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Deterministic Fuzzy Automaton on Subclasses of Fuzzy Regular ω-Languages

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Abstract. In formal language theory, we are mainly interested in the natural language computational aspects of ω-languages. Therefore in this respect it is convenient to consider fuzzy ω-languages. In this paper, we introduce two subclasses of fuzzy regular ω-languages called fuzzy n-local ω-languages and Büchi fuzzy n-local ω-languages, and give some closure properties for those subclasses. We define a deterministic fuzzy automaton acceptance conditions on fuzzy ω-languages and fuzzy n-local automaton. The relationship between deterministic fuzzy n-local automaton and two subclasses of fuzzy regular ω-languages are established and proved that every fuzzy ω-language accepted by a deterministic fuzzy automaton in 2-mode is a projection of a Büchi fuzzy 2-local ω-language.

Keywords: Fuzzy set, Local ω-Language, Deterministic fuzzy automaton, Fuzzy regular ω-Languages.

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1. Introduction

In the classical formal language theory one mostly considers languages to consist of finite words because one mostly considers finite pieces of text and finite sequences of actions. This situation has changed quite visibly in recent years. Motivated mainly by the considerations concerning semantics of programming languages and by the need to understand the behaviour of concurrent systems, the notion of an infinite word became popular and important within theoretical computer science. Finite automata are the best known computational devices. They are used to model a number of formal systems. Automata theory is closely related to formal language theory as the automata are often classified by the class of formal languages they are able to recognize. Moreover, it is widely known that finite automata can solve a large number of problems in computer science, among which are the modeling of reactive systems, the design of hardware digital systems, compilers, and the description of natural languages. Fuzzy set was introduced by Zadeh [21] and it has application in many fields of science and engineering. Fuzzy automaton was introduced by Wee in [18]. Fuzzy automata provide a reliable formal base for the theory of computing with words. Formal languages are precise while natural languages are quite imprecise. To reduce a gap between these two constructs, it becomes advantageous to introduce fuzziness into the structures of formal languages. This leads to the concept of fuzzy languages. In [18] Wee initiated the studies of fuzzy languages accepted by fuzzy automata. More recent development of algebraic theory of fuzzy automata and fuzzy languages can be found in book by Mordeson and Malik [5, 7, 8, 9, 11, 12, 14, 16, 17, 19, 20]. Berstel and Pin [3] have defined local automata and shown that a language is local if and only if it is accepted by a local automaton. Béal [2] introduced a more general definition of local automata. Caron [4] has made use of equivalent definition in order to generalize the result stated by Berstel and Pin [15]. D.S.Malik et al [13] and S.Gnanasekaran [6] studied the closure properties of fuzzy regular languages and fuzzy local languages on finitary case. In [10] studied the Fuzzy ω-Automata as an accepting device for Fuzzy ω-languages. In [1] we initiated the studies of fuzzy 2-local ω-languages, B*ü*chi fuzzy 2-local ω-languages and the relationship between deterministic fuzzy 2-local automaton and fuzzy 2-local ω-languages only.

The present paper is extended to fuzzy 2-local ω-languages to fuzzy n-local ω-languages and B*ü*chi fuzzy 2-local ω-languages to B*ü*chi fuzzy n-local ω-languages and we also establish the relationship between deterministic fuzzy n-local automaton and two subclasses of fuzzy regular ω-languages. The basic definition of this paper is given in Section 2. In Section 3, two subclasses of fuzzy regular ω-languages, namely fuzzy n-local ω-languages and B*ü*chi fuzzy n-local ω-languages are introduced. Further, some closure properties of these

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classes of languages under intersection and union are presented. In Section 4, a deterministic fuzzy automaton acceptance conditions on fuzzy ω-languages, fuzzy n-local automaton, are introduced and two subclasses of fuzzy regular ω-languages is characterized by fuzzy automata. We also proved that if $L \in \mathcal{L}_2$, then there exists a projection f and $L \in 2\text{-}\mathcal{L}_B^\omega$ such that $f(L_1) = L$. Finally, conclusion are discussed in section 5.

