfrak{A}}(A)$ becomes a Banach $A$-$\mathfrak{A}$-module, and $A$ is dense in $M_{\mathfrak{A}}(A)$ in the strict topology.

Before proving our next proposition we note that if $\varphi \in \text{Hom}_{\mathfrak{A}}(A)$, then by continuity of $\varphi$ in the strict topology, it can be extended to an $\mathfrak{A}$-homomorphism $\tilde{\varphi} : M_{\mathfrak{A}}(A) \longrightarrow M_{\mathfrak{A}}(A)$ defined by $\tilde{\varphi}(L_a, R_a) = (L_{\varphi(a)}, R_{\varphi(a)})$.

**Proposition 3.2.** *Let $A$ be an $\mathfrak{A}$-module Banach algebra with a bounded approximate identity $(e_\alpha)_\alpha$, and let $\varphi \in \mathrm{Hom}_{\mathfrak{A}}(A)$. Then $A$ is $\varphi$-$\mathfrak{A}$-module approximately amenable if and only if $M_{\mathfrak{A}}(A)$ is $\tilde{\varphi}$-$\mathfrak{A}$-module approximately amenable.*

*Proof.* Let $M_{\mathfrak{A}}(A)$ be $\tilde{\varphi}$-$\mathfrak{A}$-module approximately amenable and let $D : A \longrightarrow X^*$ be a $\varphi$-$\mathfrak{A}$-module derivation for some commutative Banach $A$-$\mathfrak{A}$-module $X$. Then by the following actions

$$T.x = \lim_\alpha T_1(e_\alpha).x, \;\; x.T = \lim_\alpha x.T_2(e_\alpha) \; (x \in X, T = (T_1, T_2) \in M_{\mathfrak{A}}(A)),$$

$X$ is a commutative Banach $M_{\mathfrak{A}}(A)$-$\mathfrak{A}$-module and by continuity of $D$ in the strict topology, it can be extended to a bounded $\tilde{\varphi}$-$\mathfrak{A}$-derivation $\tilde{D} : M_{\mathfrak{A}}(A) \longrightarrow X^*$, defined by $\tilde{D}(L_a, R_a) = D(a)$. From the $\tilde{\varphi}$-$\mathfrak{A}$-module approximate amenability of $M_{\mathfrak{A}}(A)$, it follows that there exists a net $(x_\beta^*)_\beta \subset X^*$ such that

$$\tilde{D}(T) = \lim_\beta \left( \tilde{\varphi}(T).x_\beta^* - x_\beta^*.\tilde{\varphi}(T) \right).$$

Hence for every $a \in A$ we have

$$\begin{aligned}
D(a) = \tilde{D}(L_a, R_a) &= \lim_\beta \left( \tilde{\varphi}(L_a, R_a).x_\beta^* - x_\beta^*.\tilde{\varphi}(L_a, R_a) \right) \\
&= \lim_\beta \left( (L_{\varphi(a)}, R_{\varphi(a)}).x_\beta^* - x_\beta^*.(L_{\varphi(a)}, R_{\varphi(a)}) \right) \\
&= \lim_\beta \left( \lim_\alpha L_{\varphi(a)}(e_\alpha).x_\beta^* - \lim_\alpha x_\beta^*.R_{\varphi(a)}(e_\alpha) \right) \\
&= \lim_\beta \left( \varphi(a).x_\beta^* - x_\beta^*.\varphi(a) \right).
\end{aligned}$$

This means that $D$ is approximately $\varphi$-inner and so $A$ is $\varphi$-$\mathfrak{A}$-module approximately amenable.

Conversely, Suppose that $A$ is $\varphi$-$\mathfrak{A}$-module approximately amenable. Let $X$ be a commutative Banach $M_{\mathfrak{A}}(A)$-$\mathfrak{A}$-module and let $D : M_{\mathfrak{A}}(A) \longrightarrow X^*$ be a $\tilde{\varphi}$-$\mathfrak{A}$-module derivation. We consider the module actions of $A$ on $X$ by

$$a.x = (L_a, R_a).x, \;\; x.a = x.(L_a, R_a) \; (a \in A, x \in X). \tag{3.1}$$

Thus $X$ is a commutative Banach $A$-$\mathfrak{A}$-module. Define $\tilde{D} : A \longrightarrow X^*$ by $\tilde{D}(a) = D(L_a, R_a) \; (a \in A)$. It is easy to see that $\tilde{D}$ is a $\varphi$-$\mathfrak{A}$-module derivation and from the $\varphi$-$\mathfrak{A}$-module approximate amenability of $A$, it follows that there exists a net $(x_\beta^*)_\beta \subset X^*$ such that

$$\tilde{D}(a) = \lim_\beta \left( \varphi(a).x_\beta^* - x_\beta^*.\varphi(a) \right) \; (a \in A).$$

