

## On the 2-absorbing Submodules

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**ABSTRACT.** Let  $R$  be a commutative ring and  $M$  be an  $R$ -module. In this paper, we investigate some properties of 2-absorbing submodules of  $M$ . It is shown that  $N$  is a 2-absorbing submodule of  $M$  if and only if whenever  $IJL \subseteq N$  for some ideals  $I, J$  of  $R$  and a submodule  $L$  of  $M$ , then  $IL \subseteq N$  or  $JL \subseteq N$  or  $IJ \subseteq N :_R M$ . Also, if  $N$  is a 2-absorbing submodule of  $M$  and  $M/N$  is Noetherian, then a chain of 2-absorbing submodules of  $M$  is constructed. Furthermore, the annihilation of  $E(R/\mathfrak{p})$  is studied whenever  $0$  is a 2-absorbing submodule of  $E(R/\mathfrak{p})$ , where  $\mathfrak{p}$  is a prime ideal of  $R$  and  $E(R/\mathfrak{p})$  is an injective envelope of  $R/\mathfrak{p}$ .

**Keywords:** 2-absorbing ideal, 2-absorbing submodule, A chain of 2-absorbing submodule.

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### 1. INTRODUCTION

Throughout this paper  $R$  is a commutative ring with non-zero identity and  $M$  is an unitary  $R$ -module. We defined a submodule  $N$  of  $M$  is 2-absorbing whenever  $abm \in N$  for some  $a, b \in R$ ,  $m \in M$ , then  $am \in N$  or  $bm \in N$  or  $ab \in N :_R M$ , see for instance [1, 3, 4, 6, 7, 9, 10]. It is well known that, a submodule  $N$  of  $M$  is prime if and only if  $IL \subseteq N$  for an ideal  $I$  of  $R$  and

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a submodule  $L$  of  $M$ , then either  $L \subseteq N$  or  $I \subseteq N :_R M$ . This statement persuaded us to prove that, a submodule  $N$  of  $M$  is 2-absorbing if and only if  $IJL \subseteq N$  for some ideals  $I, J$  of  $R$  and a submodule  $L$  of  $M$ , then  $IL \subseteq N$  or  $JL \subseteq N$  or  $IJ \subseteq N :_R M$ . As a corollary of this theorem, it is shown that  $L = \{m \in M : \mathfrak{p} \subseteq r(N : m)\}$  is a 2-absorbing submodule of  $M$ , where  $N$  is a 2-absorbing submodule of  $M$  with  $r(N :_R M) = \mathfrak{p} \cap \mathfrak{q}$  for some prime ideals  $\mathfrak{p}, \mathfrak{q}$  of  $R$ . Also, it is shown that if  $M/N$  is Noetherian, then there exists a chain of 2-absorbing submodules of  $M$  that begins with  $N$ . Assume that  $E(R/\mathfrak{p})$  is an injective envelope of  $R/\mathfrak{p}$ , it is shown that if  $0$  is a 2-absorbing submodule of  $E(R/\mathfrak{p})$ , then  $r(0 :_R E(R/\mathfrak{p})) = \mathfrak{p}$  and  $0 :_R x$  is determined for all nonzero element  $x$  of  $E(R/\mathfrak{p})$ .

Now, we define the concepts that we will use later. For a submodule  $L$  of  $M$  let  $L :_R M$  denote the ideal  $\{r \in R : rM \subseteq L\}$ . Similarly, for an element  $m \in M$  let  $L :_R m$  denote the ideal  $\{r \in R : rm \in L\}$ . If  $I$  is an ideal of  $R$ , then  $r(I)$  denotes the radical of  $I$ . We say that  $\mathfrak{p} \in \text{Spec}(R)$  is an associated prime ideal of  $M$  if there exists  $m \in M$  with  $0 :_R m = \mathfrak{p}$ . The set of associated prime ideals of  $M$  is denoted by  $\text{Ass}_R(M)$ , the set of integers is denoted by  $\mathbb{Z}$ .