2. Preliminaries

In this section, some basic concepts on fuzzy set, local ω-languages, deterministic fuzzy automata are recalled.

Suppose that X is a universal set. A fuzzy set A, or rather a fuzzy subset A of X, is defined by a function assigning to each element $x\ of\ X$ a value $A(x)$ in the real unit closed interval $[0, 1]$. Such a function is called a membership function, which is a generalization of the characteristic function associated to a crisp subset of X. The value $A(x)$ characterizes the degree of membership of x in A. The collection of all fuzzy subsets of X is denoted by $\mathcal{F}(X)$. For any $A, B \in \mathcal{F}(X)$, we say that A is contained in B (or B contains A), denoted by $A \subseteq B$, if $A(x) \leq B(x)$ for all $x \in X$. We say that $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$. A fuzzy set is said to be empty if its membership function is identically zero on X. We use $\emptyset$ to denote the empty fuzzy set. For any family λ_i, $i \in I$, of elements of [0, 1], we write $\vee_{i\in I}\lambda_i$ or $\vee\{\lambda_i \ \ |i \in I\}$ for the supremum of $\{\lambda_i \ \ |i \in I\}$, and $\wedge_{i\in I}\lambda_i$ or $\wedge\{\lambda_i \ \ |i \in I\}$ for the infimum. In particular, if I is finite, then $\vee_{i\in I}\lambda_i$ and $\wedge_{i\in I}\lambda_i$ are the greatest element and the least element of $\{\lambda_i \ \ |i \in I\}$ respectively. Given $A, B \in \mathcal{F}(X)$, the union of A and B, denoted $A \cup B$, is defined by the membership function $(A \cup B)(x) = A(x) \vee B(x)$ for all $x \in X$, the intersection of A and B, denoted $A \cap B$, is given by the membership function $(A \cap B)(x) = A(x) \wedge B(x)$ for all $x \in X$. Let $k \in [0, 1]$ and $A \in \mathcal{F}(X)$. The scale product kA of k and A is defined by $(kA)(x) = k \wedge A(x)$ for every $x \in X$, which is again a fuzzy subset of X.

Let Σ be a finite alphabet and Σ^* be the set of all finite words over Σ. We define the empty word by ϵ. For each $u \in \Sigma^*$, we denote by $P_n(u)$, the prefix of u of length n and by $F_n(u)$, the set of all factors of u of length n. We denote by $S_n(u)$, the suffix of u of length n. An infinite word α over Σ is a function $\alpha : N \to \Sigma$ from the set N of all positive integers to Σ. We represent the infinite word α as $\alpha = a_1a_2 \cdots$ where $\alpha(i) = a_i \in \Sigma$, for all i. We denote by Σ^ω, the set of all infinite words over Σ. For $\alpha \in \Sigma^\omega$, $inf_n(\alpha)$ denotes the set of all elements of $F_n(\alpha)$, each of which repeats infinite number of times in α. A projection f is a mapping from Σ to Γ, for some alphabets Σ and Γ, defined by $f(a) \in \Gamma$. A projection map f is extended in a usual fashion to Σ^ω as follows: $f(\epsilon) = \epsilon$, $f(au) = f(a)f(u)$, for $a \in \Sigma$ and $u \in \Sigma^\omega$.

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A deterministic fuzzy automaton is a tuple $M = (Q, \Sigma, \delta, q_0, F)$ where Q is a finite non-empty set of states, Σ is a finite alphabet, $\delta : Q \times \Sigma \to Q$ is a transition function, $q_0 \in Q$ is the initial state and F is a fuzzy subset of Q. The language accepted by M is the fuzzy subset $L(M)$ of Σ^*, defined by $L(M)(u) = F(\delta(q_0, u))$. A fuzzy language L is said to be a fuzzy regular language if there exists a deterministic fuzzy automaton M such that $L = L(M)$.