Then $D(L_a, R_a) = \lim_\beta \left( \tilde{\varphi}(L_a, R_a).x_\beta^* - x_\beta^*.\tilde{\varphi}(L_a, R_a) \right)$. Now by the continuity of $D$ and $\tilde{\varphi}$, and density of A in $M_{\mathfrak{A}}(A)$ in the strict topology, we conclude that

$$D(T) = \lim_\beta \left( \tilde{\varphi}(T).x_\beta^* - x_\beta^*.\tilde{\varphi}(T) \right) \; (T \in M_{\mathfrak{A}}(A)).$$

So $D$ is a approximately $\tilde{\varphi}$-inner. Therefore $M_{\mathfrak{A}}(A)$ is $\tilde{\varphi}$-$\mathfrak{A}$-module approximately amenable. $\qquad \square$

Let $I$ be a closed ideal of a Banach algebra $A$ with a bounded approximate identity $(e_\alpha)_\alpha$, and let $X$ be a commutative Banach $I$-$\mathfrak{A}$-module. Let $\varphi \in \text{Hom}_{\mathfrak{A}}(A)$ be such that $\varphi \mid_I \subset I$, then $X$ is a commutative Banach $A$-$\mathfrak{A}$-module with the following actions

$$a.x = \lim_\alpha \varphi(e_\alpha)a.x, \quad x.a = \lim_\alpha x.\varphi(e_\alpha)a \quad (a \in A, x \in X). \quad (3.2)$$

**Proposition 3.3.** *Let $I$ be a closed ideal of an $\mathfrak{A}$-module Banach algebra $A$ which has a bounded approximate identity $\{e_\alpha\}$, and let $I$ be $\mathfrak{A}$-invariant, i.e. $\mathfrak{A}.I \subseteq I$. Let $\varphi \in \text{Hom}_{\mathfrak{A}}(A)$ be such that $\varphi \mid_I \subset I$. If $A$ is $\varphi$-$\mathfrak{A}$-module approximately amenable, then $I$ is $\varphi \mid_I$-$\mathfrak{A}$-module approximately amenable.*

*Proof.* Let $X$ be a commutative Banach $M_{\mathfrak{A}}(I)$-$\mathfrak{A}$-module, and $D : M_{\mathfrak{A}}(I) \longrightarrow X^*$ be a $\tilde{\varphi}$-$\mathfrak{A}$-module derivation. By the same actions as (3.1), we can consider $X$ as a commutative Banach $I$-$\mathfrak{A}$-module. So, by (3.2), $X$ is a commutative Banach $A$-$\mathfrak{A}$-module. By definition of $M_{\mathfrak{A}}(I)$, there is an $\mathfrak{A}$-module morphism $h : A \longrightarrow M_{\mathfrak{A}}(I)$ and $D \circ h$ is a module derivation on $A$, so it is approximately $\varphi$-inner. Hence $D$ is approximately $\tilde{\varphi}$-inner. Since $I$ has a bounded approximate identity, by Proposition 3.2, $I$ is $\varphi \mid_I$-$\mathfrak{A}$-module approximately amenable. $\square$

Let $A$ and $B$ be $\mathfrak{A}$-module Banach algebras. It is well known that $A \oplus B$, the $l^1$-direct sum of $A$ and $B$, is a Banach algebra with respect to the canonical multiplication defined by $(a,b)(c,d) := (ac, bd)$, and is a Banach $\mathfrak{A}$-bimodule by the following actions

$$\alpha.(a,b) := (\alpha.a, \alpha.b), \quad (a,b).\alpha := (a.\alpha, b.\alpha) \quad (\alpha \in \mathfrak{A}, a \in A, b \in B).$$

We note that if $\varphi \in \text{Hom}_{\mathfrak{A}}(A)$ and $\psi \in \text{Hom}_{\mathfrak{A}}(B)$, then $\varphi \oplus \psi : A \oplus B \longrightarrow A \oplus B$ defined by $\varphi \oplus \psi(a,b) = \big(\varphi(a), \psi(b)\big)$ is an $\mathfrak{A}$-morphism on $A \oplus B$.

**Lemma 3.4.** *Let $A$ be a unital $\mathfrak{A}$-module Banach algebra, $\varphi \in \text{Hom}_{\mathfrak{A}}(A)$, and let $D : A \longrightarrow X^*$ be a $\varphi$-$\mathfrak{A}$-module derivation for some commutative Banach $A$-$\mathfrak{A}$-module $X$. If the left (resp. right, two-sided) action of $\varphi(A)$ on $X^*$ is zero, then $D$ is $\varphi$-inner.*

*Proof.* Let $e_A$ be the identity of $A$ and let the left (resp. right, two-sided) action of $\varphi(A)$ on $X^*$ is zero. We can easily show that $D = ad^\varphi_{-D(e)}$ (resp. $D = ad^\varphi_{D(e)}, D = 0$). So $D$ is $\varphi$-inner. $\square$

The proof of the following proposition is adopted from that of Proposition 2.7 of [5].

