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# Inequalities for the Derivatives of a Polynomial

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ABSTRACT. The paper presents an  $L^r-$  analogue of an inequality regarding the  $s^{th}$  derivative of a polynomial having zeros outside a circle of arbitrary radius but greater or equal to one. Our result provides improvements and generalizations of some well-known polynomial inequalities.

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## 1. Introduction and Statement of Results

Let P(z) be a polynomial of degree at most n and P'(z) be its derivative, then

$$\max_{|z|=1} |P'(z)| \le n \max_{|z|=1} |P(z)| \tag{1.1}$$

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and for every  $r \geq 1$ ,

$$\left\{ \int_{0}^{2\pi} |P'(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}} \le n \left\{ \int_{0}^{2\pi} |P(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}}.$$
(1.2)

Inequality (1.1) is a classical result of Bernstein[6] whereas inequality (1.2) is due to Zygmund[15] who proved it for all trigonometric polynomials of degree n and not only for those which are of the form  $P(e^{i\theta})$ . Arcstov[1] proved that (1.2) remains true for 0 < r < 1 as well. If  $r \to \infty$  in inequality (1.2), we get (1.1).

If we restrict ourselves to the class of polynomials having no zeros in |z| < 1, then both the inequalities (1.1) and (1.2) can be sharpened. In fact, If  $P(z) \neq 0$  in |z| < 1, then (1.1) and (1.2) can be respectively replaced by

$$\max_{|z|=1} |P'(z)| \le \frac{n}{2} \max_{|z|=1} |P(z)| \tag{1.3}$$

and

$$\left\{ \int_{0}^{2\pi} |P'(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}} \le nA_r \left\{ \int_{0}^{2\pi} |P(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}}, \tag{1.4}$$

where 
$$A_r = \left\{ \frac{1}{2\pi} \int_{0}^{2\pi} |1 + e^{i\alpha}|^r d\alpha \right\}^{\frac{-1}{r}}$$
.

Inequality (1.3) was conjectured by Erdös and later verified by Lax[11], whereas inequality (1.4) was proved by De-Bruijn[7] for  $r \geq 1$ . Rahman and Schemeisser[13] later proved that (1.4) holds for 0 < r < 1 also. If  $r \to \infty$  in (1.4), we get (1.3).

As a generalization of (1.3) Malik[12] proved that if  $P(z) \neq 0$  in  $|z| < k, k \geq 1$ , then

$$\max_{|z|=1} |P'(z)| \le \frac{n}{1+k} \max_{|z|=1} |P(z)|, \tag{1.5}$$

whereas under the same hypothesis, Govil and Rahman[9] extended inequality (1.4) by showing that

$$\left\{ \int_{0}^{2\pi} |P'(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}} \le nE_r \left\{ \int_{0}^{2\pi} |P(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}}, \tag{1.6}$$

where 
$$E_r = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |k + e^{i\alpha}|^r d\alpha \right\}^{\frac{-1}{r}}, \ r \ge 1.$$

In the same paper, Govil and Rahman[9, Theorem 4] extended inequality (1.5) to the  $s^{th}$  derivative of a polynomial and proved under the same hypothesis

for  $1 \le s < n$  that

$$\max_{|z|=1} |P^{(s)}(z)| \le \frac{n(n-1)\cdots(n-s+1)}{1+k^s} \max_{|z|=1} |P(z)|. \tag{1.7}$$

Inequality (1.7) was refined by Aziz and Rather [3, Corollary 1] by involving the binomial coefficients C(n,s),  $1 \le s < n$  and coefficients of the polynomial P(z). In fact they proved that if  $P(z) = \sum_{j=0}^{n} a_j z^j$  does not vanish in |z| < k,  $k \ge 1$ , then for  $1 \le s < n$ ,

$$\max_{|z|=1} |P^{(s)}(z)| \le \frac{n(n-1)\cdots(n-s+1)}{1+\psi_{k,s}} \max_{|z|=1} |P(z)|, \tag{1.8}$$

where

$$\psi_{k,s} = k^{s+1} \left( \frac{1 + \frac{1}{C(n,s)} \left| \frac{a_s}{a_0} \right| k^{s-1}}{1 + \frac{1}{C(n,s)} \left| \frac{a_s}{a_0} \right| k^{s+1}} \right).$$
 (1.9)

In the literature there exist various results regarding the estimates for polynomials and for general analytic functions and also the approximations of polynomials and their derivatives (for example see[8],[14]). In this paper, we prove the following result which refines the inequality (1.8).