## 2. 2-ABSORBING SUBMODULES

Let  $N$  be a proper submodule of  $M$ . We say that  $N$  is a *2-absorbing* submodule of  $M$  if whenever  $a, b \in R$ ,  $m \in M$  and  $abm \in N$ , then  $am \in N$  or  $bm \in N$  or  $ab \in N :_R M$ .

**Lemma 2.1.** *Let  $I$  be an ideal of  $R$  and  $N$  be a 2-absorbing submodule of  $M$ . If  $a \in R$ ,  $m \in M$  and  $Iam \subseteq N$ , then  $am \in N$  or  $Im \subseteq N$  or  $Ia \subseteq N :_R M$ .*

*Proof.* Let  $am \notin N$  and  $Ia \not\subseteq N :_R M$ . Then there exists  $b \in I$  such that  $ba \notin N :_R M$ . Now,  $bam \in N$  implies that  $bm \in N$ , since  $N$  is a 2-absorbing submodule of  $M$ . We have to show that  $Im \subseteq N$ . Let  $c$  be an arbitrary element of  $I$ . Thus  $(b+c)am \in N$ . Hence, either  $(b+c)m \in N$  or  $(b+c)a \in N :_R M$ . If  $(b+c)m \in N$ , then by  $bm \in N$  it follows that  $cm \in N$ . If  $(b+c)a \in N :_R M$ , then  $ca \notin N :_R M$ , but  $cam \in N$ . Thus  $cm \in N$ . Hence, we conclude that  $Im \subseteq N$ .  $\square$

**Lemma 2.2.** *Let  $I, J$  be ideals of  $R$  and  $N$  be a 2-absorbing submodule of  $M$ . If  $m \in M$  and  $IJm \subseteq N$ , then  $Im \subseteq N$  or  $Jm \subseteq N$  or  $IJ \subseteq N :_R M$ .*

*Proof.* Let  $I \not\subseteq N :_R m$  and  $J \not\subseteq N :_R m$ . We have to show that  $IJ \subseteq N :_R M$ . Assume that  $c \in I$  and  $d \in J$ . By assumption there exists  $a \in I$  such that  $am \notin N$  but  $aJm \subseteq N$ . Now, Lemma 2.1 shows that  $aJ \subseteq N :_R M$  and so  $(I \setminus N :_R m)J \subseteq N :_R M$ , similarly there exists  $b \in J \setminus N :_R m$  such that  $Ib \subseteq N :_R M$  and also  $I(J \setminus N :_R m) \subseteq N :_R M$ . Thus we have  $ab \in N :_R M$ ,  $ad \in N :_R M$  and  $cb \in N :_R M$ . By  $a + c \in I$  and  $b + d \in J$  it follows that  $(a + c)(b + d)m \in N$ . Therefore,  $(a + c)m \in N$  or  $(b + d)m \in N$  or

$(a+c)(b+d) \in N :_R M$ . If  $(a+c)m \in N$ , then  $cm \notin N$  hence,  $c \in I \setminus N :_R m$  which implies that  $cd \in N :_R M$ . Similarly by  $(b+d)m \in N$ , we can deduce that  $cd \in N :_R M$ . If  $(a+c)(b+d) \in N :_R M$ , then  $ab + ad + cb + cd \in N :_R M$  and so  $cd \in N :_R M$ . Therefore,  $IJ \subseteq N :_R M$ .  $\square$

**Theorem 2.3.** *Let  $N$  be a proper submodule of  $M$ . The following statement are equivalent:*

- (i)  $N$  is a 2-absorbing submodule of  $M$ ;
- (ii) *If  $IJL \subseteq N$  for some ideals  $I, J$  of  $R$  and a submodule  $L$  of  $M$ , then  $IL \subseteq N$  or  $JL \subseteq N$  or  $IJ \subseteq N :_R M$ .*