**Proposition 3.5.** *Let $A$ and $B$ be unital $\mathfrak{A}$-module Banach algebras with identities $e_A$ and $e_B$, respectively, and let $\varphi \in \text{Hom}_{\mathfrak{A}}(A)$ and $\psi \in \text{Hom}_{\mathfrak{A}}(B)$ such that $\varphi(e_A).\alpha = \alpha.\varphi(e_A)$, and $\psi(e_B).\alpha = \alpha.\psi(e_B)$ ($\alpha \in \mathfrak{A}$). If $A$ is $\varphi$-$\mathfrak{A}$-module approximately amenable and $B$ is $\psi$-$\mathfrak{A}$-module approximately amenable, then $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable.*

*Proof.* Let $X$ be a commutative Banach $A \oplus B$-$\mathfrak{A}$-module and let $D : A \oplus B \longrightarrow X^*$ be a $\varphi \oplus \psi$-$\mathfrak{A}$-module derivation. Write $Y_1 = \varphi(e_A).X^*.\varphi(e_A)$, $Y_2 = \psi(e_B).X^*.\psi(e_B)$, $Y_3 = \varphi(e_A).X^*.\psi(e_B)$, $Y_4 = \psi(e_B).X^*.\varphi(e_A)$, $Y_5 = \big(1 - \varphi(e_A)\big)\big(1 - \psi(e_B)\big).X^*.\varphi(e_A)$, $Y_6 = \big(1 - \varphi(e_A)\big)\big(1 - \psi(e_B)\big).X^*.\psi(e_B)$ , $Y_7 = \varphi(e_A).X^*.\big(1-\varphi(e_A)\big)\big(1-\psi(e_B)\big)$, $Y_8 = \psi(e_B).X^*.\big(1-\varphi(e_A)\big)\big(1-\psi(e_B)\big)$, $Y_9 = \big(1 - \varphi(e_A)\big)\big(1 - \psi(e_B)\big).X^*.\big(1 - \varphi(e_A)\big)\big(1 - \psi(e_B)\big)$ and let $\pi_j : X^* \longrightarrow Y_j$ be the associated projections. Thus $X^* = Y_1 \oplus Y_2 \oplus Y_3 \oplus Y_4 \oplus Y_5 \oplus Y_6 \oplus Y_7 \oplus Y_8 \oplus Y_9$. Consider the derivations $D_j = \pi_j \circ D$, so $D = D_1 + D_2 + D_3 + D_4 + D_5 + D_6 + D_7 + D_8 + D_9$. From the fact that $\varphi(e_A).\alpha = \alpha.\varphi(e_A)$ ($\alpha \in \mathfrak{A}$), and $\psi(e_B).\alpha = \alpha.\psi(e_B)$ ($\alpha \in \mathfrak{A}$), one can easily check that $Y_j$ for $j = 1, ..., 9$ is a commutative Banach $A \oplus B$-$\mathfrak{A}$-module. Since the action of $\varphi(A) \oplus \psi(B)$ on (at least) one side on $Y_5$ (resp. $Y_6, Y_7, Y_8, Y_9$) is zero, by Lemma 3.4, we conclude that $D_5$ (resp. $D_6, D_7, D_8, D_9$ ) is approximately $\varphi \oplus \psi$-inner.

From the $\varphi$-$\mathfrak{A}$-module approximate amenability of $A$, it follows that the $\varphi \oplus \psi$-$\mathfrak{A}$-module derivation $A \oplus 0 \longrightarrow \varphi(e_A).X^*.\varphi(e_A)$ is approximately $\varphi \oplus \psi$-inner and since the action of $0 \oplus \psi(B)$ on $\varphi(e_A).X^*.\varphi(e_A)$ is zero, we conclude that $D_1$ is approximately $\varphi \oplus \psi$-inner. Similarly, the $\varphi \oplus \psi$-$\mathfrak{A}$-module derivation $D_2 : A \oplus B \longrightarrow \psi(e_B).X^*.\psi(e_B)$ is approximately $\varphi \oplus \psi$-inner.

The right action of $\varphi(A) \oplus 0$ on $\varphi(e_A).X^*.\psi(e_B)$ is zero. Hence, by Lemma 3.4, $D_3 \mid_{A \oplus 0}$ is $\varphi \oplus \psi$-inner. So there exists $\xi \in \varphi(e_A).X^*.\psi(e_B)$ such that

$$D_3 \mid_{A \oplus 0} (a, 0) = \varphi(a).\xi - \xi.\varphi(a) = \big(\varphi(a), \psi(b)\big)\varphi(e_A).\xi.\psi(e_B),$$

for every $a \in A$ and $b \in B$. Similarly, there exists $\eta \in \varphi(e_A).X^*.\psi(e_B)$ such that

$$D_3 \mid_{0 \oplus B} (0, b) = \psi(b).\eta - \eta.\psi(b) = -\varphi(e_A).\eta.\psi(e_B)\big(\varphi(a), \psi(b)\big),$$

for every $a \in A$ and $b \in B$. Hence

$$D_3(a, b) = \big(\varphi(a), \psi(b)\big)\varphi(e_A).\xi.\psi(e_B) - \varphi(e_A).\eta.\psi(e_B)\big(\varphi(a), \psi(b)\big).$$