3. Sub Classes of Fuzzy Regular ω-Languages

In this section, we define two subclasses of fuzzy regular ω-languages called fuzzy n-local ω-languages and Büchi fuzzy n-local ω-languages, and some closure properties for those subclasses are studied.

Definition 3.1. A fuzzy ω-language L over Σ is called a fuzzy n-local, if there exists a pair or fuzzy local system $S = (\lambda_1, \lambda_2)$ where λ_1 is a fuzzy subset of Σ^{n-1} and λ_2 is fuzzy subsets of Σ^n such that $L(\alpha) = \lambda_1(P_{n-1}(\alpha)) \bigwedge (\bigwedge_{x \in F_n(\alpha)} \lambda_2(x))$, $\forall \alpha \in \Sigma^\omega$ and we write $L = L^\omega_L(S)$. The class of all fuzzy n-local ω-languages is denoted by $n\text{-}\mathcal{L}^\omega_L$. The language $L \subseteq \Sigma^\omega$ is called a local, if $L \in n\text{-}\mathcal{L}^\omega_L$, for some positive integer n. The class of all fuzzy local ω-languages is denoted by $\mathcal{L}^\omega_L$.

Remark 3.2. The class of all local ω-languages is a proper subset of the class of all fuzzy local ω-languages.

Theorem 3.3. *If L_1 and L_2 are fuzzy n-local ω-languages over Σ, then $L_1 \cap L_2$ is a fuzzy n-local ω-language over Σ.*

Proof. If L_1 and L_2 are fuzzy n-local ω-languages, then $L_1 = L^\omega(S_1)$ for some fuzzy local system $S_1 = (\lambda_1^{'}, \lambda_2^{'})$ and $L_2 = L^\omega(S_2)$ for some fuzzy local system $S_2 = (\lambda_1^{''}, \lambda_2^{''})$. Consider the local system $S = (\lambda_1, \lambda_2)$ where $\lambda_1 = \lambda_1^{'} \wedge \lambda_1^{''}$ and $\lambda_2 = \lambda_2^{'} \wedge \lambda_2^{''}$. Let us show that $L^\omega(S) = L^\omega(S_1) \cap L^\omega(S_2)\,(= L_1 \cap L_2)$. For $\alpha \in \Sigma^\omega$,

$$
\begin{aligned}
L^\omega(S)(\alpha) &= \lambda_1(P_{n-1}(\alpha)) \wedge \left(\wedge_{x \in F_n(\alpha)} \lambda_2(x)\right) \\
&= \left((\lambda_1^{'} \wedge \lambda_1^{''})(P_{n-1}(\alpha))\right) \wedge \left(\wedge_{x \in F_n(\alpha)} (\lambda_2^{'} \wedge \lambda_2^{''})(x)\right) \\
&= \left(\lambda_1^{'}(P_{n-1}(\alpha)) \wedge \lambda_1^{''}(P_{n-1}(\alpha))\right) \wedge \left(\wedge_{x \in F_n(\alpha)} (\lambda_2^{'}(x) \wedge \lambda_2^{''}(x))\right) \\
&= \left(\lambda_1^{'}(P_{n-1}(\alpha)) \wedge (\wedge_{x \in F_n(\alpha)} \lambda_2^{'}(x))\right) \\
&\wedge \left(\lambda_2^{''}(P_{n-1}(\alpha)) \wedge (\wedge_{x \in F_n(\alpha)} \lambda_2^{''}(x))\right) \\
&= L^\omega(S_1)(\alpha) \wedge L^\omega(S_2)(\alpha) \\
&= L_1(\alpha) \wedge L_2(\alpha) \\
&= (L_1 \cap L_2)(\alpha) \\
\text{Thus } L^\omega(S) &= L_1 \cap L_2.
\end{aligned}
$$

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Therefore $L_1 \cap L_2$ is a fuzzy n-local ω-language. □

Note that union of two fuzzy n-local ω-languages over Σ need not be a fuzzy n-local ω-languages.