**Theorem 1.1.** If  $P(z) = \sum_{j=0}^{n} a_j z^j$  is a polynomial of degree n having no zeros in |z| < k,  $k \ge 1$ , and  $m = \min_{|z|=k} |P(z)|$  then for  $1 \le s < n$ ,

$$\max_{|z|=1} |P^{(s)}(z)| \le \frac{n(n-1)\cdots(n-s+1)}{1+\psi_{k,s}} \Big(\max_{|z|=1} |P(z)| - \frac{m\psi_{k,s}}{k^n}\Big),$$
 (1.10)

where  $\psi_{k,s}$  is defined by (1.9).

The result is best possible for k = 1 and equality holds for  $P(z) = z^n + 1$ .

Remark 1.2. For s = 1 and m = 0, Theorem 1.1 reduces to a result of Govil et. al.[10, Theorem 1] and for k = s = 1, inequality (1.10) reduces to a result of Aziz and Dawood[2, Theorem A].

Remark 1.3. Note by inequality (2.2) of Lemma 2.1 (stated in section 2) that  $\frac{1}{C(n,s)} \left| \frac{a_s}{a_0} \right| k^s \leq 1$ , which can easily be shown to be equivalent to  $\psi_{k,s} \geq k^s$ ,  $1 \leq s < n$ . Using this fact in inequality (1.10), we get the following improvement of inequality (1.7).

**Corollary 1.4.** If  $P(z) = \sum_{j=0}^{n} a_j z^j$  is a polynomial of degree n having no zeros in |z| < k,  $k \ge 1$ , and  $m = \min_{|z|=k} |P(z)|$  then for  $1 \le s < n$ ,

$$\max_{|z|=1} |P^{(s)}(z)| \le \frac{n(n-1)\cdots(n-s+1)}{1+k^s} \left(\max_{|z|=1} |P(z)| - \frac{m}{k^{n-s}}\right). \tag{1.11}$$

In order to prove the Theorem 1.1, we prove the following more general result which extends Theorem 1.1 to its corresponding  $L^r$  – analogue.

**Theorem 1.5.** If  $P(z) = \sum_{j=0}^{n} a_j z^j$  is a polynomial of degree n having no zeros in |z| < k,  $k \ge 1$ , and  $m = \min_{|z|=k} |P(z)|$ , then for every complex number  $\beta$  with  $|\beta| \le 1$  and  $1 \le s < n$ , we have

$$\left\{ \int_{0}^{2\pi} \left| P^{(s)}(e^{i\theta}) + \frac{\beta m n(n-1) \cdots (n-s+1) \psi_{k,s}}{k^{n} (1+\psi_{k,s})} \right|^{r} d\theta \right\}^{\frac{1}{r}} \\
\leq n(n-1) \cdots (n-s+1) C_{r} \left\{ \int_{0}^{2\pi} |P(e^{i\theta})|^{r} d\theta \right\}^{\frac{1}{r}}, \quad (1.12)$$

where 
$$C_r = \left\{ \frac{1}{2\pi} \int_{0}^{2\pi} |\psi_{k,s} + e^{i\alpha}|^r d\alpha \right\}^{\frac{-1}{r}}, \ r > 0 \ and \ \psi_{k,s} \ is \ defined \ by \ (1.9).$$

Remark 1.6. Using the fact that  $\psi_{k,s} \geq k^s$  and take  $\beta = 0$  in inequality (1.12), we obtain a result of Aziz and Shah[5].

### 2. Lemmas

We need the following lemmas for the proofs of Theorems. Here, throughout this paper we write  $Q(z) = z^n \overline{P(\frac{1}{z})}$ .