Since $D_3(e_A, e_B) = 0$, it follows that

$$0 = D_3(e_A, e_B) = \varphi(e_A).\xi.\psi(e_B) - \varphi(e_A).\eta.\psi(e_B).$$

Then for every $a \in A$ and $b \in B$, we have

$$D_3(a, b) = \big(\varphi(a), \psi(b)\big)\varphi(e_A).\xi.\psi(e_B) - \varphi(e_A).\xi.\psi(e_B)\big(\varphi(a), \psi(b)\big).$$

Thus $D_3$ is $\varphi \oplus \psi$-inner. The same argument holds for the $\varphi \oplus \psi$-$\mathfrak{A}$-module derivation $D_4 : A \oplus B \longrightarrow \psi(e_B).X^*.\varphi(e_A)$. Therefore $D$ is approximately $\varphi \oplus \psi$-inner, and so $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable. $\quad\square$

**Lemma 3.6.** *Let $A$ and $B$ be $\mathfrak{A}$-module Banach algebras, $\varphi \in \mathrm{Hom}_{\mathfrak{A}}(A)$ and $\psi \in \mathrm{Hom}_{\mathfrak{A}}(B)$. If there is a $h$ in $\mathrm{Hom}_{\mathfrak{A}}(A, B)$ such that $h \circ \varphi = \psi \circ h$ and the range of $h$ is a dense subset of $B$, then $\varphi$-$\mathfrak{A}$-module approximate amenability of $A$ implies $\psi$-$\mathfrak{A}$-module approximate amenability of $B$.*

*Proof.* Let $D : B \longrightarrow X^*$ be a $\psi$-$\mathfrak{A}$-module derivation for some commutative Banach $B$-$\mathfrak{A}$-module $X$. Then by the following actions

$$a \bullet x = h(a).x, \ x \bullet a = x.h(a) \qquad (a \in A, x \in X),$$

$X$ is a commutative Banach $A$-$\mathfrak{A}$-module. Let $\tilde{D} = D \circ h : A \longrightarrow X^*$. One can easily prove that $D$ is a $\varphi$-$\mathfrak{A}$-module derivation. From the $\varphi$-$\mathfrak{A}$-module approximate amenability of $A$, it follows that there exists a net $(x_\alpha^*)_\alpha$ in $X^*$ such that $\tilde{D}(a) = \lim_\alpha \left( \varphi(a) \bullet x_\alpha^* - x_\alpha^* \bullet \varphi(a) \right)$ $(a \in A)$. Now continuity and density of $h(A)$ in $B$, imply that $D$ is approximately $\psi$-inner. Therefore $B$ is $\psi$-$\mathfrak{A}$-module approximately amenable. $\qquad \square$

**Proposition 3.7.** *Let $A$ and $B$ be $\mathfrak{A}$-module Banach algebras, $\varphi \in \mathrm{Hom}_{\mathfrak{A}}(A)$ and $\psi \in \mathrm{Hom}_{\mathfrak{A}}(B)$. If $A$ is not $\varphi$-$\mathfrak{A}$-module approximately amenable or $B$ is not $\psi$-$\mathfrak{A}$-module approximately amenable, then $A \oplus B$ is not $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable.*

*Proof.* Suppose that $A$ is not $\varphi$-$\mathfrak{A}$-module approximately amenable. The projection map $\pi : A \oplus B \longrightarrow A$ determines an $\mathfrak{A}$-module epimorphism of $A \oplus B$ onto $A$ such that $\pi \circ (\varphi \oplus \psi) = \varphi \circ \pi$. So, if $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable, then by Lemma 3.6, $A$ is $\varphi$-$\mathfrak{A}$-module approximately amenable. This contradicts the fact that $A$ is not $\varphi$-$\mathfrak{A}$-module approximately amenable. Therefore $A \oplus B$ is not $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable. Similarly, we can prove the result for $B$. $\qquad \square$

Let $\mathfrak{A}$ be a non-unital Banach algebra. Then $\mathfrak{A}^{\#} = \mathfrak{A} \oplus \mathbb{C}$, the unitization of $\mathfrak{A}$ is a unital Banach algebra which contains $\mathfrak{A}$ as a closed ideal. Let $A$ be a Banach $\mathfrak{A}$-bimodule. Then $A$ is a Banach $\mathfrak{A}^{\#}$-module with the following module actions:

$$(\alpha, \lambda).a = \alpha.a + \lambda a, \ a.(\alpha, \lambda) = a.\alpha + \lambda a \ (\lambda \in \mathbb{C}, \alpha \in \mathfrak{A}, a \in A).$$

