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# A Graphical Characterization for SPAP-Rings

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ABSTRACT. Let R be a commutative ring and I an ideal of R. The zero-divisor graph of R with respect to I, denoted by  $\Gamma_I(R)$ , is the simple graph whose vertex set is  $\{x \in R \setminus I \mid xy \in I, \text{ for some } y \in R \setminus I\}$ , with two distinct vertices x and y are adjacent if and only if  $xy \in I$ . In this paper, we state a relation between zero-divisor graph of R with respect to an ideal and almost prime ideals of R. We then use this result to give a graphical characterization for SPAP-rings.

**Keywords:** SPAP-ring, Almost prime ideal, Zero-divisor graph with respect to an ideal.

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## 1. Introduction

Throughout this paper, all rings are assumed to be commutative with identity. A graph (simple graph) G is an ordered pair of disjoint sets (V, E) such that V = V(G) is the vertex set of G and E = E(G) is its edge set. A graph F is called a subgraph of a graph G if  $V(F) \subseteq V(G)$  and  $E(F) \subseteq E(G)$ . A subgraph F of G is said to be an induced subgraph of G if each edge of G having its ends in V(F) is also an edge of F. A graph in which each pair of distinct vertices is joined by an edge is called complete.

There have been several studies concerning the assignment a graph to a ring, a group, a semigroup or a module, for more information see [1], [8] and [12]. The

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concept of the zero-divisor graph of a commutative ring R was first introduced by Beck [6]. The zero-divisor graph of a commutative ring R is defined to be the graph  $\Gamma(R)$ , whose vertices are the non-zero zero-divisors of R, and where x is adjacent to y if xy = 0. In [10] Redmond has generalized the notion of the zero-divisor graph. For a given ideal I of a commutative ring R, he defined the zero-divisor graph of R with respect to I, denoted by  $\Gamma_I(R)$ , is the simple graph whose vertex set is  $\{x \in R \setminus I \mid xy \in I, \text{ for some } y \in R \setminus I\}$ , with two distinct vertices x and y joined by an edge when  $xy \in I$ . Clearly  $\Gamma_0(R) = \Gamma(R)$ . Bhatwadekar and Sharma [7] defined a proper ideal I of an integral domain Rto be almost prime if for  $a, b \in R$ ,  $ab \in I \setminus I^2$ , then either  $a \in I$  or  $b \in I$ . Anderson and Bataineh [3], use this definition for an arbitrary commutative ring and stated a necessary and sufficient condition for a commutative Noetherin ring under which every proper ideal of R is a product of almost prime ideals. Then Rostami and Nekooei [11], considered SPAP-rings and characterized the structure of SPAP-rings, in special cases. Also, they showed that SPAP-rings are quasi – Frobenius (a Noetherian self-injective ring), and SPAP-rings are an applicative class of rings in Coding Theory, for more information see [11]. In the next section, we state a relation between zero-divisor graph with respect to an ideal of R and almost prime ideals of R. Then we state the concept of the intersection graph of ideals of R, and we give a graphical characterization for SPAP-rings.

## 2. Main Results

A proper ideal I in a ring R is called almost prime if for all  $a, b \in R$ ,  $ab \in I \setminus I^2$  either  $a \in I$  or  $b \in I$ . Also, a proper ideal I of a ring R is called weakly prime if for all  $a, b \in R$  with  $0 \neq ab \in I$ , either  $a \in I$  or  $b \in I$ . Clearly, every weakly prime ideal is almost prime. The following lemma which plays an important role in this paper gives a graphical characterization for almost prime ideals.

**Lemma 2.1.** Let I be a proper ideal of R. Then I is an almost prime ideal of R if and only if  $\Gamma_I(R)$  is an induced subgraph of  $\Gamma_{I^2}(R)$ .

