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A Graphical Characterization for $SPAP$-Rings

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Abstract. Let R be a commutative ring and I an ideal of R. The zero-divisor graph of R with respect to I, denoted by $\Gamma_I(R)$, is the simple graph whose vertex set is $\{x \in R \setminus I \mid xy \in I, \text{ for some } y \in R \setminus I\}$, with two distinct vertices x and y are adjacent if and only if $xy \in I$. In this paper, we state a relation between zero-divisor graph of R with respect to an ideal and almost prime ideals of R. We then use this result to give a graphical characterization for $SPAP$-rings.

Keywords: $SPAP$-ring, Almost prime ideal, Zero-divisor graph with respect to an ideal.

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1. Introduction

Throughout this paper, all rings are assumed to be commutative with identity. A graph (simple graph) G is an ordered pair of disjoint sets (V, E) such that $V = V(G)$ is the vertex set of G and $E = E(G)$ is its edge set. A graph F is called a subgraph of a graph G if $V(F) \subseteq V(G)$ and $E(F) \subseteq E(G)$. A subgraph F of G is said to be an induced subgraph of G if each edge of G having its ends in $V(F)$ is also an edge of F. A graph in which each pair of distinct vertices is joined by an edge is called complete.

There have been several studies concerning the assignment a graph to a ring, a group, a semigroup or a module, for more information see [1], [8] and [12]. The

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concept of the zero-divisor graph of a commutative ring R was first introduced by Beck [6]. The zero-divisor graph of a commutative ring R is defined to be the graph $\Gamma(R)$, whose vertices are the non-zero zero-divisors of R, and where x is adjacent to y if $xy = 0$. In [10] Redmond has generalized the notion of the zero-divisor graph. For a given ideal I of a commutative ring R, he defined the zero-divisor graph of R with respect to I, denoted by $\Gamma_I(R)$, is the simple graph whose vertex set is $\{x \in R \setminus I \mid xy \in I, \text{ for some } y \in R \setminus I\}$, with two distinct vertices x and y joined by an edge when $xy \in I$. Clearly $\Gamma_0(R) = \Gamma(R)$. Bhatwadekar and Sharma [7] defined a proper ideal I of an integral domain R to be almost prime if for $a, b \in R$, $ab \in I \setminus I^2$, then either $a \in I$ or $b \in I$. Anderson and Bataineh [3], use this definition for an arbitrary commutative ring and stated a necessary and sufficient condition for a commutative Noetherin ring under which every proper ideal of R is a product of almost prime ideals. Then Rostami and Nekooei [11], considered $SPAP$-rings and characterized the structure of $SPAP$-rings, in special cases. Also, they showed that $SPAP$-rings are $quasi - Frobenius$ (a Noetherian self-injective ring), and $SPAP$-rings are an applicative class of rings in Coding Theory, for more information see [11].
In the next section, we state a relation between zero-divisor graph with respect to an ideal of R and almost prime ideals of R. Then we state the concept of the intersection graph of ideals of R, and we give a graphical characterization for $SPAP$-rings.

2. Main Results

A proper ideal I in a ring R is called almost prime if for all $a, b \in R$, $ab \in I \setminus I^2$ either $a \in I$ or $b \in I$. Also, a proper ideal I of a ring R is called weakly prime if for all $a, b \in R$ with $0 \neq ab \in I$, either $a \in I$ or $b \in I$. Clearly, every weakly prime ideal is almost prime. The following lemma which plays an important role in this paper gives a graphical characterization for almost prime ideals.

