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# Distance-Balanced Closure of Some Graphs

N. Ghareghani<sup>a,\*</sup>, B. Manoochehrian<sup>b</sup>, M. Mohammad-Noori<sup>c</sup>

<sup>a</sup>Department of Engineering Science, College of Engineering, University of Tehran, P.O. Box 11165-4563, Tehran, Iran.

<sup>b</sup>Academic Center for Education, Culture and Research (ACECR), Tehran Branch, P.O. Box 19395-5746, Tehran, Iran.

<sup>c</sup>Department of Computer Science, School of Mathematics, Statistics and Computer Science, College of Science, University of Tehran, P.O. Box 14155-6455, Tehran, Iran.

E-mail: ghareghani@ut.ac.ir
E-mail: behzad@khayam.ut.ac.ir

E-mail: mnoori@khayam.ut.ac.ir, morteza@ipm.ir

ABSTRACT. In this paper we prove that any distance-balanced graph G with  $\Delta(G) \geq |V(G)| - 3$  is regular. Also we define notion of distance-balanced closure of a graph and we find distance-balanced closures of trees T with  $\Delta(T) \geq |V(T)| - 3$ .

**Keywords:** Distances in graphs, Distance-balanced graphs, Distance-balanced closure.

2000 Mathematics subject classification: 05B20, 05E30.

# 1. Introduction

Let G be a graph with vertex set V(G) and edge set E(G). We denote |V(G)| by n. The set of neighbors of a vertex  $v \in V(G)$  is denoted by  $N_G(v)$ , and  $N_G[v] = N_G(v) \cup \{v\}$ . The degree of a vertex v is denoted by  $\deg_G(v)$  and minimum degree and maximum degree of G denoted by  $\delta(G)$  and  $\Delta(G)$ ,

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<sup>\*</sup>Corresponding Author

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respectively. The distance  $d_G(u,v)$  between vertices u and v is the length of a shortest path between u and v in G. The diameter  $\operatorname{diam}(G)$  of graph G is defined as  $\max\{d_G(u,v): u,v\in V(G)\}$ . The notion of distance is studied in several works in graph theory (See [2] and the references therein) and many research works are based on the concepts related to this notion (See for instance [8] and [10]).

For an edge xy of a graph G,  $W_{xy}^G$  is the set of vertices which are closer to x than y, more formally

$$W_{xy}^G = \{ u \in V(G) | d_G(u, x) < d_G(u, y) \}.$$

Moreover,  ${}_xW_y^G$  is the set of vertices of G that have equal distances to x and y, that is

$$_{x}W_{u}^{G} = \{u \in V(G) | d_{G}(u, x) = d_{G}(u, y)\}.$$

These sets play important roles in metric graph theory, see for instance [1, 3, 4, 5]. Since x always belongs to  $W_{xy}^G$ , for convenience we let  $U_{xy}^G = W_{xy}^G \setminus \{x\}$ . Distance-balanced graphs are introduced in [9] as graphs for which  $|W_{xy}^G|$  $|W_{yx}^G|$  (or equivalently  $|U_{xy}^G| = |U_{yx}^G|$ ) for every pair of adjacent vertices  $x, y \in$ V(G).

In [9], the parameter b(G) of a graph G is introduced as the smallest number of the edges which can be added to G such that the obtained graph is distancebalanced. Since the complete graph is distance-balanced, this parameter is well-defined. We call graph G a distance-balanced closure of H if G is distancebalanced and H is a spanning subgraph of G with |E(G)| = b(H) + |E(H)|; in other words, a distance-balanced closure of H is a distance-balanced graph Gwhich contains H as a spanning subgraph and has minimum number of edges. As mentioned in [9], the computation of b(G) is quite hard in general but it might be interesting in some special cases. In this paper we compute b(G) for all trees T with  $\Delta(T) \geq |V(T)| - 3$ . In Section 2, we compute that distancebalanced closure of graphs G with  $\Delta(G) = n - 1$ . In Section 3, and Section 4, we concern graphs G with  $\Delta(G) = n - 2$  and  $\Delta(G) = n - 3$ , respectively. Then we compute b(T) for all trees T with  $\Delta(T) = n - 2$  and  $\Delta(T) = n - 3$ .