Proof. Let I be an almost prime ideal of R and  $x \in V(\Gamma_I(R))$ , then  $x \in R \setminus I$ , and there exists  $y \in R \setminus I$  such that  $xy \in I$ . Thus  $x, y \notin I^2$ . Now if  $xy \notin I^2$ , then we have  $xy \in I \setminus I^2$ , this gives  $x \in I$  or  $y \in I$ , a contradiction. Thus  $xy \in I^2$ , and so  $x \in V(\Gamma_{I^2}(R))$ . Now let  $x, y \in V(\Gamma_I(R))$  be adjacent in  $\Gamma_I(R)$ , so  $xy \in I$ , if  $xy \notin I^2$ , then we have  $xy \in I \setminus I^2$ , this gives  $x \in I$  or  $y \in I$ , a contradiction. Therefore, x and y are adjacent in  $\Gamma_{I^2}(R)$ . Thus  $E(\Gamma_I(R)) \subseteq E(\Gamma_{I^2}(R))$ . Clearly, each edge of  $\Gamma_{I^2}(R)$  having its ends in  $\Gamma_I(R)$  is also an edge of  $\Gamma_I(R)$ . Therefore,  $\Gamma_I(R)$  is an induced subgraph of  $\Gamma_{I^2}(R)$ . Conversely, let  $\Gamma_I(R)$  be an induced subgraph of  $\Gamma_{I^2}(R)$  and  $ab \in I \setminus I^2$ , if

 $a, b \notin I$  then, a and b are adjacent in  $\Gamma_I(R)$  and so a and b are adjacent in  $\Gamma_{I^2}(R)$ , thus  $ab \in I^2$ , a contradiction. Therefore, either  $a \in I$  or  $b \in I$ .

The following lemma is a similar result for weakly prime ideals.

**Lemma 2.2.** Let I be a proper ideal of R. Then I is a weakly prime ideal of R if and only if  $\Gamma_I(R)$  is an induced subgraph of  $\Gamma(R)$ .

Proof. Let I be a weakly prime ideal of R and  $x \in V(\Gamma_I(R))$ . Then  $x \in R \setminus I$ , and there exists  $y \in R \setminus I$  such that  $xy \in I$ . If  $xy \neq 0$ , we have  $0 \neq xy \in I$ , this gives  $x \in I$  or  $y \in I$ , a contradiction. Thus xy = 0, and so  $x \in V(\Gamma(R))$ . Now, let  $x, y \in V(\Gamma_I(R))$  be adjacent in  $\Gamma_I(R)$ , thus  $xy \in I$ . Repeating the previous argument leads to xy = 0. Hence x, y are adjacent in  $\Gamma(R)$ . Clearly, each edge of  $\Gamma(R)$  having its ends in  $\Gamma_I(R)$  is also an edge of  $\Gamma_I(R)$ . Therefore  $\Gamma_I(R)$  is an induced subgraph of  $\Gamma(R)$ . Conversely, let  $\Gamma_I(R)$  be an induced subgraph of  $\Gamma(R)$  and  $0 \neq ab \in I$  for  $a, b \in R$ , if  $a \notin I$  and  $b \notin I$  then, a and b are adjacent in  $\Gamma_I(R)$ , thus a and b are adjacent in  $\Gamma(R)$ . This gives ab = 0, a contradiction. Thus either  $a \in I$  or  $b \in I$ .

**Lemma 2.3.** Let I be a proper ideal of R. Then I is a prime ideal of R if and only if  $\Gamma_I(R) = \emptyset$ .

*Proof.* The proof is straightforward.

Now let I be a prime ideal of R. Thus  $\Gamma_I(R) = \emptyset$  and so  $\Gamma_I(R) = \emptyset$  is an induced subgraph of  $\Gamma_{I^2}(R)$  and  $\Gamma(R)$ , this is a graphical verification for the fact that "prime ideals are almost prime and weakly prime".

**Definition 2.4.** A local ring (R, m) is called special product of almost prime ideals ring (SPAP-ring), if for each  $x \in m \setminus m^2$ ,  $\langle x^2 \rangle = m^2$  and  $m^3 = 0$ .

SPAP-rings were first introduced in [3] by D. D. Anderson and M. Bataineh. In [3], D. D. Anderson and M. Bataineh used SPAP-rings to characteriz Noetherian rings whose proper ideals are a product of almost prime ideals. In general, an SPAP-ring is not Noetherian, see [3, Example 20]. For an SPAP-ring (R, m), m is the unique prime ideal of R, thus R is a Noetherian ring if and only if R is an Artinian ring if and only if R is a finitely generated ideal of R.