*Proof.* (ii)  $\Rightarrow$  (i) is obvious. To prove (i)  $\Rightarrow$  (ii), assume that  $IJL \subseteq N$  for some ideals  $I, J$  of  $R$  and a submodule  $L$  of  $M$  and  $IJ \not\subseteq N :_R M$ . Then by Lemma 2.2 for all  $m \in L$  either  $Im \subseteq N$  or  $Jm \subseteq N$ . If  $Im \subseteq N$ , for all  $m \in L$  we are done. Similarly if  $Jm \subseteq N$ , for all  $m \in L$  we are done. Suppose that  $m, m' \in L$  are such that  $Im \not\subseteq N$  and  $Jm' \not\subseteq N$ . Thus  $Jm \subseteq N$  and  $Im' \subseteq N$ . Since  $IJ(m + m') \subseteq N$  we have either  $I(m + m') \subseteq N$  or  $J(m + m') \subseteq N$ . By  $I(m + m') \subseteq N$ , it follows that  $Im \subseteq N$  which is a contradiction, similarly by  $J(m + m') \subseteq N$  we get a contradiction. Therefore either  $IL \subseteq N$  or  $JL \subseteq N$ .  $\square$

A submodule  $N$  of  $M$  is called strongly 2-absorbing if it satisfies in condition (ii), see [5]. Therefore, Theorem 2.3 shows that  $N$  is a 2-absorbing submodule of  $M$  if and only if  $N$  is a strongly 2-absorbing submodule of  $M$ .

**Corollary 2.4.** *Let  $M$  be an  $R$ -module and  $N$  be a 2-absorbing submodule of  $M$ . Then  $N :_M I = \{m \in M : Im \subseteq N\}$  is a 2-absorbing submodules of  $M$  for all ideal  $I$  of  $R$ . Furthermore  $N :_M I^n = N :_M I^{n+1}$ , for all  $n \geq 2$ .*

*Proof.* Let  $I$  be an ideal of  $R$ ,  $a, b \in R$ ,  $m \in M$  and  $abm \in N :_M I$ . Thus  $Iabm \subseteq N$ . Hence,  $Im \subseteq N$  or  $Iab \subseteq N :_R M$  or  $abm \in N$ , by Lemma 2.2. If  $Im \subseteq N$  we are done. If  $Iab \subseteq N :_R M$ , then  $ab \in (N :_R M) :_R I = (N :_M I) :_R M$ . If  $abm \in N$ , then  $am \in N$  or  $bm \in N$  or  $ab \in N :_R M$ . Thus  $Iam \subseteq N$  or  $Ibm \subseteq N$  or  $Iab \subseteq N :_R M$  which complete the proof.

For the second statement, it is enough to show that  $N :_M I^2 = N :_M I^3$ . It is clear that  $N :_M I^2 \subseteq N :_M I^3$ . Let  $m \in N :_M I^3$ . Then  $I^3m \subseteq N$ . Now, by Lemma 2.2, we have  $I^2m \subseteq N$  or  $Im \subseteq N$  or  $I^3 \subseteq N :_R M$ . If  $I^2m \subseteq N$  or  $Im \subseteq N$ , we are done. If  $I^3 \subseteq N :_R M$ , then  $I^2 \subseteq N :_R M$  since  $N :_R M$  is a 2-absorbing ideal of  $R$  by [9, Theorem 2.3].  $\square$

It is clear that,  $n\mathbb{Z}$  is a 2-absorbing ideal of  $\mathbb{Z}$  if and only if  $n = 0, p, p^2, pq$ , where  $p, q$  are distinct prime integers. It is easy to see that  $4\mathbb{Z} :_{\mathbb{Z}} 6\mathbb{Z} = 2\mathbb{Z}$  but  $4\mathbb{Z} :_{\mathbb{Z}} 36\mathbb{Z} = \mathbb{Z}$ . Hence, the equality mentioned in the Corollary 2.4, is not necessarily true for  $n = 1$ .