Here we mention some more definitions and notations about trees. Let  $P_n$ denoted the path with n vertices. A tree which has exactly one vertex of degree greater than two is said to be starlike. The vertex of maximum degree is called the central vertex. We denote by  $S(n_1, n_2, \ldots, n_k)$  a starlike tree in which removing the central vertex leaves disjoint paths  $P_{n_1}, P_{n_2}, \dots, P_{n_k}$ . We say that  $S(n_1, n_2, \ldots, n_k)$  has branches of length  $n_1, n_2, \ldots, n_k$ . It is obvious that  $S(n_1, n_2, \ldots, n_k)$  has  $n_1 + n_2 + \ldots + n_k + 1$  vertices. For simplicity a starlike with  $\alpha_i$  branches of length  $n_i$   $(1 \le i \le k)$  is denoted by  $S(n_1^{\alpha_1}, n_2^{\alpha_2}, \dots, n_k^{\alpha_k})$ .

## 2. Distance-Balanced Graphs with Maximum Degree n-1

In this section we prove that for any graph G with  $\Delta(G) = n - 1$ , the only distance-balanced closure of G is the complete graph  $K_n$ . The following result is very useful in this paper. It is in fact a slight modification of Corollary 2.3 of [9].

**Theorem 2.1.** Let G be a graph with diameter at most 2 and H be a distance-balanced graph such that G is a spanning subgraph of H. Then H is a regular graph. Moreover, every regular graph with diameter at most 2 is distance-balanced.

**Corollary 2.2.** For every integer  $m \ge 1$ , the graph  $K_{1,m}$  has a unique distance-balanced closure which is isomorphic to  $K_{m+1}$ , hence,  $b(K_{1,m}) = {m+1 \choose 2} - m$ .

*Proof.* Let G be a distance-balanced closure of  $K_{1,m}$ . By Theorem 2.1, G is a regular graph and since  $K_{1,m}$  has a vertex of degree m, G should be m-regular, hence  $G \cong K_{m+1}$ .

The following is an immediate conclusion of Theorem 2.1.

Corollary 2.3. Let G be a graph with n vertices and  $\Delta(G) = n - 1$ . Then the graph G has a unique distance balanced closure. Moreover, this closure is isomorphic to  $K_n$ .

### 3. Distance-Balanced Graphs with Maximum Degree n-2

In this section, we prove that any distance-balanced graph G with  $\Delta(G) = n-2$  is a regular graph using this, we construct a distance-balanced closure of T where T is a tree with this property (that is  $\Delta(T) = n-2$ ) and then compute b(T).

The following Lemma will be used occasionally in this paper and the proof is easily deduced from the definition of  $U_{xy}^G$ .

**Lemma 3.1.** Let x and y be two adjacent vertices of a graph G, then  $U_{xy}^G \cap N_G(y) = \emptyset$  (or  $U_{xy}^G \subseteq V(G) \setminus N_G[y]$ ). Furthermore,  $N_G[y] \setminus U_{yx}^G \subseteq N_G[x]$ .

**Theorem 3.2.** Let  $T = S(2, 1^{m-1})$  be a starlike tree and H be a distance-balanced graph containing T as a spanning subgraph. Then  $\operatorname{diam}(H) \leq 2$ , hence, H is an r-regular graph for some  $m \leq r \leq m+1$ .

*Proof.* Suppose that the vertices of T are labeled as shown in Figure 1. If  $oy \in E(H)$ , then H contains  $K_{1,m+1}$  as a spanning subgraph, so by Corollary 2.2,  $H \cong K_{m+2}$ .