Lemma 2.1. *Let I be a proper ideal of R. Then I is an almost prime ideal of R if and only if $\Gamma_I(R)$ is an induced subgraph of $\Gamma_{I^2}(R)$.*

Proof. Let I be an almost prime ideal of R and $x \in V(\Gamma_I(R))$, then $x \in R \setminus I$, and there exists $y \in R \setminus I$ such that $xy \in I$. Thus $x, y \notin I^2$. Now if $xy \notin I^2$, then we have $xy \in I \setminus I^2$, this gives $x \in I$ or $y \in I$, a contradiction. Thus $xy \in I^2$, and so $x \in V(\Gamma_{I^2}(R))$. Now let $x, y \in V(\Gamma_I(R))$ be adjacent in $\Gamma_I(R)$, so $xy \in I$, if $xy \notin I^2$, then we have $xy \in I \setminus I^2$, this gives $x \in I$ or $y \in I$, a contradiction. Therefore, x and y are adjacent in $\Gamma_{I^2}(R)$. Thus $E(\Gamma_I(R)) \subseteq E(\Gamma_{I^2}(R))$. Clearly, each edge of $\Gamma_{I^2}(R)$ having its ends in $\Gamma_I(R)$ is also an edge of $\Gamma_I(R)$. Therefore, $\Gamma_I(R)$ is an induced subgraph of $\Gamma_{I^2}(R)$. Conversely, let $\Gamma_I(R)$ be an induced subgraph of $\Gamma_{I^2}(R)$ and $ab \in I \setminus I^2$, if

$a, b \notin I$ then, a and b are adjacent in $\Gamma_I(R)$ and so a and b are adjacent in $\Gamma_{I^2}(R)$, thus $ab \in I^2$, a contradiction. Therefore, either $a \in I$ or $b \in I$. □

The following lemma is a similar result for weakly prime ideals.

Lemma 2.2. *Let I be a proper ideal of R. Then I is a weakly prime ideal of R if and only if $\Gamma_I(R)$ is an induced subgraph of $\Gamma(R)$.*

Proof. Let I be a weakly prime ideal of R and $x \in V(\Gamma_I(R))$. Then $x \in R \setminus I$, and there exists $y \in R \setminus I$ such that $xy \in I$. If $xy \neq 0$, we have $0 \neq xy \in I$, this gives $x \in I$ or $y \in I$, a contradiction. Thus $xy = 0$, and so $x \in V(\Gamma(R))$. Now, let $x, y \in V(\Gamma_I(R))$ be adjacent in $\Gamma_I(R)$, thus $xy \in I$. Repeating the previous argument leads to $xy = 0$. Hence x, y are adjacent in $\Gamma(R)$. Clearly, each edge of $\Gamma(R)$ having its ends in $\Gamma_I(R)$ is also an edge of $\Gamma_I(R)$. Therefore $\Gamma_I(R)$ is an induced subgraph of $\Gamma(R)$. Conversely, let $\Gamma_I(R)$ be an induced subgraph of $\Gamma(R)$ and $0 \neq ab \in I$ for $a, b \in R$, if $a \notin I$ and $b \notin I$ then, a and b are adjacent in $\Gamma_I(R)$, thus a and b are adjacent in $\Gamma(R)$. This gives $ab = 0$, a contradiction. Thus either $a \in I$ or $b \in I$. □

Lemma 2.3. *Let I be a proper ideal of R. Then I is a prime ideal of R if and only if $\Gamma_I(R) = \emptyset$.*

Proof. The proof is straightforward. □

Now let I be a prime ideal of R. Thus $\Gamma_I(R) = \emptyset$ and so $\Gamma_I(R) = \emptyset$ is an induced subgraph of $\Gamma_{I^2}(R)$ and $\Gamma(R)$, this is a graphical verification for the fact that "prime ideals are almost prime and weakly prime".

Definition 2.4. A local ring (R, m) is called special product of almost prime ideals ring ($SPAP$-ring), if for each $x \in m \setminus m^2$, $< x^2 >= m^2$ and $m^3 = 0$.

$SPAP$-rings were first introduced in [3] by D. D. Anderson and M. Bataineh. In [3], D. D. Anderson and M. Bataineh used $SPAP$-rings to characteriz Noetherian rings whose proper ideals are a product of almost prime ideals. In general, an $SPAP$-ring is not Noetherian, see [3, Example 20]. For an $SPAP$-ring (R, m), m is the unique prime ideal of R, thus R is a Noetherian ring if and only if R is an Artinian ring if and only if m is a finitely generated ideal of R.