**Theorem 2.5.** *Let  $N$  be a 2-absorbing submodule of  $M$  such that  $r(N :_R M) = \mathfrak{p} \cap \mathfrak{q}$  where  $\mathfrak{p}$  and  $\mathfrak{q}$  are the only distinct prime ideals of  $R$  that are minimal over  $N :_R M$ . Then  $L = \{m \in M : \mathfrak{p} \subseteq r(N :_R m)\}$  is a 2-absorbing submodule of  $M$  containing  $N$ . Also, either  $r(L :_R M) = \mathfrak{q}$  or  $r(L :_R M) = \mathfrak{p}' \cap \mathfrak{q}$ , where  $\mathfrak{p}'$  is a prime ideal of  $R$  containing  $\mathfrak{p}$ .*

*Proof.* It is clear that  $L$  is a submodule of  $M$  containing  $N$ . Assume that  $a, b \in R$ ,  $m \in M$  and  $abm \in L$ . We have to show that  $am \in L$  or  $bm \in L$  or  $ab \in L :_R M$ . Since  $\mathfrak{p} \subseteq r(N :_R abm)$ , thus  $\mathfrak{p}^2 abm \subseteq N$ , by [9, Theorem 2.4] and [2, Theorem 2.4]. Therefore, by Lemma 2.1, we have  $abm \in N$  or  $\mathfrak{p}^2 m \subseteq N$  or  $\mathfrak{p}^2 ab \subseteq N :_R M$ . If  $abm \in N$ , then  $am \in N$  or  $bm \in N$  or  $ab \in N :_R M$  which implies that  $am \in L$  or  $bm \in L$  or  $ab \in L :_R M$ . If  $\mathfrak{p}^2 m \subseteq N$ , then  $\mathfrak{p}^2 \subseteq N :_R m$  and so  $\mathfrak{p} \subseteq r(N :_R m)$  thus  $m \in L$  and we are done. If  $\mathfrak{p}^2 ab \subseteq N :_R M$ , then by [2, Theorem 2.13], we have  $\mathfrak{p}^2 a \subseteq N :_R M$  or  $\mathfrak{p}^2 b \subseteq N :_R M$  or  $ab \in N :_R M$ . In the first case we conclude that  $\mathfrak{p}^2 \subseteq N :_R am$  and so  $am \in L$ . By a similar argument in the second case we can deduced that  $bm \in L$ . If  $ab \in N :_R M$ , then  $ab \in L :_R M$ . Therefore, the result follows.

For the second statement, first we show that  $r(N :_R M) = r(L :_R M) \cap \mathfrak{p}$ . It is clear  $r(N :_R M) \subseteq r(L :_R M) \cap \mathfrak{p}$ . Assume that  $a \in (L :_R M) \cap \mathfrak{p}$ . Thus  $aM \subseteq L$  and so, by definition of  $L$ ,  $\mathfrak{p} \subseteq r(N :_R am)$ , for all  $m \in M$ . Hence, [2, Theorem 2.4] shows that  $\mathfrak{p}^2 \subseteq N :_R am$ , for all  $m \in M$ . Therefore,  $a^3 \in N :_R m$ , for all  $m \in M$ . So that  $a^3 \in N :_R M$  and then  $a \in r(N :_R M)$ . Thus  $r(L :_R M) \cap \mathfrak{p} \subseteq r(N :_R M)$ . Now,  $L :_R M$  is a 2-absorbing ideal of  $R$ , therefore either  $r(L :_R M) = \mathfrak{p}'$  or  $r(L :_R M) = \mathfrak{p}' \cap \mathfrak{q}'$ , for some prime ideals  $\mathfrak{p}', \mathfrak{q}'$  of  $R$ . In the first case we have  $r(N :_R M) = \mathfrak{p} \cap \mathfrak{p}'$  which implies that  $\mathfrak{p}' = \mathfrak{q}$  and in the second case we have  $r(N :_R M) = \mathfrak{p} \cap \mathfrak{p}' \cap \mathfrak{q}'$  which implies that either  $\mathfrak{p}' = \mathfrak{q}$  or  $\mathfrak{q}' = \mathfrak{q}$ .  $\square$