EXAMPLE 3.4. Consider the fuzzy 2-local ω-languages L_1 and L_2 over $\Sigma = \{a, b, c\}$ with membership function,

$$L_1(\alpha) = \begin{cases} 0.3 & \text{if } \alpha = a(bc)^\omega, \\ 0 & \text{otherwise.} \end{cases}$$

and

$$L_2(\alpha) = \begin{cases} 0.4 & \text{if } \alpha = a^\omega, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore

$$(L_1 \cup L_2)(\alpha) = \begin{cases} 0.4 & \text{if } \alpha = a^\omega, \\ 0.3 & \text{if } \alpha = a(bc)^\omega, \\ 0 & \text{otherwise.} \end{cases}$$

If $L_1 \cup L_2$ is a fuzzy local ω-language, then there exists a fuzzy local system $S = (\lambda_1, \lambda_2)$ such that $L_1 \cup L_2 = L^\omega(S)$. Here $\lambda_1(a)$,$\lambda_2(aa)$,$\lambda_2(ab)$, $\lambda_2(bc)$, $\lambda_2(cb)$ are all grater than zero, but $L_1(a^n(bc)^\omega = 0$ and $L_2(a^n(bc)^\omega = 0$ which is a contradiction. Therefore $(L_1 \cup L_2)(a^n(bc)^\omega \neq \phi, n \geq 1$.

Theorem 3.5. *If Σ_1 and Σ_2 are two disjoint subsets of the alphabet Σ whose union is Σ and if $L_1 \subseteq \Sigma_1^\omega$ and $L_2 \subseteq \Sigma_2^\omega$ are fuzzy n-local ω-languages, then $L_1 \cup L_2$ is a fuzzy n-local ω-language over Σ.*

Proof. Since L_1 and L_2 are fuzzy n-local ω-languages over Σ_1^ω and Σ_2^ω, we have $L_1 = L^\omega(S_1)$ for some fuzzy local system $S_1 = (\lambda_1^{'}, \lambda_2^{'})$ and $L_2 = L^\omega(S_2)$ for some fuzzy local system $S_2 = (\lambda_1^{''}, \lambda_2^{''})$. Consider the local system $S = (\lambda_1, \lambda_2)$ where $\lambda_1 = \lambda_1^{'} \vee \lambda_1^{''}$ and $\lambda_2 = \lambda_2^{'} \vee \lambda_2^{''}$. Here L_1 and L_2 are defined on disjoint domains Σ_1^ω and Σ_2^ω, respectively. We can view them as having same domain Σ^ω by defining $L_1(\alpha) = 0$ for every $\alpha \in \Sigma^\omega - \Sigma_1^\omega$ and $L_2(\alpha) = 0$ for every $\alpha \in \Sigma^\omega - \Sigma_2^\omega$. Let us show that $L^\omega(S) = L^\omega(S_1) \cup L^\omega(S_2)\,(= L_1 \cup L_2)$. For

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$\alpha \in \Sigma^{\omega}$,

$$\begin{aligned}
L^{\omega}(S)(\alpha) &= \lambda_1(P_{n-1}(\alpha)) \wedge (\wedge_{x\in F_n(\alpha)}\lambda_2(x)) \\
&= \Big((\lambda_1^{'} \vee \lambda_1^{''})(P_{n-1}(\alpha))\Big) \wedge \Big(\wedge_{x\in F_n(\alpha)}(\lambda_2^{'} \vee \lambda_2^{''})(x)\Big) \\
&= \Big(\lambda_1^{'}(P_{n-1}(\alpha)) \vee \lambda_1^{''}(P_{n-1}(\alpha))\Big) \wedge \Big(\wedge_{x\in F_n(\alpha)}(\lambda_2^{'}(x) \vee \lambda_2^{''}(x))\Big) \\
&= \Big(\lambda_1^{'}(P_{n-1}(\alpha)) \wedge (\wedge_{x\in F_n(\alpha)}\lambda_2^{'}(x))\Big) \\
&\vee \Big(\lambda_1^{''}(P_{n-1}(\alpha)) \wedge (\wedge_{x\in F_n(\alpha)}\lambda_2^{''}(x)\Big) \\
&= L^{\omega}(S_1)(\alpha) \vee L^{\omega}(S_2)(\alpha) \\
&= L_1(\alpha) \vee L_2(\alpha) \\
&= (L_1 \cup L_2)(\alpha) \\
\text{Thus } L^{\omega}(S) &= L_1 \cup L_2.
\end{aligned}$$