Before proceeding, we mention the definition of the intersection graph of ideals of a ring which helps us to give a characterization for SPAP-rings.

**Definition 2.5.** Let R be a ring, the intersection graph of ideals of R, denoted by G(R), is the graph whose vertices are proper non-trivial ideals of R and two distinct vertices are adjacent if and only if the corresponding ideals of R have a non-trivial (non-zero)intersection.

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**Lemma 2.6.** [5, Theorem 2.11.] Let (R, m) be an Artinian local ring. Then the intersection graph of ideals of R is complete if and only if R has a unique minimal ideal.

For more information about intersection graph of ideals of R, see [2, 5]. In the remainder of this section, we characterize Artinian local rings which  $\Gamma_I(R)$  is an induced subgraph of  $\Gamma_{I^2}(R)$  for all non-minimal ideals I of R, and the intersection graph of ideal of R is complete.

**Lemma 2.7.** Let (R,m) be an Artinian local ring and  $\Gamma_I(R)$  is an induced subgraph of  $\Gamma_{I^2}(R)$ , for every non-minimal ideal I of R. Then  $m^2$  is a minimal ideal of R or  $m^2 = 0$ .

*Proof.* Let  $m^2$  be a non-minimal ideal of R. Then by Lemma 2.1,  $m^2$  is an almost prime ideal of R. We show that  $m^2$  must be zero in this case. For this purpose, we show that  $m^2 = m^3 = m^4$  and the Nakayama's Lemma gives  $m^2 = 0$ . If for all  $x, y \in m$ ,  $xy \in m^4$ , we have  $m^2 \subseteq m^4$ , thus  $m^2 = m^3 = m^4$ . Now let there exist  $x, y \in m$  such that  $xy \notin m^4$ , so  $xy \in m^2 \setminus m^4 = m^2 \setminus (m^2)^2$ , since  $m^2$  is almost prime, only one of the following cases happens;

 $x \in m^2$  and  $y \notin m^2$  or  $x \notin m^2$  and  $y \in m^2$ . Suppose  $x \in m^2$  and  $y \notin m^2$ . Since  $y^2 \in m^2$ ,  $y \notin m^2$  and  $m^2$  is almost prime, we must have  $y^2 \in m^4$ . Repeating the previous argument and  $y, x + y \notin m^2$  and  $y(x + y) \in m^2$  leads to  $y(x + y) \in m^4$ . Thus  $xy + y^2 = y(x + y), y^2 \in m^4$ , so  $xy \in m^4$ , a contradiction. Thus  $m^2$  is zero or a minimal ideal.

Now we mention the definition of a class of rings which are important in the rest of this paper.

**Definition 2.8.** A commutative ring R is called special principal ideal ring (SPIR), if it is a principal ideal ring with unique prime ideal and that prime ideal is nilpotent.

Mori [9] has shown that a ring has the property that every ideal is a product of prime ideals if and only if it is a finite direct product of Dedekind domains and special principal ideal rings (SPIRs) (For more information about special principal ideal ring see [9]). In the next lemma, we state a relation between SPAP-rings and SPIR rings.

**Lemma 2.9.** Let (R, m) be an SPIR ring such that  $\Gamma_I(R)$  is an induced subgraph of  $\Gamma_{I^2}(R)$ , for every non-minimal ideal I and  $m^2$  is the unique minimal ideal of R. Then (R, m) is an SPAP-ring.

*Proof.* Since R is an SPIR ring,  $m = \langle x \rangle$  for some  $x \in m$ . Now let  $0 \neq J \neq m^2$  be an ideal of R. If  $J = J^2$ , Nakayama's Lemma gives J = 0, a contradiction. So  $J \neq J^2$ , thus we can select  $y \in J \setminus J^2 \subseteq m$  such that  $J = \langle y \rangle$ . Thus  $y = rx \in J \setminus J^2$ , for some  $r \in R$ . Since  $J \neq m^2$ , Lemma

2.1 gives J is an almost prime ideal of R and since  $y=rx\in J\setminus J^2$ , we have  $x\in J$  or  $r\in J$ . If  $x\in J$ , then J=m and if  $r\in J\subseteq m$ , then we have  $J=< y>=< rx>\subseteq m^2$  and since  $m^2$  is the unique minimal ideal of R, J=0 or  $J=m^2$ , a contradiction. This means, the set of all ideals of R is  $\{0,m^2,m,R\}$ .