So, we may assume that  $oy \notin E(H)$  (and consequently diam $(H) \neq 1$ ). We prove diam(H) = 2. For this, it is enough to show that  $d(y, x_i) \leq 2$  for  $i = 2, \dots, m$ . Let i be an integer with  $1 \leq i \leq m$ ; using the fact that i is the only vertex which is not adjacent to i in i and using Lemma 3.1, we

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conclude that  $U_{x_io}^H \subseteq \{y\}$ ; If  $U_{x_io}^H = \{y\}$ , then  $x_iy \in E(H)$  and  $d_H(x_i,y) = 1$ , hence in this case  $d_H(x_i,y) \leq 2$ . Otherwise,  $U_{x_io}^H = \emptyset$ , in this case, for every  $z \in V(H) \setminus \{o,x_i\}$  we have  $d_H(z,x_i) = d_H(z,o)$ , particularly,  $d_H(z,x_i) = d_H(y,o) = 2$ . Hence, diam(H) = 2, as required. The result is now concluded using Theorem 2.1.

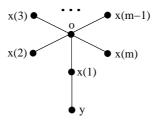


Figure 1

**Theorem 3.3.** Let  $T = S(2, 1^{m-1})$  be a starlike of order m + 2. Then

$$b(T) = \begin{cases} \frac{m^2}{2} - 1 & if m is even; \\ \binom{m+1}{2} & otherwise. \end{cases}$$

*Proof.* Suppose that the vertices of T be labeled as in Figure 1, and let  $\overline{T}$  be a distance-balanced closure of T. First, suppose that m is an odd integer; Since there is no m-regular graph of order m+2, by Theorem 3.2,  $\overline{T} \cong K_{m+2}$ .

Now, suppose that m is an even integer. Let  $H=K_{m+2}$  be a complete graph with vertex set V(T) and M be a complete matching of H which contains the edge oy. Then  $H\setminus M$ , is an m-regular graph with diameter 2 and contains T as a spanning subgraph. Hence, by Theorem 2.1 and Theorem 3.2,  $\overline{T}=H\setminus M$ , is a distance-balanced closure of T and  $b(T)=\frac{m^2}{2}-1$ .

Corollary 3.4. Let G be a connected graph of order n, with  $\Delta(G) = n - 2$  and H be a distance-balanced graph which contains G as a spanning subgraph. Then H is either an (n-2)-regular graph or the complete graph  $K_n$ .

*Proof.* In this case  $S(2,1^{n-3})$  is an spanning subgraph of G. So, by Theorem 3.2, H is either an (n-2)-regular graph or the complete graph  $K_n$ .

## 4. Distance-Balanced Graphs with Maximum Degree n-3

In this section we will prove that every distance-balanced graph with  $\Delta(G) = n-3$  is regular. Moreover, by constructing distance-balanced closure of trees with  $\Delta(T) = n-3$  we compute b(T) for these trees.

**Theorem 4.1.** Let  $T = S(2^2, 1^{m-2})$  be a starlike of order m+3 and H be a distance-balanced graph which contains T as a spanning subgraph. Then  $\operatorname{diam}(H) \leq 2$ . Moreover, H is an r-regular graph with  $r \geq m$ .

*Proof.* Suppose the vertices of T are labeled as in Figure 2. If either oy or oz be an edge of H, then by Theorem 3.2, H is an r-regular graph with  $r \ge m+1$ , which proves this theorem. So, suppose that oy,  $oz \notin E(H)$ .

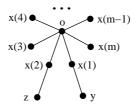


Figure 2

By Lemma 3.1, for each  $1 \leq i \leq m$ ,  $U_{x_io}^H \subseteq \{y, z\}$ . Now, we prove that  $\deg_H(x_i) = m$ , for each  $i = 1, \dots, m$ . For this, we consider three possible cases:

Case 1.  $|U_{x_io}^H| = 0$ : Then  $U_{x_io}^H = \emptyset$  and  $U_{ox_i}^H = \emptyset$ . Hence, by Lemma 3.1,  $N_H(x_i) = \{o, x_1, x_2, \dots, x_m\} \setminus \{x_i\}$  and  $\deg_H(x_i) = m$ .