Therefore $L_1 \cup L_2$ is a fuzzy n-local ω-language. □

Definition 3.6. A fuzzy ω-language L over Σ is called a Büchi fuzzy n-local, if there exists a triple $S = (\lambda_1, \lambda_2, \lambda_3)$ where λ_1 is a fuzzy subset of Σ^{n-1} and λ_2, λ_3 are fuzzy subsets of Σ^n such that $\lambda_3 \leq \lambda_2$ and $L(\alpha) = \lambda_1(P_{n-1}(\alpha)) \bigwedge(\bigwedge_{x\in F_n(\alpha)} \lambda_2(x)) \bigwedge(\bigvee_{x\in inf_n(\alpha)} \lambda_3(x)), \forall \alpha \in \Sigma^{\omega}$ and we write $L = L_B^{\omega}(S)$. The class of all Büchi fuzzy n-local ω-languages is denoted by $n\text{-}\mathcal{L}_B^{\omega}$. The language $L \subseteq \Sigma^{\omega}$ is called a Büchi fuzzy local, if $L \in n\text{-}\mathcal{L}_B^{\omega}$, for some positive integer n. The class of all Büchi fuzzy local ω-languages is denoted by $\mathcal{L}_B^{\omega}$.

Example 3.7. Consider a fuzzy ω-language L whose membership function is given by

$$L(\alpha) = \begin{cases} 0.5 & \text{if } \alpha = a^{+}b^{\omega}, \\ 0 & \text{otherwise.} \end{cases}$$

Let us consider the fuzzy local system $S = (\lambda_1, \lambda_2, \lambda_3)$, where

$$\lambda_1(x) = \begin{cases} 0.5 & \text{if } x = a, \\ 0 & \text{otherwise.} \end{cases}$$

$$\lambda_2(x) = \begin{cases} 0.6 & \text{if } x = ab, \\ 0.5 & \text{if } x \in \{bb, aa\}, \\ 0 & \text{otherwise.} \end{cases}$$

and

$$\lambda_3(x) = \begin{cases} 0.5 & \text{if } x = bb, \\ 0 & \text{otherwise.} \end{cases}$$

Then $L = L_B^{\omega}(S)$ and therefore L is a Büchi fuzzy local ω-language.

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Remark 3.8. The class $\mathcal{L}_L^\omega$ of all fuzzy local ω-languages is a subset of the class of all B*ü*chi fuzzy local ω-languages $\mathcal{L}_B^\omega$.

Example 3.9. The language L in Example 3.7 is a B*ü*chi fuzzy local ω-language. But L is not a fuzzy local ω-language, otherwise, $a^\omega \in L$. Therefore the class $\mathcal{L}_L^\omega$ of all fuzzy local ω-languages is a subset of the class of all B*ü*chi fuzzy local ω-languages $\mathcal{L}_B^\omega$..

4. Fuzzy Local Automata

In this section we define two different modes of acceptance for fuzzy ω-language by deterministic fuzzy automata and also establish relationships between the classes of fuzzy ω-languages.