Let $A^{\sharp} = (A \oplus \mathfrak{A}^{\#}, \bullet)$, where the multiplication $\bullet$ is defined through

$$(a, u) \bullet (b, v) = (ab + a.v + u.b, uv) \ (a, b \in A, u, v \in \mathfrak{A}^{\#}).$$

Then with the actions defined by

$$u.(a, v) = (u.a, uv), \ (a, v).u = (a.u, vu) \ (a \in A, u, v \in \mathfrak{A}^{\#}),$$

$A^{\sharp}$ is a unital $\mathfrak{A}^{\#}$-module Banach algebra with the identity $1_{A^{\sharp}} = (0, 1_{\mathfrak{A}^{\#}})$ (see [4] ).

Before we turn to our next result we note that if for every $\varphi \in \mathrm{Hom}_{\mathfrak{A}^{\#}}(A)$, one defines $\varphi^{\sharp} : A^{\sharp} \longrightarrow A^{\sharp}$ by $\varphi^{\sharp}(a, u) = \left( \varphi(a), u \right)$ $\left( (a, u) \in A^{\sharp} \right)$, then $\varphi^{\sharp} \in \mathrm{Hom}_{\mathfrak{A}^{\#}}(A^{\sharp})$.

The following proposition generalizes Proposition 2.7 of [5].

**Theorem 3.8.** *Let $A$ and $B$ be $\mathfrak{A}$-module Banach algebras and each has a bounded approximate identity. Let $\varphi \in \mathrm{Hom}_{\mathfrak{A}^{\#}}(A)$ and $\psi \in \mathrm{Hom}_{\mathfrak{A}^{\#}}(B)$. Then $A$*

*is $\varphi$-$\mathfrak{A}^\#$-module approximately amenable and $B$ is $\psi$-$\mathfrak{A}^\#$-module approximately amenable if and only if $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}^\#$-module approximately amenable.*

*Proof.* Suppose that $A$ is $\varphi$-$\mathfrak{A}^\#$-module approximately amenable and $B$ is $\psi$-$\mathfrak{A}^\#$-module approximately amenable. By Proposition 12 of [13], $A^\sharp$ is $\varphi^\sharp$-$\mathfrak{A}^\#$-module approximately amenable and $B^\sharp$ is $\psi^\sharp$-$\mathfrak{A}^\#$-module approximately amenable, so by Proposition 3.5, $A^\sharp \oplus B^\sharp$ is $\varphi^\sharp \oplus \psi^\sharp$-$\mathfrak{A}^\#$-module approximately amenable. Since $A \oplus B$ is a closed $\mathfrak{A}^\#$-invariant ideal in $A^\sharp \oplus B^\sharp$, the result follows from Proposition 3.3.

For the converse, suppose that $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}^\#$-module approximately amenable. Then by Proposition 3.7, $A$ is $\varphi$-$\mathfrak{A}^\#$-module approximately amenable and $B$ is $\psi$-$\mathfrak{A}^\#$-module approximately amenable. $\square$

## 4. $\varphi \oplus \psi$-Module Approximate Amenability and $\varphi \oplus \psi$-Amenability of Direct Sum of Banach Algebras

We start this section with the following definition:

**Definition 4.1.** We say the Banach algebra $\mathfrak{A}$ acts trivially on $A$ from the left (right) if for every $\alpha \in \mathfrak{A}$ and $a \in A$, $\alpha.a = f(\alpha)a$ (resp. $a.\alpha = f(\alpha)a$), where $f$ is a multiplicative linear functional on $\mathfrak{A}$.

We assume that $J_{A,\mathfrak{A}}$ is the closed linear span of

$$\{(a.\alpha)b - a(\alpha.b) \mid \alpha \in \mathfrak{A}, a, b \in A\},$$

in $A$. It follows immediately that $J_{A,\mathfrak{A}}$ is both $A$-submodule and $\mathfrak{A}$-submodule of $A$. So $\frac{A}{J_{A,\mathfrak{A}}}$ is both Banach $A$-module and $\mathfrak{A}$-module (see page 346 of [14]).

To prove our next result we need to quote the following lemma from [2].

**Lemma 4.2.** *Let $A$ be a Banach algebra and Banach $\mathfrak{A}$-module with compatible actions, and $J_0$ be a closed ideal of $A$ such that $J_{A,\mathfrak{A}} \subseteq J_0$. If $\frac{A}{J_0}$ has a left or right identity $e + J_0$, then for each $\alpha \in \mathfrak{A}$ and $a \in A$ we have $a.\alpha - \alpha.a \in J_0$, i.e, $\frac{A}{J_0}$ is commutative Banach $\mathfrak{A}$-module.*

Before we turn to our next result we note that if for every $\varphi \in \text{Hom}_\mathfrak{A}(A)$, one defines $\overline{\varphi} : \frac{A}{J_{A,\mathfrak{A}}} \longrightarrow \frac{A}{J_{A,\mathfrak{A}}}$ by $\overline{\varphi}(a + J_{A,\mathfrak{A}}) = \varphi(a) + J_{A,\mathfrak{A}}$, then $\overline{\varphi} \in \text{Hom}_\mathfrak{A}(\frac{A}{J_{A,\mathfrak{A}}})$.