Now if  $m=m^2$ , we have m=0, a contradiction. Thus  $m\neq m^2$ . If  $a\in m\setminus m^2$ , since the set of all ideals of R is  $\{0,m^2,m,R\}$ , we have m=< a>, so  $m^2=< a^2>$ . Now if  $m^3\neq 0$ , we have  $m^2=m^3$ , and Nakayama's Lemma gives m=0, a contradiction. Thus  $m^3=0$ . This completes the proof.

D. D. Anderson and M. Bataineh in [3], by using *SPAP*-rings, characterized Noetherian rings whose proper ideals are a product of almost prime ideals. Actually, they stated the following theorem.

**Theorem 2.10.** [3, Theorem 22]. Let R be a Noetherian ring. Then every proper ideal of R is a product of almost prime ideals if and only if R is a finite direct product of Dedekind domains, SPIRs, and (Noetherian) SPAP-rings.

**Proposition 2.11.** Let (R,m) be an Artinian local ring such that  $\Gamma_I(R)$  is an induced subgraph of  $\Gamma_{I^2}(R)$ , for every non-minimal ideal I of R and the intersection graph of ideal of R is complete. If  $m^2 \neq 0$  then R is an SPAP-ring.

Proof. Since the intersection graph of ideals of R is complete, by Lemma 2.6, R has a unique minimal ideal. Since  $m^2 \neq 0$ , Lemma 2.7 gives,  $m^2$  is the unique minimal ideal of R. Now let I be an arbitrary proper ideal of R, if I is a non-minimal ideal of R, then  $\Gamma_I(R)$  is an induced subgraph of  $\Gamma_{I^2}(R)$ , so I is an almost prime ideal of R, by Lemma 2.1, and if I is a minimal ideal of R, then  $I = m^2$ . Therefore, in all cases I is finite product of almost prime ideals (note that m is prime and so is almost prime), thus by Theorem 2.10, R is a finite direct product of Dedekind domains, SPIR rings, and SPAP-rings. Since R is a local ring, this direct product must have a single ring. Let R be a Dedekind domain. Since  $m^2$  is a minimal ideal of R, we have  $m^3 = 0$ 

An R-module M is said to be a multiplication R-module if for each submodule N of M there exists an ideal I of R such that N = IM. Clearly, every cyclic module is multiplication module, see [4] for more information. After stating

or  $m^2 = m^3$ , in both cases, we have  $m^2 = 0$ . Thus R is not a Dedekind domain

and Lemma 2.9, completes the proof.

the main result, we require the following three lemmas.

**Lemma 2.12.** Let (R, m) be an SPAP-ring. If  $m^2 \neq 0$ , then  $m^2$  is a minimal ideal of R.

Proof. If  $m=m^2$ , then  $m^2=m^3=0$ , a contradiction. Therefore  $m\neq m^2$ , thus there exists  $y\in m\setminus m^2$ . So  $m^2=< y^2>$ . Therefore,  $m^2$  is a cyclic R-module and so it is a multiplication R-module. Now if J is a submodule (ideal of R) of  $m^2$ , there exists an ideal K of R, such that  $J=Km^2$ . If K=R, then  $J=m^2$  and if  $K\neq R$  then  $J=Km^2\subseteq m^3=0$ , hence J=0. Therefore  $m^2$  is a minimal ideal of R.

**Lemma 2.13.** Let (R, m) be an SPAP-ring. If  $m^2 \neq 0$  and I is a proper ideal of R, then I = 0 or  $I = m^2$  or  $I^2 = m^2$ .

*Proof.* Since  $m^2 \neq 0$ , by Lemma 2.12,  $m^2$  is a minimal ideal of R. Now let I be a proper ideal of (R,m). If  $I \subseteq m^2$ , then I=0 or  $I=m^2$ . If  $I \nsubseteq m^2$ , then there exists  $y \in I \setminus m^2$ . So  $m^2 = \langle y^2 \rangle$ , hence  $m^2 = \langle y^2 \rangle \subseteq I^2 \subseteq m^2$ . Thus  $I^2 = m^2$ .