Definition 4.1. A deterministic fuzzy automaton is a tuple $M = (Q, \Sigma, \delta, q_0, F)$ where Q is a finite non-empty set of states, Σ is a finite alphabet, $\delta : Q \times \Sigma \to Q$ is a transition function, $q_0 \in Q$ is the initial state and F is a fuzzy subset of Q. If $\alpha = a_1a_2a_3 \cdots \in \Sigma^\omega$, the sequence $\rho = \{q_n\}_{n=0}^\infty$ of states from Q is called a run or path of M for α, if for $n \geq 1$, $q_n = \delta(q_{n-1}, a_n)$. We write $\rho : q_0 \overset{a_1}{\to} q_1 \overset{a_2}{\to} q_2 \to \cdots$. The range of ρ , denoted by $\mathrm{ran}(\rho)$, is the set $\mathrm{ran}(\rho) = \{q_0, q_1, q_2 \cdots\}$ and $\inf(\rho)$ denotes the set of states which appear infinitely often in ρ. We say that for $i \in \{1', 2\}$, the acceptance value of ρ on α in i- mode is $acc_i(\rho, \alpha)$ where, $(i.)$ $acc_{1'}(\rho, \alpha) = \bigwedge_{q \in \mathrm{ran}(\rho)} F(q)$ and $(ii.)$ $acc_2(\rho, \alpha) = \bigvee_{q \in \inf(\rho)} F(q)$.
The fuzzy ω-language $L \subseteq \Sigma^\omega$ is said to be accepted by M in i-mode, is the fuzzy subset $L_i^\omega(M)$ of Σ^ω defined by $L_i^\omega(M)(\alpha) = acc_i(\rho, \alpha)$. We denote the class of all fuzzy ω-languages accepted by deterministic fuzzy automata by $\mathcal{L}_i$. A fuzzy ω-language L is said to be a fuzzy regular ω-language if there exists a deterministic fuzzy automaton M such that $L = L_2^\omega(M)$.

Definition 4.2. A deterministic fuzzy automaton $M = (Q, \Sigma, \delta, q_0, F)$ is said to be n-local if for every $u \in \Sigma^n$, the set $\{\delta(q, u) : q \in Q\}$ contains at most one element. We denote the class of all fuzzy ω-languages accepted by deterministic fuzzy n-local automata in i-mode by n-$\mathcal{L}_i$.

Theorem 4.3. *$L \subseteq \Sigma^\omega$ is a fuzzy n-local ω-languages if and only if L is accepted by a fuzzy n-local automaton in $1'$-mode.*

Proof. Let $L \in n$-$\mathcal{L}_L^\omega$. Then there exists a pair $S = (\lambda_1, \lambda_2)$ where λ_1 is a fuzzy subset of Σ^{n-1} and λ_2 is a fuzzy subset of Σ^n such that $L(\alpha) = \lambda_1(P_{n-1}(\alpha)) \bigwedge(\bigwedge_{x \in F_n(\alpha)} \lambda_2(x)), \forall \alpha \in \Sigma^\omega$. Consider the deterministic fuzzy automaton $M = (Q, \Sigma, \delta, q_0, F)$ where

$$Q = \{\{[\lambda]\} \cup \{[a_1], [a_1a_2], \cdots, [a_1a_2 \cdots a_{n-1}] : \lambda_1(a_1a_2 \cdots a_{n-1}) \neq 0\} \cup \{[u] : \lambda_2(u) \neq 0\}\},$$

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$q_0 = \{[\lambda]\}$,

δ is defined as follows:

For all $a_1a_2\cdots a_{n-1} \in \Sigma^{n-1}$ such that $\lambda_1(a_1a_2\cdots a_{n-1}) \neq 0$ and $a \in \Sigma$,

$$\begin{aligned}\delta([\lambda], a_1) &= [a_1],\\ \delta([a_1a_2\cdots a_i], a_{i+1}) &= [a_1a_2\cdots a_{i+1}], i = 1, \cdots, n-2 \text{ and}\\ \delta([a_1a_2\cdots a_{n-1}], a) &= [a_1a_2\cdots a_{n-1}a], \text{if } \lambda_2(a_1a_2\cdots a_{n-1}a) \neq 0.\end{aligned}$$