**Corollary 2.6.** *Let  $N$  be a 2-absorbing submodule of  $M$  such that  $r(N :_R M) = \mathfrak{p} \cap \mathfrak{q}$  where  $\mathfrak{p}$  and  $\mathfrak{q}$  are the only distinct prime ideals of  $R$  that are minimal over  $N :_R M$ . If  $M/N$  is a Noetherian  $R$ -module, then*

- (i) *there exists a chain  $N = L_0 \subseteq L_1 \subseteq \dots \subseteq L_{n-1} \subseteq L_n = M$  of 2-absorbing submodules of  $M$ . Furthermore,  $\text{Ass}(M) \subseteq \text{Ass}(M/L_{n-1}) \cup \text{Ass}(L_{n-1}/L_{n-2}) \cup \text{Ass}(L_{n-2}/L_{n-3}) \cup \dots \cup \text{Ass}(L_1/N)$ , where  $\text{Ass}(L_i/N)$  is the union of at most two totally ordered set, for all  $i$ .*
- (ii) *there exists a chain  $N \subseteq L_n \subseteq L_{n-1} \subseteq \dots \subseteq L_1 \subseteq L_0 = M$  of submodules of  $M$  such that  $L_i$  is a 2-absorbing submodule of  $L_{i+1}$ , for all  $0 \leq i \leq n-1$ .*

*Proof.* (i) Let  $L_1 = \{m \in M : \mathfrak{p} \subseteq r(N :_R m)\}$ . Then by Corollary 2.4,  $L_1$  is a 2-absorbing submodule of  $M$  and so either  $r(L_1 :_R M) = \mathfrak{q}$  or  $r(L_1 :_R M) = \mathfrak{p}_1 \cap \mathfrak{q}$ , where  $\mathfrak{p}_1$  is a prime ideal of  $R$  containing  $\mathfrak{p}$ . If  $r(L_1 :_R M) = \mathfrak{q}$ , then choose  $L_2 = \{m \in M : \mathfrak{q} \subseteq r(L_1 :_R m)\} = M$ . Hence,  $N \subseteq L_1 \subseteq L_2 = M$  is requested chain. If  $r(L_1 :_R M) = \mathfrak{p}_1 \cap \mathfrak{q}$ , set  $L_2 = \{m \in M : \mathfrak{p}_1 \subseteq r(L_1 :_R m)\}$

and so either  $r(L_2 :_R M) = \mathfrak{q}$  or  $r(L_2 :_R M) = \mathfrak{p}_2 \cap \mathfrak{q}$ , where  $\mathfrak{p}_2$  is a prime ideal of  $R$  containing  $\mathfrak{p}_1$ . Proceeding in this way, we can achieve  $N \subseteq L_0 \subseteq L_1 \subseteq \cdots \subseteq L_{n-1} \subseteq L_n = M$  of 2-absorbing submodules of  $M$ . The last statement is obvious, by [9, Theorem 2.6].

(ii) Let  $L_1 = \{m \in M : \mathfrak{p} \subseteq r(N :_R m)\}$ . Then  $N$  is a 2-absorbing submodule of  $L_1$ . So that either  $r(N :_R L_1) = \mathfrak{p}_1$  or  $r(N :_R L_1) = \mathfrak{p}_1 \cap \mathfrak{q}_1$ , for some prime ideals  $\mathfrak{p}_1, \mathfrak{q}_1$  of  $R$ . If  $r(N :_R L_1) = \mathfrak{p}_1$ , then choose  $L_2 = \{x \in L_1 : \mathfrak{p}_1 \subseteq r(N :_R x)\} = N$ . Hence, in this case  $N \subseteq L_1 \subseteq L_0 = M$  is the requested chain. If  $r(N :_R L_1) = \mathfrak{p}_1 \cap \mathfrak{q}_1$ , then set  $L_2 = \{x \in L_1 : \mathfrak{p}_1 \subseteq r(N :_R x)\}$  and continue the same way to achieve the chain  $N \subseteq L_n \subseteq L_{n-1} \subseteq \cdots \subseteq L_1 \subseteq L_0 = M$  of 2-absorbing submodules of  $M$ .  $\square$