**Lemma 2.1.** If  $P(z) = \sum_{j=0}^{n} a_j z^j$  is a polynomial of degree n which does not vanish in  $|z| < k, k \ge 1$ , then for  $1 \le s < n$  and |z| = 1,

$$|Q^{(s)}(z)| \ge \psi_{k,s} |P^{(s)}(z)|,$$
 (2.1)

and

$$\frac{1}{C(n,s)} \left| \frac{a_s}{a_0} \right| k^s \le 1, \tag{2.2}$$

where  $\psi_{k,s}$  is defined by (1.9).

The above lemma is due to Aziz and Rather[3].

**Lemma 2.2.** If P(z) is a polynomial of degree n, then for each  $\alpha$ ,  $0 \le \alpha < 2\pi$  and r > 0, we have

$$\int_{0}^{2\pi} \int_{0}^{2\pi} \left| Q'(e^{i\theta}) + e^{i\alpha} P'(e^{i\theta}) \right|^{r} d\theta d\alpha \le 2\pi n^{r} \int_{0}^{2\pi} |P(e^{i\theta})|^{r} d\theta. \tag{2.3}$$

The above lemma is due to Aziz and Shah[4].

**Lemma 2.3.** If  $P(z) = \sum_{j=0}^{n} a_j z^j$  is a polynomial of degree n which does not vanish in  $|z| < k, k \ge 1$ , then for  $1 \le s < n$  and |z| = 1,

$$|Q^{(s)}(z)| \ge \psi_{k,s} |P^{(s)}(z)| + \frac{mn(n-1)\cdots(n-s+1)}{k^n} \psi_{k,s},$$
 (2.4)

where  $m = \min_{|z|=k} |P(z)|$ .

*Proof.* Since  $m \leq |P(z)|$  for |z| = k, we have for every  $\beta$  with  $|\beta| < 1$ ,

$$\left| \frac{m\beta z^n}{k^n} \right| < |P(z)| \ for \ |z| = k.$$

Therefore by Rouche's theorem  $P(z) + \frac{m\beta z^n}{k^n}$  has no zero in  $|z| < k, \ k \ge 1$ . Applying Lemma 2.1 to the polynomial  $P(z) + \frac{m\beta z^n}{k^n}$ , we get for  $1 \le s < n$  and |z| = 1,

$$|Q^{(s)}(z)| \ge \psi_{k,s} |P^{(s)}(z) + \frac{mn(n-1)\cdots(n-s+1)\beta}{k^n}|.$$
 (2.5)

Choose the argument of  $\beta$  so that

$$\left| P^{(s)}(z) + \frac{mn(n-1)\cdots(n-s+1)\beta z^{n-s}}{k^n} \right| = \left| P^{(s)}(z) \right| + \frac{mn(n-1)\cdots(n-s+1)|\beta z^{n-s}|}{k^n},$$

it follows from (2.5) that for |z| = 1,

$$|Q^{(s)}(z)| \ge \psi_{k,s} |P^{(s)}(z)| + \frac{mn(n-1)\cdots(n-s+1)|\beta z^{n-s}|}{k^n} \psi_{k,s}.$$
 (2.6)

Letting  $|\beta| \to 1$  in inequality (2.6), we get

$$|Q^{(s)}(z)| \ge \psi_{k,s} |P^{(s)}(z)| + \frac{mn(n-1)\cdots(n-s+1)}{k^n} \psi_{k,s}.$$

This completes the proof of Lemma 2.3.

**Lemma 2.4.** If A, B, C are non-negative real numbers such that  $B + C \leq A$ . Then for every real  $\alpha$ ,

$$\left| (A - C) + e^{i\alpha}(B + C) \right| \le \left| A + e^{i\alpha}B \right|. \tag{2.7}$$

The above lemma is due to Aziz and Shah[4].