For all $a_1a_2\cdots a_n \in \Sigma^n$ such that $\lambda_2(a_1a_2\cdots a_n) \neq 0$ and $a \in \Sigma$,

$$\delta([a_1a_2\cdots a_n], a) = [a_2a_3\cdots a_na], \text{if } \lambda_2(a_2a_3\cdots a_na) \neq 0.$$

and F is defined by

$$F([u]) = \begin{cases}\lambda_1(u) & \text{if } u \in \Sigma^{n-1},\\ \lambda_1(u) & \text{if } u \in \Sigma^n.\end{cases}$$

Then M is n-local and $L = L^{\omega}_{1'}(M)$. Therefore $L \in n\text{-}\mathcal{L}_{1'}$.
Conversely, assume that $L \in \mathcal{L}_{1'}$. Consider the pair $S = (\lambda_1, \lambda_2)$ where, for each $u \in \Sigma^n$,

$$\lambda_1(u) = \begin{cases}1 & \text{if } \delta(q_0, u) \in Q,\\ 0 & \text{otherwise.}\end{cases}$$

and for each $u \in \Sigma^{n+1}$,

$$\lambda_2(u) = \begin{cases}F(\delta(q, u)) & \text{if } \delta(q, u) \in Q,\\ 0 & \text{otherwise.}\end{cases}$$

Then $L = L^{\omega}_{L}(S)$ and therefore $L \in n\text{-}\mathcal{L}^{\omega}_{L}$. Hence $L \in n\text{-}\mathcal{L}_{1'} = L \in n\text{-}\mathcal{L}^{\omega}_{L}$, $n \geq 1$. □

Theorem 4.4. *$L \subseteq \Sigma^{\omega}$ is a Büchi fuzzy local ω-languages if and only if L is accepted by a fuzzy local automaton in 2-mode.*

Proof. Let $L \in n\text{-}\mathcal{L}^{\omega}_{B}$. Then there exists a triple $S = (\lambda_1, \lambda_2, \lambda_3)$ where λ_1 is a fuzzy subset of Σ^{n-1} and λ_2, λ_3 are fuzzy subset of Σ^n such that $\lambda_3 \leq \lambda_2$ and $L(\alpha) = \lambda_1(P_{n-1}(\alpha)) \bigwedge(\bigwedge_{x\in F_n(\alpha)} \lambda_2(x)) \bigwedge(\bigvee_{x\in inf_n(\alpha)} \lambda_3(x)), \forall\alpha \in \Sigma^{\omega}$. Consider the deterministic fuzzy automaton $M = (Q, \Sigma, \delta, q_0, F)$ where

$Q = \{\{[\lambda]\} \cup \{[a_1], [a_1a_2], \cdots, [a_1a_2\cdots a_{n-1}] : \lambda_1(a_1a_2\cdots a_{n-1}) \neq 0\} \cup \{[u] : \lambda_2(u) \neq 0\}\}$,

$q_0 = \{[\lambda]\}$,

δ is defined as follows:

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For all $a_1a_2\cdots a_{n-1} \in \Sigma^{n-1}$ such that $\lambda_1(a_1a_2\cdots a_{n-1}) \neq 0$ and $a \in \Sigma$,

$$\begin{aligned}\delta([\lambda], a_1) &= [a_1],\\ \delta([a_1a_2\cdots a_i], a_{i+1}) &= [a_1a_2\cdots a_{i+1}], i = 1, \cdots, n-2 \text{ and}\\ \delta([a_1a_2\cdots a_{n-1}], a) &= [a_1a_2\cdots a_{n-1}a], \text{if } \lambda_2(a_1a_2\cdots a_{n-1}a) \neq 0.\end{aligned}$$

For all $a_1a_2\cdots a_n \in \Sigma^n$ such that $\lambda_2(a_1a_2\cdots a_n) \neq 0$ and $a \in \Sigma$,

$$\delta([a_1a_2\cdots a_n], a) = [a_2a_3\cdots a_na], \text{if } \lambda_2(a_2a_3\cdots a_na) \neq 0.$$

and F is defined by

$$F([u]) = \begin{cases}\lambda_3(p) & \text{if } u \in \Sigma^n,\\ 0 & \text{otherwise.}\end{cases}$$