**Theorem 4.3.** *Let $A$ and $B$ be $\mathfrak{A}$-module Banach algebras and let $\varphi \in \text{Hom}_\mathfrak{A}(A)$ and $\psi \in \text{Hom}_\mathfrak{A}(B)$. Then the following statements are valid:*

(i) *$A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}$-module amenable (resp. $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable) if and only if $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ is $\overline{\varphi} \oplus \overline{\psi}$-$\mathfrak{A}$-module amenable (resp. $\overline{\varphi} \oplus \overline{\psi}$-$\mathfrak{A}$-module approximately amenable).*

(ii) *Let $\mathfrak{A}$ acts on $A$ and $B$ trivially from the left by $f \in \text{Hom}_\mathbb{C}(\mathfrak{A})$. Suppose that $\frac{A}{J_{A,\mathfrak{A}}}$ and $\frac{B}{J_{B,\mathfrak{A}}}$ are unital, and $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}$-module amenable (resp. $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable), then $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ is $\overline{\varphi} \oplus \overline{\psi}$-amenable (resp. $\overline{\varphi} \oplus \overline{\psi}$-approximately amenable).*

(iii) *Let $\mathfrak{A}$ have a bounded approximately identity and $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ is $\overline{\varphi} \oplus \overline{\psi}$-amenable (resp. $\overline{\varphi} \oplus \overline{\psi}$-approximately amenable). Then $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}$-module amenable (resp. $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable).*

*Proof.* (i) Let $A \oplus B$ be $\varphi \oplus \psi$-$\mathfrak{A}$-module amenable, and let $D : \frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}} \longrightarrow X^*$ be $\overline{\varphi} \oplus \overline{\psi}$-$\mathfrak{A}$-module derivation for some commutative Banach $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$-$\mathfrak{A}$-module $X$. Then $X$ becomes a $A \oplus B$-bimodule through the following actions

$$(a,b).x := (a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}).x \ \ (a \in A, b \in B, x \in X), \tag{4.1}$$

and

$$x.(a,b) := x.(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}) \ \ (a \in A, b \in B, x \in X). \tag{4.2}$$

Hence $X$ is a commutative Banach $A \oplus B$-$\mathfrak{A}$-module. Define $\tilde{D} : A \oplus B \longrightarrow X^*$ by

$$\tilde{D}(a,b) = D(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}) \ (a \in A, b \in B).$$

It is easy to check that, $\tilde{D}$ is a $\varphi \oplus \psi$-$\mathfrak{A}$-module derivation. From the $\varphi \oplus \psi$-$\mathfrak{A}$-module amenability of $A \oplus B$, it follows that there exists $x^* \in X^*$ such that

$$\tilde{D}(a,b) = \varphi \oplus \psi(a,b).x^* - x^*.\varphi \oplus \psi(a,b) \ \ (a \in A, b \in B).$$

Thus

$$D(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}) = \overline{\varphi} \oplus \overline{\psi}(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}).x^*$$
$$- x^*.\overline{\varphi} \oplus \overline{\psi}(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}).$$

This means that $D$ is $\overline{\varphi} \oplus \overline{\psi}$-inner. Therefore $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ is $\overline{\varphi} \oplus \overline{\psi}$-$\mathfrak{A}$-module amenable.

Conversely, suppose that $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ is $\overline{\varphi} \oplus \overline{\psi}$-$\mathfrak{A}$-module amenable. Let $D : A \oplus B \longrightarrow X^*$ be a $\varphi \oplus \psi$-$\mathfrak{A}$-module derivation for some commutative Banach $A \oplus B$-$\mathfrak{A}$-module $X$. We consider the following module actions of $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ on $X$,

$$(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}).x := (a,b).x, \ \ x.(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}) := x.(a,b),$$

for all $a \in A, b \in B$ and $x \in X$. Using (2.1) and the commutativity of $X$, we have $J_{A,\mathfrak{A}}X = J_{B,\mathfrak{A}}X = XJ_{A,\mathfrak{A}} = XJ_{B,\mathfrak{A}} = 0$. Thus $(J_{A,\mathfrak{A}} \oplus J_{B,\mathfrak{A}})X = X(J_{A,\mathfrak{A}} \oplus J_{B,\mathfrak{A}}) = 0$. So $X$ is a commutative Banach $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$-$\mathfrak{A}$-module. Define $\tilde{D} : \frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}} \longrightarrow X^*$ by

$$\tilde{D}(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}) = D(a,b) \ \ \ (a \in A, b \in B).$$

Also using (2.2) and (2.3) we see that $D$ vanishes on $J_{A,\mathfrak{A}} \oplus J_{B,\mathfrak{A}}$. Hence $\tilde{D}$ is well defined. One can easily check that $\tilde{D}$ is a $\overline{\varphi} \oplus \overline{\psi}$-$\mathfrak{A}$-module derivation.