### 3. Proofs of Theorems

**Proof of the Theorem 1.5.** Since P(z) is a polynomial of degree n,  $P(z) \neq 0$  in  $|z| < k, k \geq 1$ , and  $Q(z) = z^n P(\frac{1}{z})$ . Therefore, for each  $\alpha, 0 \leq \alpha < 2\pi, F(z) = Q(z) + e^{i\alpha}P(z)$  is a polynomial of degree n and we have

$$F^{(s)}(z) = Q^{(s)}(z) + e^{i\alpha} P^{(s)}(z),$$

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which is clearly a polynomial of degree  $n-s, 1 \leq s < n$ . By the repeated application of inequality (1.2), we have for each r > 0,

$$\int_{0}^{2\pi} |Q^{(s)}(e^{i\theta}) + e^{i\alpha} P^{(s)}(e^{i\theta})|^{r} d\theta 
\leq (n - s + 1)^{r} \int_{0}^{2\pi} |Q^{(s-1)}(e^{i\theta}) + e^{i\alpha} P^{(s-1)}(e^{i\theta})|^{r} d\theta 
\leq (n - s + 1)^{r} (n - s + 2)^{r} \int_{0}^{2\pi} |Q^{(s-2)}(e^{i\theta}) + e^{i\alpha} P^{(s-2)}(e^{i\theta})|^{r} d\theta 
\cdot 
\cdot 
\leq (n - s + 1)^{r} (n - s + 2)^{r} \dots (n - 1)^{r} \int_{0}^{2\pi} |Q'(e^{i\theta}) + e^{i\alpha} P'(e^{i\theta})|^{r} d\theta.$$
(3.1)

Integrating inequality (3.1) with respect to  $\alpha$  over  $[0, 2\pi]$  and using inequality (2.3) of Lemma 2.2, we get

$$\int_{0}^{2\pi} \int_{0}^{2\pi} \left| Q^{(s)}(e^{i\theta}) + e^{i\alpha} P^{(s)}(e^{i\theta}) \right|^{r} d\theta d\alpha 
\leq 2\pi (n - s + 1)^{r} (n - s + 2)^{r} \dots (n - 1)^{r} n^{r} \int_{0}^{2\pi} \left| P(e^{i\theta}) \right|^{r} d\theta.$$
(3.2)

Now, from inequality (2.4) of Lemma 2.3, it easily follows that

$$\psi_{k,s} \left\{ \left| P^{(s)}(e^{i\theta}) \right| + \frac{mn(n-1)\dots(n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})} \right\} \\
\leq \left| Q^{(s)}(e^{i\theta}) \right| - \frac{mn(n-1)\dots(n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})}.$$
(3.3)

Taking  $A = |Q^{(s)}(e^{i\theta})|, B = |P^{(s)}(e^{i\theta})|, C = \frac{mn(n-1)...(n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})}$ and noting that  $\psi_{k,s} \ge k^s \ge 1, 1 \le s < n$ , so that by (3.3)

$$B+C \le \psi_{k,s}(B+C) \le A-C \le A$$
,

we get from Lemma 2.4 that

$$\left| \left\{ |Q^{(s)}(e^{i\theta})| - \frac{mn(n-1)\dots(n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})} \right\} \right. \\
+ e^{i\alpha} \left\{ |P^{(s)}(e^{i\theta})| + \frac{mn(n-1)\dots(n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})} \right\} \\
\leq \left| \left| Q^{(s)}(e^{i\theta}) \right| + e^{i\alpha} \left| P^{(s)}(e^{i\theta}) \right| \right|.$$

This implies for each r > 0,

$$\int_0^{2\pi} \left| F(\theta) + e^{i\alpha} G(\theta) \right|^r d\alpha \le \int_0^{2\pi} \left| \left| Q^{(s)}(e^{i\theta}) \right| + e^{i\alpha} \left| P^{(s)}(e^{i\theta}) \right| \right|^r d\alpha, \quad (3.4)$$

where

$$F(\theta) = \left| Q^{(s)}(e^{i\theta}) \right| - \frac{mn(n-1)\dots(n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})}$$

and

$$G(\theta) = \left| P^{(s)}(e^{i\theta}) \right| + \frac{mn(n-1)\dots(n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})}.$$