Before proceeding, we mention the definition of the intersection graph of ideals of a ring which helps us to give a characterization for $SPAP$-rings.

Definition 2.5. Let R be a ring, the intersection graph of ideals of R, denoted by $G(R)$, is the graph whose vertices are proper non-trivial ideals of R and two distinct vertices are adjacent if and only if the corresponding ideals of R have a non-trivial (non-zero)intersection.

Lemma 2.6. [5, Theorem 2.11.] *Let (R, m) be an Artinian local ring. Then the intersection graph of ideals of R is complete if and only if R has a unique minimal ideal.*

For more information about intersection graph of ideals of R, see [2, 5]. In the remainder of this section, we characterize Artinian local rings which $\Gamma_I(R)$ is an induced subgraph of $\Gamma_{I^2}(R)$ for all non-minimal ideals I of R, and the intersection graph of ideal of R is complete.

Lemma 2.7. *Let (R, m) be an Artinian local ring and $\Gamma_I(R)$ is an induced subgraph of $\Gamma_{I^2}(R)$, for every non-minimal ideal I of R. Then m^2 is a minimal ideal of R or $m^2 = 0$.*

Proof. Let m^2 be a non-minimal ideal of R. Then by Lemma 2.1, m^2 is an almost prime ideal of R. We show that m^2 must be zero in this case. For this purpose, we show that $m^2 = m^3 = m^4$ and the Nakayama's Lemma gives $m^2 = 0$. If for all $x, y \in m$, $xy \in m^4$, we have $m^2 \subseteq m^4$, thus $m^2 = m^3 = m^4$. Now let there exist $x, y \in m$ such that $xy \notin m^4$, so $xy \in m^2 \setminus m^4 = m^2 \setminus (m^2)^2$, since m^2 is almost prime, only one of the following cases happens;

$x \in m^2$ and $y \notin m^2$ or $x \notin m^2$ and $y \in m^2$. Suppose $x \in m^2$ and $y \notin m^2$. Since $y^2 \in m^2$, $y \notin m^2$ and m^2 is almost prime, we must have $y^2 \in m^4$. Repeating the previous argument and $y, x+y \notin m^2$ and $y(x+y) \in m^2$ leads to $y(x+y) \in m^4$. Thus $xy + y^2 = y(x+y), y^2 \in m^4$, so $xy \in m^4$, a contradiction. Thus m^2 is zero or a minimal ideal. □

Now we mention the definition of a class of rings which are important in the rest of this paper.

Definition 2.8. A commutative ring R is called special principal ideal ring ($SPIR$), if it is a principal ideal ring with unique prime ideal and that prime ideal is nilpotent.

Mori [9] has shown that a ring has the property that every ideal is a product of prime ideals if and only if it is a finite direct product of Dedekind domains and special principal ideal rings ($SPIRs$)(For more information about special principal ideal ring see [9]). In the next lemma, we state a relation between $SPAP$-rings and $SPIR$ rings.

Lemma 2.9. *Let (R, m) be an $SPIR$ ring such that $\Gamma_I(R)$ is an induced subgraph of $\Gamma_{I^2}(R)$, for every non-minimal ideal I and m^2 is the unique minimal ideal of R. Then (R, m) is an $SPAP$-ring.*

Proof. Since R is an $SPIR$ ring, $m =< x >$ for some $x \in m$. Now let $0 \neq J \neq m^2$ be an ideal of R. If $J = J^2$, Nakayama's Lemma gives $J = 0$, a contradiction. So $J \neq J^2$, thus we can select $y \in J \setminus J^2 \subseteq m$ such that $J =< y >$. Thus $y = rx \in J \setminus J^2$, for some $r \in R$. Since $J \neq m^2$, Lemma

2.1 gives J is an almost prime ideal of R and since $y = rx \in J \setminus J^2$, we have $x \in J$ or $r \in J$. If $x \in J$, then $J = m$ and if $r \in J \subseteq m$, then we have $J =< y >=< rx >\subseteq m^2$ and since m^2 is the unique minimal ideal of R, $J = 0$ or $J = m^2$, a contradiction. This means, the set of all ideals of R is $\{0, m^2, m, R\}$.