 $N_H(x_i) = \{o, x_1, x_2, \dots, x_m\} \setminus \{x_i\}$  and  $\deg_H(x_i) = m$ . **Case 2.**  $|U_{x_io}^H| = 1$ : Without loss of generality we can assume that  $U_{x_io}^H = \{y\}$ . Then there is an integer  $1 \leq j \leq m$  such that  $U_{ox_i}^H = \{x_j\}$ . Since  $d_H(o, y) = 2$ ,  $yx_i \in E(H)$ . Hence, using Lemma 3.1,  $N_H(x_i) = \{o, y, x_1, x_2, \dots, x_m\} \setminus \{x_j\}$  and  $\deg_H(x_i) = m$ .

 $\{o, y, x_1, x_2, \dots, x_m\} \setminus \{x_j\}$  and  $\deg_H(x_i) = m$ . **Case 3.**  $|U_{x_io}^H| = 2$ : We have  $U_{x_io}^H = \{y, z\}$ . Since  $d_H(o, y) = d_H(o, z) = 2$ , we conclude that  $yx_i, zx_i \in E(H)$ . Since  $|U_{ox_i}^H| = 2$ , there are integers j and k such that  $U_{ox_i}^H = \{x_j, x_k\}$ . Hence, by Lemma 3.1, we have  $N_H(x_i) = \{o, y, z, x_1, x_2, x_3, \dots, x_m\} \setminus \{x_i, x_j, x_k\}$  and  $\deg_H(x_i) = m$ .

Next, we prove that  $\deg_H(y) \geq m-3$  and  $\deg_H(z) \geq m-3$ . From  $\deg_H(x_1) = m$  and Lemma 3.1, it concludes that  $|U_{yx_1}^H| \leq 2$ , hence,  $|U_{x_1y}^H| \leq 2$ , which means that there are at most two elements in  $N_H(x_1) \setminus N_H[y]$ . Using this and Lemma 3.1, we provide  $\deg_H(y) \geq m-3$ . With a similar argument, the inequality  $\deg_H(z) \geq m-3$  is concluded.

Now, by using  $\deg_H(y), \deg_H(z) \geq m-3$ ,  $\deg_H(x_i) \geq m$ ,  $(i=1,\cdots,m)$ , and  $oy, oz \notin E(H)$ , hence every two nonadjacent vertices have a common neighbor, provided that  $m \geq 7$ . This means that  $\operatorname{diam}(H) = 2$ , which proves the result in case  $m \geq 7$ , using Theorem 2.1. For the cases,  $3 \leq m \leq 6$ , through a case by case inspection (by using  $\deg_H(x_i) \geq m$ ,  $i=1,\cdots,m$ ,) the same result is obtained.

**Theorem 4.2.** For the starlike tree  $T = S(2^2, 1^{m-2})$  of order m + 3,  $b(G) = \frac{m^2 + m - 4}{2}$ .

Proof. Let the vertices of T be labeled as in Figure 2 and  $\overline{T}$  be a distance-balanced closure of T. Now, we are going to construct  $\overline{T}$ . Let  $H=K_{m+3}$  be a complete graph with the same vertex set as H. Omit the edges of cycles  $C_1=x_1x_2x_3\ldots x_mx_1$  and  $C_2=oyzo$  from H to obtain  $\overline{T}=H\setminus (C_1\cup C_2)$ . Now,  $\overline{G}$  is an m-regular graph with diameter 2, which contains T as a spanning subgraph, so by Theorem 4.1 and Theorem 2.1,  $\overline{T}$  is a distance-balanced closure of T and  $b(T)=\frac{m^2+m-4}{2}$ .

**Theorem 4.3.** Let T be the tree of Figure 3 and H be a distance-balanced graph which contains T as a spanning subgraph. Then  $\operatorname{diam}(H) \leq 2$ , hence, H is a regular graph. Moreover,  $b(T) = \frac{m^2 + m - 4}{2}$ .