Integrating inequality (3.4) with respect to  $\theta$  on  $[0, 2\pi]$  and using inequality (3.2), we obtain

$$\frac{1}{2\pi} \int_{0}^{2\pi} \int_{0}^{2\pi} \left| F(\theta) + e^{i\alpha} G(\theta) \right|^{r} d\alpha d\theta$$

$$\leq \frac{1}{2\pi} \int_{0}^{2\pi} \int_{0}^{2\pi} \left| \left| Q^{(s)}(e^{i\theta}) \right| + e^{i\alpha} \left| P^{(s)}(e^{i\theta}) \right| \right|^{r} d\alpha d\theta$$

$$= \frac{1}{2\pi} \int_{0}^{2\pi} \int_{0}^{2\pi} \left| Q^{(s)}(e^{i\theta}) + e^{i\alpha} P^{(s)}(e^{i\theta}) \right|^{r} d\alpha d\theta$$

$$\leq (n - s + 1)^{r} (n - s + 2)^{r} \dots (n - 1)^{r} n^{r} \int_{0}^{2\pi} \left| P(e^{i\theta}) \right|^{r} d\theta.$$
(3.5)

Now for every real number  $\alpha$  and  $t_1 \geq t_2 \geq 1$ , we have

$$|t_1 + e^{i\alpha}| \ge |t_2 + e^{i\alpha}|,$$

which implies for every r > 0,

$$\int_0^{2\pi} |t_1 + e^{i\alpha}|^r d\alpha \ge \int_0^{2\pi} |t_2 + e^{i\alpha}|^r d\alpha.$$

If  $G(\theta) \neq 0$ , we take  $t_1 = \left| \frac{F(\theta)}{G(\theta)} \right|$  and  $t_2 = \psi_{k,s}$ , then from (3.3) and noting that  $\psi_{k,s} \geq 1$ , we have  $t_1 \geq t_2 \geq 1$ , hence

$$\int_{0}^{2\pi} \left| F(\theta) + e^{i\alpha} G(\theta) \right|^{r} d\alpha = |G(\theta)|^{r} \int_{0}^{2\pi} \left| \frac{F(\theta)}{G(\theta)} + e^{i\alpha} \right|^{r} d\alpha$$

$$= |G(\theta)|^{r} \int_{0}^{2\pi} \left| \left| \frac{F(\theta)}{G(\theta)} \right| + e^{i\alpha} \right|^{r} d\alpha$$

$$\ge |G(\theta)|^{r} \int_{0}^{2\pi} \left| \psi_{k,s} + e^{i\alpha} \right|^{r} d\alpha$$

$$= \left\{ \left| P^{(s)}(e^{i\theta}) \right| + \frac{mn(n-1)\dots(n-s+1)\psi_{k,s}}{k^{n}(1+\psi_{k,s})} \right\}^{r}$$

$$\int_{0}^{2\pi} \left| \psi_{k,s} + e^{i\alpha} \right|^{r} d\alpha. \tag{3.6}$$

For  $G(\theta) = 0$ , this inequality is trivially true. Using this in (3.5), it follows for each r > 0,

$$\int_{0}^{2\pi} \left\{ \left| P^{(s)}(e^{i\theta}) \right| + \frac{mn(n-1)\dots(n-s+1)\psi_{k,s}}{k^{n}(1+\psi_{k,s})} \right\}^{r} d\theta \\
\leq \frac{(n-s+1)^{r}(n-s+2)^{r}\dots(n-1)^{r}n^{r}}{\frac{1}{2\pi} \int_{0}^{2\pi} \left| \psi_{k,s} + e^{i\alpha} \right|^{r} d\alpha} \int_{0}^{2\pi} \left| P(e^{i\theta}) \right|^{r} d\theta. \tag{3.7}$$

Now using the fact that for every  $\beta$  with  $|\beta| \leq 1$ ,

$$\left| P^{(s)}(e^{i\theta}) + \frac{\beta mn(n-1)\dots(n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})} \right| \le \left| P^{(s)}(e^{i\theta}) \right| + \frac{mn(n-1)\dots(n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})},$$

the desired result follows from (3.7).

**Proof of the Theorem 1.1** Making  $r \to \infty$  and choosing the argument of  $\beta$  suitably with  $|\beta| = 1$  in (1.12), Theorem 1.1 follows.

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