Now from the $\overline{\varphi} \oplus \overline{\psi}$-$\mathfrak{A}$-module amenability of $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$, it follows that there exists $x^* \in X^*$ such that

$$\tilde{D}(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}) = \overline{\varphi} \oplus \overline{\psi}(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}).x^*$$
$$- x^*.\overline{\varphi} \oplus \overline{\psi}(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}) \ (a \in A, b \in B).$$

It follows that

$$D(a,b) = \varphi \oplus \psi(a,b).x^* - x^*.\varphi \oplus \psi(a,b) \ (a \in A, b \in B).$$

Thus $D$ is $\varphi \oplus \psi$-inner. So $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}$-module amenable.

Similarly, we can show that $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable if and only if $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ is $\overline{\varphi} \oplus \overline{\psi}$-$\mathfrak{A}$-module approximately amenable.

(ii) Let $A \oplus B$ be $\varphi \oplus \psi$-$\mathfrak{A}$-module amenable and let $D : \frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}} \longrightarrow X^*$ be a derivation for some Banach $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$-bimodule $X$. Then $X$ becomes a $A \oplus B$-bimodule through the actions as (4.1) and (4.2). Also $X$ is an $\mathfrak{A}$-bimodule with $f$-trivial actions, that is

$$\alpha.x = x.\alpha = f(\alpha)x \ (\alpha \in \mathfrak{A}, \ x \in X).$$

Then $X$ is a commutative Banach $A \oplus B$-$\mathfrak{A}$-module. Define

$$\Gamma : \frac{A \oplus B}{I} \longrightarrow \frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}, \ (a,b) + I \longmapsto (a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}),$$

where $I = J_{A,\mathfrak{A}} \oplus J_{B,\mathfrak{A}}$. It is routinely checked that $\Gamma$ defines an $\mathfrak{A}$-bimodule morphism. Let $\Pi : A \oplus B \longrightarrow \frac{A \oplus B}{I}$ be the quotient map, and let $\tilde{D} := D \circ \Gamma \circ \Pi : A \oplus B \longrightarrow X^*$. For every $(a,b), (a',b') \in A \oplus B$, we may easily prove that

$$\tilde{D}\big((a,b)(a',b')\big) = \tilde{D}(a,b).\varphi \oplus \psi(a',b') + \varphi \oplus \psi(a,b).\tilde{D}(a',b'),$$

and for every $(a,b) \in A \oplus B$, and $\alpha \in \mathfrak{A}$ , we have

$$\tilde{D}\big(\alpha.(a,b)\big) = \tilde{D}\big((\alpha.a, \alpha.b)\big) = \tilde{D}\big((f(\alpha)a, f(\alpha)b)\big)$$
$$= D\big((f(\alpha)a + J_{A,\mathfrak{A}}, f(\alpha)b + J_{B,\mathfrak{A}})\big)$$
$$= D\big(f(\alpha)(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}})\big)$$
$$= f(\alpha)D\big((a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}})\big)$$
$$= \alpha.D\big((a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}})\big)$$
$$= \alpha.\tilde{D}(a,b),$$

and using Lemma 4.2, we have

$$\tilde{D}\Big((a,b).\alpha\Big) = \tilde{D}\Big((a.\alpha, b.\alpha)\Big) = D\Big((a.\alpha + J_{A,\mathfrak{A}}, b.\alpha + J_{B,\mathfrak{A}})\Big)$$

$$= D\Big((\alpha.a + J_{A,\mathfrak{A}}, \alpha.b + J_{B,\mathfrak{A}})\Big)$$

$$= D\Big(f(\alpha)(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}})\Big)$$

$$= f(\alpha)D\Big((a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}})\Big)$$

$$= D\Big((a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}})\Big).\alpha$$

$$= \tilde{D}(a,b).\alpha.$$

Thus $\tilde{D}$ is a $\varphi \oplus \psi$-$\mathfrak{A}$-module derivation and from the $\varphi \oplus \psi$-$\mathfrak{A}$-module amenability of $A \oplus B$, it follows that there exists $x^* \in X^*$ such that

$$\tilde{D}(a,b) = \varphi \oplus \psi(a,b).x^* - x^*.\varphi \oplus \psi(a,b) \ \ (a \in A, b \in B).$$

It follows that

$$D(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}) = \overline{\varphi} \oplus \overline{\psi}(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}).x^*$$
$$- x^*.\overline{\varphi} \oplus \overline{\psi}(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}).$$

So $D$ is $\overline{\varphi} \oplus \overline{\psi}$-inner. Therefore $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ is $\overline{\varphi} \oplus \overline{\psi}$-amenable.