**Theorem 2.7.** *Let  $N$  be a 2-absorbing submodule of  $M$ . Then  $N :_R M$  is a prime ideal of  $R$  if and only if  $N :_R m$  is a prime ideal of  $R$  for all  $m \in M \setminus N$ .*

*Proof.* Assume that  $a, b \in R$ ,  $m \in M \setminus N$  and  $ab \in N :_R m$ . Then  $abm \subseteq N$ . We have  $am \in N$  or  $bm \in N$  or  $ab \in N :_R M$  since  $N$  is a 2-absorbing submodule of  $M$ . If  $am \in N$  or  $bm \in N$  we are done. If  $ab \in N :_R M$ , then by assumption either  $a \in N :_R M$  or  $b \in N :_R M$ . Thus either  $a \in N :_R m$  or  $b \in N :_R m$ . So  $N :_R m$  is a prime ideal.

Conversely, suppose that  $ab \in N :_R M$  for some  $a, b \in R$  and assume that there exist  $m, m' \in M$  such that  $am \notin N$  and  $bm' \notin N$ . By  $abm, abm' \in N$  it follows that  $bm \in N$  and  $am' \in N$  since  $N :_R m$  and  $N :_R m'$  are prime ideals of  $R$ . If  $m+m' \in N$ , then  $am \in N$  which is a contradiction. Thus  $m+m' \notin N$ . Now by  $ab(m'+m'') \in N$  we have  $a(m'+m'') \in N$  or  $b(m'+m'') \in N$  which is a contradiction. Thus  $aM \subseteq N$  or  $bM \subseteq N$  which implies that  $N :_R M$  is prime.  $\square$

**Corollary 2.8.** *Let  $N$  be a 2-absorbing submodule of  $M$ . Then  $N :_R M$  is a prime ideal of  $R$  if and only if  $N :_R K$  is a prime ideal of  $R$  for all submodules  $K$  of  $M$  containing  $N$ .*

*Proof.* By Theorem 2.7 and [9, Theorem 2.6] it follows that  $\{N :_R x : x \in K \setminus N\}$  is a totally ordered set of prime ideals of  $R$ . Hence,  $N :_R K = \bigcap_{x \in K} N :_R x$  is a prime ideal of  $R$ .  $\square$

**Theorem 2.9.** *Let  $\mathfrak{p}$  be a prime ideal of  $R$  and  $E(R/\mathfrak{p})$  be an injective envelop of  $R/\mathfrak{p}$ . If  $0$  is a 2-absorbing submodule of  $E(R/\mathfrak{p})$ , then*

- (i)  $\mathfrak{p}^2 \subseteq 0 :_R E(R/\mathfrak{p}) \subseteq \mathfrak{p}$  so that  $r(0 :_R E(R/\mathfrak{p})) = \mathfrak{p}$ .
- (ii)  $\mathfrak{p}^2 \subseteq 0 :_R x \subset 0 :_R ax = \mathfrak{p}$ , for all non-zero element  $x$  of  $E(R/\mathfrak{p})$  and all  $a \in \mathfrak{p} \setminus 0 :_R x$ .
- (iii)  $\mathfrak{p}^2 \subseteq 0 :_R x = 0 :_R a^n x \subseteq \mathfrak{p}$ , for all  $a \notin \mathfrak{p}$ .

*Proof.* (i) We have  $r(0 :_R x) = \mathfrak{p}$  for all non-zero element  $x$  of  $E(R/\mathfrak{p})$ , by [8, Theorem 18.4]. Also it is obvious  $0 :_R E(R/\mathfrak{p}) \subseteq 0 :_R x$ . Thus  $0 :_R E(R/\mathfrak{p}) \subseteq \mathfrak{p}$ .