Now if $m = m^2$, we have $m = 0$, a contradiction. Thus $m \neq m^2$. If $a \in m \setminus m^2$, since the set of all ideals of R is $\{0, m^2, m, R\}$, we have $m =< a >$, so $m^2 =< a^2 >$. Now if $m^3 \neq 0$, we have $m^2 = m^3$, and Nakayama's Lemma gives $m = 0$, a contradiction. Thus $m^3 = 0$. This completes the proof. □

D. D. Anderson and M. Bataineh in [3], by using $SPAP$-rings, characterized Noetherian rings whose proper ideals are a product of almost prime ideals. Actually, they stated the following theorem.

Theorem 2.10. [3, Theorem 22]. *Let R be a Noetherian ring. Then every proper ideal of R is a product of almost prime ideals if and only if R is a finite direct product of Dedekind domains, SPIRs, and (Noetherian) SPAP-rings.*

Proposition 2.11. *Let (R, m) be an Artinian local ring such that $\Gamma_I(R)$ is an induced subgraph of $\Gamma_{I^2}(R)$, for every non-minimal ideal I of R and the intersection graph of ideal of R is complete. If $m^2 \neq 0$ then R is an $SPAP$-ring.*

Proof. Since the intersection graph of ideals of R is complete, by Lemma 2.6, R has a unique minimal ideal. Since $m^2 \neq 0$, Lemma 2.7 gives, m^2 is the unique minimal ideal of R. Now let I be an arbitrary proper ideal of R, if I is a non-minimal ideal of R, then $\Gamma_I(R)$ is an induced subgraph of $\Gamma_{I^2}(R)$, so I is an almost prime ideal of R, by Lemma 2.1, and if I is a minimal ideal of R, then $I = m^2$. Therefore, in all cases I is finite product of almost prime ideals (note that m is prime and so is almost prime), thus by Theorem 2.10, R is a finite direct product of Dedekind domains, $SPIR$ rings, and $SPAP$-rings. Since R is a local ring, this direct product must have a single ring.
Let R be a Dedekind domain. Since m^2 is a minimal ideal of R, we have $m^3 = 0$ or $m^2 = m^3$, in both cases, we have $m^2 = 0$. Thus R is not a Dedekind domain and Lemma 2.9, completes the proof. □

An R-module M is said to be a multiplication R-module if for each submodule N of M there exists an ideal I of R such that $N = IM$. Clearly, every cyclic module is multiplication module, see [4] for more information. After stating the main result, we require the following three lemmas.

Lemma 2.12. *Let (R, m) be an $SPAP$-ring. If $m^2 \neq 0$, then m^2 is a minimal ideal of R.*

Proof. If $m = m^2$, then $m^2 = m^3 = 0$, a contradiction. Therefore $m \neq m^2$, thus there exists $y \in m \setminus m^2$. So $m^2 =< y^2 >$. Therefore, m^2 is a cyclic R-module and so it is a multiplication R-module. Now if J is a submodule (ideal of R) of m^2, there exists an ideal K of R, such that $J = Km^2$. If $K = R$, then $J = m^2$ and if $K \neq R$ then $J = Km^2 \subseteq m^3 = 0$, hence $J = 0$. Therefore m^2 is a minimal ideal of R. □

Lemma 2.13. *Let (R, m) be an $SPAP$-ring. If $m^2 \neq 0$ and I is a proper ideal of R, then $I = 0$ or $I = m^2$ or $I^2 = m^2$.*

Proof. Since $m^2 \neq 0$, by Lemma 2.12, m^2 is a minimal ideal of R . Now let I be a proper ideal of (R, m). If $I \subseteq m^2$, then $I = 0$ or $I = m^2$. If $I \nsubseteq m^2$, then there exists $y \in I \setminus m^2$. So $m^2 =< y^2 >$, hence $m^2 =< y^2 >\subseteq I^2 \subseteq m^2$. Thus $I^2 = m^2$. □

By combining the above two lemmas, we have the following lemma.