Then M is n-local and $L = n\text{-}L_2^{\omega}(M)$. Therefore $L \in n\text{-}\mathcal{L}_2$.
Conversely, assume that $L \in n\text{-}\mathcal{L}_2$. Consider the fuzzy local system $S = (\lambda_1, \lambda_2, \lambda_2)$ where, for each $u \in \Sigma^n$,

$$\lambda_1(u) = \begin{cases}1 & \text{if } \delta(q_0, u) \in Q,\\ 0 & \text{otherwise.}\end{cases}$$

and for each $u \in \Sigma^{n+1}$,

$$\lambda_2(u) = \begin{cases}1 & \text{if u is a factor of } \alpha \in \Sigma^{\omega} \text{ for which } L^{\omega}(M)(\alpha) \neq 0\ ,\\ 0 & \text{otherwise.}\end{cases}$$

and $\lambda_3(u) = F(\delta(q, u))$, for some $q \in Q$. Then $L = L_B^{\omega}(S)$ and therefore $L \in (n+1)\text{-}\mathcal{L}_B^{\omega}$. Hence $L \in n\text{-}\mathcal{L}_2 = L \in n\text{-}\mathcal{L}_B^{\omega}$, $n \geq 1$. □

Theorem 4.5. *Every fuzzy regular ω-languages is a projection of a Büchi fuzzy local ω-languages.*

Proof. Let $L \in \mathcal{L}_2$. Consider the alphabet $\Gamma = Q \times \Sigma \times Q$. Consider the fuzzy local system $S = (\lambda_1, \lambda_2, \lambda_3)$ where λ_1 is a fuzzy subset of Γ be defined by

$$\lambda_1(q_1, a, q_2) = \begin{cases}1 & \text{if } q_1 = q_0,\\ 0 & \text{otherwise.}\end{cases}$$

λ_2 and λ_3 are the fuzzy subsets of Γ^2 be defined by

$$\lambda_2((q_1, a, q_2)(q_3, a, q_4)) = \begin{cases}1 & \text{if } q_2 = q_3,\\ 0 & \text{otherwise.}\end{cases}$$

$$\lambda_3((q_1, a, q_2)(q_3, a, q_4)) = \begin{cases}F(q_4) & \text{if } q_2 = q_3,\\ 0 & \text{otherwise.}\end{cases}$$

Let $L_1 = L^{\omega}(S)$, where S is a fuzzy local system over Σ. Then L_1 is a fuzzy Büchi 2-local ω-language. Define the projection map $f : \Gamma \longrightarrow \Sigma$ by

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$f(q_1, a, q_2) = a$. This map can be extended to Γ^ω as $f((q_1, a, q_2)(q_3, b, q_4)\ldots) = ab\ldots$. Then $f(L_1) = L$. □

5. Conclusion

In this paper, extension of fuzzy regular languages to fuzzy regular ω-languages and fuzzy 2-local ω-languages to fuzzy n-local ω-languages, B$\ddot{u}$chi fuzzy 2-local ω languages to B$\ddot{u}$chi fuzzy n-local ω-languages are carried out. Some closure properties of fuzzy n-local ω-languages under union and intersection are discussed. Deterministic fuzzy automaton acceptance conditions on fuzzy ω-languages are defined and shown that $L \in \mathcal{L}_L^\omega$ is the class of fuzzy local ω-languages if and only if L is accepted by fuzzy local automaton in $1^{'}$-mode and also proved that $L \in \mathcal{L}_B^\omega$ is the class of B$\ddot{u}$chi fuzzy local ω-languages if and only if L is accepted by fuzzy local automaton in 2-mode. Finally every fuzzy regular ω-languages is a projection of a B$\ddot{u}$chi fuzzy local ω-languages. This work can be further investigated by applying fuzzy learning automata for optimization problems.

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