Now, assume that  $a \in \mathfrak{p}^2$  and  $x$  is a non-zero element of  $E(R/\mathfrak{p})$ . Since  $0$  is a 2-absorbing submodule of  $M$ ,  $0 :_R x$  is a 2-absorbing ideal of  $R$ , by [9, Theorem 2.4]. Therefore we have  $\mathfrak{p}^2$  is a subset of  $0 :_R x$ , by [2, Theorem 2.4]. Hence,  $ax = 0$  and therefore  $aE(R/\mathfrak{p}) = 0$  and  $\mathfrak{p}^2 \subseteq 0 :_R E(R/\mathfrak{p})$ .

(ii) Let  $x$  be a non-zero element of  $E(R/\mathfrak{p})$ . Then we have  $\mathfrak{p}^2 \subseteq 0 :_R x \subseteq \mathfrak{p}$ . Assume that  $a \in \mathfrak{p} \setminus 0 :_R x$ . Thus  $ax \neq 0$  but  $a^2x = 0$  which shows that  $0 :_R x \subset 0 :_R ax$ . If  $b \in \mathfrak{p}$ , then  $ab \in \mathfrak{p}^2$  and  $abx = 0$ . Thus  $b \in 0 :_R ax$  and so  $\mathfrak{p} \subseteq 0 :_R ax$ .

(iii) Assume that  $a \notin \mathfrak{p}$ . It is obvious that  $0 :_R x \subseteq 0 :_R a^n x$ , for all  $n \in \mathbb{N}$ . Let  $b \in \text{Ann}_R(a^n x)$ . Thus  $ba^n x = 0$ . But multiplication by  $a^n$  is an automorphism on  $E(R/\mathfrak{p})$ , so that  $bx = 0$  and  $b \in 0 :_R x$ . Therefore,  $0 :_R x = 0 :_R a^n x$ .

□

**Corollary 2.10.** *Let  $R$  be a principal ideal domain and  $\mathfrak{p}$  is a prime ideal of  $R$ . If  $0$  is a 2-absorbing submodule of  $E(R/\mathfrak{p})$ , then for all non-zero element  $x$  of  $E(R/\mathfrak{p})$  either  $0 :_R x = \mathfrak{p}^2$  or  $0 :_R x = \mathfrak{p}$ .*

*Proof.* Let  $\mathfrak{p} = (a)$ . Then  $\mathfrak{p}^2 = (a^2)$ . Let  $x$  be a non-zero element of  $E(R/\mathfrak{p})$ . Then  $\mathfrak{p}^2 \subseteq 0 :_R x = (b) \subseteq \mathfrak{p}$  by Theorem 2.9(ii). Thus  $a^2 = bc$  and  $b = ae$  for some  $c, e \in R$ . Hence,  $a^2 = aec$ . So  $a = ec \in \mathfrak{p}$ . Therefore, either  $c \in \mathfrak{p}$  or  $e \in \mathfrak{p}$ . If  $c \in \mathfrak{p}$ , then  $c = ac'$  and so  $a = eac'$  which implies that  $1 = ec'$  and  $a = bc'$  thus  $0 :_R x = \mathfrak{p}$ . If  $e \in \mathfrak{p}$ , then  $e = ae'$  and so  $a = ae'c$  which implies that  $1 = e'c$  and  $b = a^2e'$  thus  $0 :_R x = \mathfrak{p}^2$ . □

The following example shows that the condition “ $0$  is a 2-absorbing submodule of  $E(R/\mathfrak{p})$ ” is essential. It is well-known that  $E(\mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}(p^\infty) = \{m/n + \mathbb{Z} : m, n \in \mathbb{Z}, n \neq 0\}$ , where  $p$  is a prime integer. But neither  $p^2\mathbb{Z} = 0 :_{\mathbb{Z}} 1/p^3 + \mathbb{Z}$  nor  $0 :_{\mathbb{Z}} 1/p^3 + \mathbb{Z} = p\mathbb{Z}$ . Hence,  $0$  is not a 2-absorbing submodule of  $E(\mathbb{Z}/p\mathbb{Z})$ . Also, this example shows that if  $0$  is a 2-absorbing submodule of  $M$ , then it is not necessarily a 2-absorbing submodule of  $E(M)$ .

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