(iii) Suppose that $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ is $\overline{\varphi} \oplus \overline{\psi}$-amenable. Since $\mathfrak{A}$ has a bounded approximate identity, by Proposition 2.1 of [1], we conclude that $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ is $\overline{\varphi} \oplus \overline{\psi}$-$\mathfrak{A}$-module amenable. So by (i), $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}$-module amenable.

Similar relations can be obtained between the $\varphi \oplus \psi$-$\mathfrak{A}$-module approximate amenability of $A \oplus B$ and $\overline{\varphi} \oplus \overline{\psi}$-approximate amenability of $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$. $\square$

**Proposition 4.4.** *Let $A$ be an $\mathfrak{A}$-module Banach algebra, where $\mathfrak{A}$ acts on $A$ trivially from the left by $f \in \mathrm{Hom}_{\mathbb{C}}(\mathfrak{A})$. Let $\varphi \in \mathrm{Hom}_{\mathfrak{A}}(A)$ and $\frac{A}{J_{A,\mathfrak{A}}}$ be unital. If $A$ is $\varphi$-$\mathfrak{A}$-module approximately amenable, then $\frac{A}{J_{A,\mathfrak{A}}}$ is $\overline{\varphi}$-approximately amenable.*

*Proof.* Let $X$ be a Banach $\frac{A}{J_{A,\mathfrak{A}}}$-bimodule and $D : \frac{A}{J_{A,\mathfrak{A}}} \longrightarrow X^*$ be a $\overline{\varphi}$-derivation. Then $X$ becomes a $A$-bimodule through the following actions

$$a.x = (a + J_{A,\mathfrak{A}}).x, \ x.a = x.(a + J_{A,\mathfrak{A}}) \ (a \in A, x \in X),$$

and $X$ is an $\mathfrak{A}$-bimodule with $f$-trivial actions, that is $\alpha.x = x.\alpha = f(\alpha)x$ ($\alpha \in \mathfrak{A}, x \in X$). By Lemma 4.2, $f(\alpha)a - a.\alpha \in J_{A,\mathfrak{A}}$ ($\alpha \in \mathfrak{A}, a \in A$). So, $f(\alpha)a + J_{A,\mathfrak{A}} = a.\alpha + J_{A,\mathfrak{A}}$ ($\alpha \in \mathfrak{A}, a \in A$), and the actions of $\mathfrak{A}$ and $A$ on $X$ are compatible. Thus $X$ is a commutative Banach $A$-$\mathfrak{A}$-module. Let $\tilde{D} : A \longrightarrow X^*$ be defined by $\tilde{D}(a) = D(a + J_{A,\mathfrak{A}})$ ($a \in A$). A similar argument as in the proof of Theorem 3.2 of [2], shows that $\tilde{D}$ is approximately $\varphi$-inner. So, $D$ is approximately $\overline{\varphi}$-inner. Therefore $\frac{A}{J_{A,\mathfrak{A}}}$ is $\overline{\varphi}$-approximately amenable. $\square$

**Theorem 4.5.** *Let $\mathfrak{A}$ have a bounded approximate identity, and let $A$ and $B$ be $\mathfrak{A}$-module Banach algebras, where $\mathfrak{A}$ acts on $A$ and $B$ trivially from the left. Let $\varphi \in \text{Hom}_{\mathfrak{A}}(A)$, $\psi \in \text{Hom}_{\mathfrak{A}}(B)$, and let $\frac{A}{J_{A,\mathfrak{A}}}$ and $\frac{B}{J_{B,\mathfrak{A}}}$ be unital. Then $A$ is $\varphi$-$\mathfrak{A}$-module approximately amenable and $B$ is $\psi$-$\mathfrak{A}$-module approximately amenable if and only if $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable.*

*Proof.* Suppose that $A$ is $\varphi$-$\mathfrak{A}$-module approximately amenable and $B$ is $\psi$-$\mathfrak{A}$-module approximately amenable. By Proposition 4.4, $\frac{A}{J_{A,\mathfrak{A}}}$ and $\frac{B}{J_{B,\mathfrak{A}}}$ are $\overline{\varphi}$-approximately amenable and $\overline{\psi}$-approximately amenable, respectively. Now by using Proposition 3.5 for $\mathfrak{A} = \mathbb{C}$, we conclude that $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ is $\overline{\varphi} \oplus \overline{\psi}$-approximately amenable. So, Theorem 4.3, implies that $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable.

Conversely, suppose that $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable. Then by Proposition 3.7, $A$ is $\varphi$-$\mathfrak{A}$-module approximately amenable and $B$ is $\psi$-$\mathfrak{A}$-module approximately amenable.                                    $\square$

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