By combining the above two lemmas, we have the following lemma.

**Lemma 2.14.** Let (R, m) be an SPAP-ring. If  $m^2 \neq 0$ , then  $m^2$  is the unique minimal ideal of R.

Now we can state the main result of this paper.

**Theorem 2.15.** Let (R, m) be an Artinian local ring with  $m^2 \neq 0$ . Then  $\Gamma_I(R)$  is an induced subgraph of  $\Gamma_{I^2}(R)$ , for every non-minimal ideal I of R and the intersection graph of ideals of R is complete if and only if R is an SPAP-ring.

Proof. Let R be an SPAP-ring by Lemma 2.14,  $m^2$  is the unique minimal ideal of R, so by Lemma 2.6, the intersection graph of ideals of R is complete. Now let I be a proper ideal of R, Lemma 2.13 gives I = 0 or  $I = m^2$  or  $I^2 = m^2$ . If I is a non-zero non-minimal ideal of R and  $ab \in I \setminus I^2$ , for  $a, b \in R$ , then  $ab \notin I^2 = m^2$ , so a or b is not in m, thus a or b is unit. Thus a or b must be in I. This shows that I is an almost prime ideal of R. Hence, by Lemma 2.1,  $\Gamma_I(R)$  is an induced subgraph of  $\Gamma_{I^2}(R)$ . In general, the zero ideal is an almost prime of R. Thus every non-minimal ideal of R is almost prime and so  $\Gamma_I(R)$  is an induced subgraph of  $\Gamma_{I^2}(R)$ , for every non-minimal ideal I of R. The converse of theorem is valid by Proposition 2.11.

EXAMPLE 2.16. Let k be an ordered field. Then for a non-empty set  $\{x_{\alpha}\}_{\alpha\in\Delta}$  of indeterminates. Define  $R=k[[\{x_{\alpha}\}_{\alpha\in\Lambda}]],\ m=<\{x_{\alpha}\}_{\alpha\in\Delta}>,\ \text{and}\ J=<\{x_{\alpha}x_{\beta},x_{\alpha}^2-x_{\beta}^2\}_{\alpha\neq\beta},\{x_{\alpha}^3\}_{\alpha}>.$  Let  $\overline{R}=\frac{R}{J}.$  Then  $\overline{R}$  is an SPAP-ring with  $\overline{m}^2\neq 0$  and  $\overline{m}$  is not principal for  $|\Delta|>1$ , see [3, Example 20]. If  $\Delta$  is a finite set, then  $\overline{R}$  is a Noethrian SPAP-ring with  $\overline{m}^2\neq 0$ , and thus  $\Gamma_{\overline{I}}(\overline{R})$  is an induced subgraph of  $\Gamma_{\overline{I}^2}(\overline{R})$ , for every non-minimal ideal  $\overline{I}$  of  $\overline{R}$  and the intersection graph of ideals of  $\overline{R}$  is complete.

**Theorem 2.17.** Let (R,m) be an Artinian local ring with  $m^2 \neq 0$ , such that  $\Gamma_I(R)$  is an induced subgraph of  $\Gamma_{I^2}(R)$ , for every non-minimal ideal I of R and the intersection graph of ideals of R is complete. If  $\operatorname{char}(R) \neq p^2$ , for any prime number p and  $\operatorname{char}(\frac{R}{m}) \neq 2$ , then there exists a regular local ring (S,n), a positive integer number h, and subset  $\{x_a\}_{a=1,\ldots,h}$  of n such that  $R \cong \frac{S}{K}$  in which K is minimally generated by the elements  $\{x_ix_j\}_{1\leq i< j\leq h}$ ,  $\{x_j^2\}_{2\leq j\leq \tau}$  and  $\{x_i^2u_ix_1^2\}_{\tau+1\leq i\leq h}$ , where the  $u_i$  are unit in R and  $\tau$  is the Cohen-Macaulay type of R.

*Proof.* By Theorem 2.15 and [11, Proposition 6.3.].

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