Lemma 2.14. *Let (R, m) be an $SPAP$-ring. If $m^2 \neq 0$, then m^2 is the unique minimal ideal of R.*

Now we can state the main result of this paper.

Theorem 2.15. *Let (R, m) be an Artinian local ring with $m^2 \neq 0$. Then $\Gamma_I(R)$ is an induced subgraph of $\Gamma_{I^2}(R)$, for every non-minimal ideal I of R and the intersection graph of ideals of R is complete if and only if R is an SPAP-ring.*

Proof. Let R be an $SPAP$-ring by Lemma 2.14, m^2 is the unique minimal ideal of R, so by Lemma 2.6, the intersection graph of ideals of R is complete. Now let I be a proper ideal of R, Lemma 2.13 gives $I = 0$ or $I = m^2$ or $I^2 = m^2$. If I is a non-zero non-minimal ideal of R and $ab \in I \setminus I^2$, for $a, b \in R$, then $ab \notin I^2 = m^2$, so a or b is not in m, thus a or b is unit. Thus a or b must be in I. This shows that I is an almost prime ideal of R. Hence, by Lemma 2.1, $\Gamma_I(R)$ is an induced subgraph of $\Gamma_{I^2}(R)$. In general, the zero ideal is an almost prime of R. Thus every non-minimal ideal of R is almost prime and so $\Gamma_I(R)$ is an induced subgraph of $\Gamma_{I^2}(R)$, for every non-minimal ideal I of R.
The converse of theorem is valid by Proposition 2.11.

□

Example 2.16. Let k be an ordered field. Then for a non-empty set $\{x_\alpha\}_{\alpha\in\Delta}$ of indeterminates. Define $R = k[[\{x_\alpha\}_{\alpha\in\Lambda}]]$, $m =< \{x_\alpha\}_{\alpha\in\Delta} >$, and $J =< \{x_\alpha x_\beta, x_\alpha^2 - x_\beta^2\}_{\alpha\neq\beta}, \{x_\alpha^3\}_\alpha >$. Let $\overline{R} = \frac{R}{J}$. Then $\overline{R}$ is an $SPAP$-ring with $\overline{m}^2 \neq 0$ and $\overline{m}$ is not principal for $| \Delta |> 1$, see [3, Example 20]. If Δ is a finite set, then $\overline{R}$ is a Noethrian $SPAP$-ring with $\overline{m}^2 \neq 0$, and thus $\Gamma_{\overline{I}}(\overline{R})$ is an induced subgraph of $\Gamma_{\overline{I}^2}(\overline{R})$, for every non-minimal ideal $\overline{I}$ of $\overline{R}$ and the intersection graph of ideals of $\overline{R}$ is complete.

Theorem 2.17. *Let (R, m) be an Artinian local ring with $m^2 \neq 0$, such that $\Gamma_I(R)$ is an induced subgraph of $\Gamma_{I^2}(R)$, for every non-minimal ideal I of R and the intersection graph of ideals of R is complete. If $char(R) \neq p^2$, for any prime number p and $char(\frac{R}{m}) \neq 2$, then there exists a regular local ring (S, n), a positive integer number h, and subset $\{x_a\}_{a=1,\ldots,h}$ of n such that $R \cong \frac{S}{K}$ in which K is minimally generated by the elements $\{x_i x_j\}_{1 \leq i < j \leq h}$, $\{x_j^2\}_{2 \leq j \leq \tau}$ and $\{x_i^2 u_i x_1^2\}_{\tau+1 \leq i \leq h}$, where the u_i are unit in R and τ is the Cohen- Macaulay type of R.*

Proof. By Theorem 2.15 and [11, Proposition 6.3.]. □

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