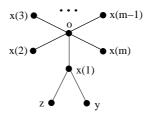
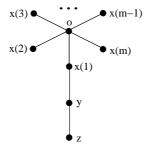


Figure 3

Proof. If either  $oy \in E(H)$  or  $oz \in E(H)$ , then by Theorem 3.2,  $\operatorname{diam}(H) \leq 2$  and H is a regular graph. So, suppose that neither oy nor oz is in E(H). Since  $|W^H_{x_1y}| = |W^H_{yx_1}|$  and  $o \in W^H_{x_1y}$ , there exists a vertex  $x_i, i \neq 1$ , such that  $yx_i \in E(H)$ . Therefore, graph H contains graph  $S(2^2, 1^{m-2})$  as a spanning subgraph and using Theorem 4.1,  $\operatorname{diam}(H) \leq 2$  and H is a regular graph. Furthermore, the graph introduced in the proof of Theorem 4.2, is also distance-balanced closure of T. Hence  $b(T) = \frac{m^2 + m - 4}{2}$ .

**Theorem 4.4.** Consider the starlike tree  $T = S(3, 1^{m-1})$  of order m + 3 and let H be a distance-balanced graph which contains T as a spanning subgraph. Then H is an r-regular graph for some  $m \le r \le m + 2$ .

*Proof.* Let the vertices of T be labeled as in Figure 4.



#### Figure 4

If either  $oy \in E(H)$  or  $oz \in E(H)$ , then by Theorem 3.2,  $\operatorname{diam}(H) \leq 2$  and H is a regular graph. If  $zx_1 \in E(H)$ , then H contains the graph shown in Figure 3 as a spanning subgraph, so by Theorem 4.3, H is a regular graph. So we may assume that  $\{oy, oz, x_1z\} \cap E(H) = \emptyset$ . Since  $|W_{yz}^H| = |W_{zy}^H|$  and  $x_1 \in W_{yz}^H$ , the vertex z is adjacent to at least one vertex in  $\{x_2, x_3, \ldots, x_m\}$  (because otherwise according to the structure of T we have  $V \setminus \{y, z\} \subseteq U_{yz}$ ). Hence, H contains the graph  $S(2^2, 1^{m-2})$ , as a spanning subgraph. So, by Theorem 4.1,  $\operatorname{diam} H \leq 2$  and H is a regular graph, as desired.

**Corollary 4.5.** Let G be a connected graph of order n with  $\Delta(G) = n-3$ . Then every distance-balanced graph H which contains G as a spanning subgraph, is regular.

*Proof.* Since  $\Delta(G) = n - 3$ , G contains at least one of the graphs  $S(2^2, 1^{n-2})$ ,  $S(3, 1^{n-1})$  or the graph shown in Figure 3, as a spanning subgraph. Hence, the result follows from Theorem 4.2, Theorem 4.6 and Theorem 4.3.

**Theorem 4.6.** For the starlike tree  $G = S(3, 1^{m-1})$  of order m+3,  $b(G) = \frac{m^2+m-4}{2}$ .

Proof. Let the vertices of G be labeled as in Figure 4 and let  $\overline{G}$  be a distance-balanced closure of G. Now, we are going to construct  $\overline{G}$ . Let  $H = K_{m+3}$  be a complete graph with the same vertex set as G. Omit the edges of cycles  $C_1 = x_3x_4 \dots x_mx_3$  and  $C_2 = oyx_2x_1z_0$  from H to obtain  $\overline{G} = H \setminus (C_1 \cup C_2)$ . Then the graph  $\overline{G}$  is an n-regular graph with diameter 2, which contains G as a spanning subgraph. So by Theorem 4.4,  $\overline{G}$  is a distance-balanced closure of G and  $b(G) = \frac{m^2 + m - 4}{2}$ .

**Conclusion.** In previous sections, we have proved that any connected distance-balanced graph G with  $\Delta(G) \geq |V(G)| - 3$ , is a regular graph, moreover, distanced-closure of such a graph G is a smallest regular graph which contains G. This helped us to find a distance-balanced closure of trees T with  $\Delta(T) \geq |V(T)| - 3$  and to compute b(T) for such trees.

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