Iranian Journal of Mathematical Sciences and Informatics

Vol. 10, No. 2 (2015), pp 55-66 DOI: 10.7508/ijmsi.2015.02.006

p-Analog of the Semigroup Fourier-Steiltjes Algebras

Marzieh Shams Yousefi

Department of Pure Mathematics, Faculty of Mathematical Science, University of Guilan, Rasht, Iran.

E-mail: m.shams@guilan.ac.ir

ABSTRACT. In this paper we define the p-analog of the restricted representations and the p-analog of the Fourier–Stieltjes algebras on inverse semigroups. Also we improve some results about Herz algebras on Clifford semigroups and we give a necessary and sufficient condition for amenability of these algebras on Clifford semigroups.

Keywords: Restricted fourier–Stieltjes algebras, Restricted inverse semigroup, Restricted representations, QSL_p -spaces, p-Analog of the Fourier–Stieltjes algebras.

2000 Mathematics subject classification: 20M18, 43A15.

1. Introduction and Preliminaries

An inverse semigroup S is a discrete semigroup such that for each $s \in S$ there exists a unique element $s^* \in S$ such that $ss^*s = s, s^*ss^* = s^*$. The set E(S) of idempotents of S consists of elements of the form ss^* , $s \in S$. Actually for each abstract inverse semigroup S there is a *-semigroup homomorphism from S into the inverse semigroup of partial isometries on some Hilbert space[18].

Dunkl and Ramirez in [8] and T. M. Lau in [15] attempted to define a suitable substitution for Fourier and Fourier–Stieltjes algebras on semigroups. Each definition has its own difficulties. Amini and Medghalchi introduced and

Received 30 October 2013; Accepted 25 May 2015 © 2015 Academic Center for Education, Culture and Research TMU

^{*}Corresponding Author

extensively studied the theory of restricted semigroups and restricted representations and restricted Fourier and Fourier–Stieltjes algebras, $A_{r,e}(S)$, $B_{r,e}(S)$ in [2] and [3]. Also they studied the spectrum of the Fourier Stieltjes algebra for a unital foundation topological *-semigroup in [4]. In this section we mention some of their results.

Throughout this paper S is an inverse semigroup. Given $x, y \in S$, the restricted product of x, y is xy if $x^*x = yy^*$, and undefined, otherwise. The set S with its restricted product forms a groupoid [16, 3.1.4] which is called the associated groupoid of S. If we adjoin a zero element 0 to this groupoid, and put $0^* = 0$, we will have an inverse semigroup S_r with the multiplication rule

$$x \bullet y = \begin{cases} xy & \text{if } x^*x = yy^*, \\ 0 & \text{otherwise} \end{cases}$$

for $x, y \in S \cup \{0\}$, which is called the restricted semigroup of S. A restricted representation $\{\pi, \mathcal{H}_{\pi}\}$ of S is a map $\pi : S \longrightarrow \mathcal{B}(\mathcal{H}_{\pi})$ such that $\pi(x^*) = \pi(x)^*$ $(x \in S)$ and

$$\pi(x)\pi(y) = \begin{cases} \pi(xy) & \text{if } x^*x = yy^*, \\ 0 & \text{otherwise} \end{cases}$$

for $x,y \in S$. Let $\Sigma_r = \Sigma_r(S)$ be the family of all restricted representations π of S with $\|\pi\| \leq 1$. Now it is clear that, via a canonical identification, $\Sigma_r(S) = \Sigma_0(S_r)$, consist of all $\pi \in \Sigma(S_r)$ with $\pi(0) = 0$, where the notation Σ has been used for all *-homomorphism from S into $\mathcal{B}(\mathcal{H})$ [2]. One of the central concepts in the analytic theory of inverse semigroups is the left regular representation $\lambda: S \longrightarrow \mathcal{B}(\ell^2(S))$ defined by

$$\lambda(x)\xi(y) = \begin{cases} \xi(x^*y) & \text{if } xx^* \ge yy^*, \\ 0 & \text{otherwise} \end{cases}$$

for $\xi \in \ell^2(S), x, y \in S$. The restricted left regular representation $\lambda_r : S \longrightarrow \mathcal{B}(\ell^2(S))$ is defined in [2] by

$$\lambda_r(x)\xi(y) = \begin{cases} \xi(x^*y) & \text{if } xx^* = yy^*, \\ 0 & \text{otherwise} \end{cases}$$

for $\xi \in \ell^2(S), x, y \in S$. The main objective of [2] is to change the convolution product on the semigroup algebra to restore the relation with the left regular representation.

For each $f, g \in \ell^1(S)$, define

$$(f \bullet g)(x) = \sum_{x^*x = yy^*} f(xy)g(y^*) \quad (x \in S),$$

and for all $x \in S$, $\tilde{f}(x) = \overline{f(x^*)}$. $\ell_r^1(S) := (\ell^1(S), \bullet, \tilde{\ })$ is a Banach *-algebra with an approximate identity . The left regular representation λ_r lifts to a faithful representation $\tilde{\lambda}$ of $\ell_r^1(S)$. We call the completion $C^*_{\lambda_r}(S)$ of $\ell_r^1(S)$ with the

norm $\|\cdot\|_{\lambda_r} := \|\tilde{\lambda}_r(\cdot)\|$ which is a C^* -norm on $\ell^1_r(S)$, the restricted reduced C^* -algebra and its completion with the norm $\|\cdot\|_{\Sigma_r} := \sup\{\|\tilde{\pi}(\cdot)\|, \pi \in \Sigma(S_r)\}$ the restricted full C^* -algebra and show it by $C^*_r(S)$. The dual space of C^* -algebra $C^*_r(S)$ is a unital Banach algebra which is called the restricted Fourier-Stieltjes algebra and is denoted by $B_{r,e}(S)$. The closure of the set of finitely support functions in $B_{r,e}(S)$ is called the restricted Fourier algebra and is denoted by $A_{r,e}(S)[2]$.

In [10], Figà-Talamanca introduced a natural generalization of the Fourier algebra, for a compact abelian group G, by replacing $L_2(G)$ by $L_p(G)$. In [11], Herz extended the notion to an arbitrary group, to get the commutative Banach algebra $A_p(G)$, called the Figà-Talamanca-Herz algebra. Figà-Talamanca-Herz algebra and Eymard's Fourier algebra have very similar behavior. For example, Leptin's theorem is valid: G is amenable if and only if $A_p(G)$ has a bounded approximate identity [12]. The p-analog, $B_p(G)$ of the Fourier-Stieltjes algebra is defined as the multiplier algebra of $A_p(G)$, by some authors, as mentioned in [5] and [19]. Runde in [20] defined and studied $B_p(G)$, the p-analog of the Fourier-Stieltjes algebra on the locally compact group G. He developed the theory of representations and defined the suitable coefficient functions on them.

For $p \in (1, \infty)$, Medghalchi and Pourmahmood Aghababa developed the theory of restricted representations on $\ell_p(S)$ and defined the Banach algebra of p-pseudomeasures $PM_p(S)$ and the Figa- Talamanca-Herz algebras $A_p(S)$. They showed that $A_q(S)^* = PM_p(S)$ for dual pairs p,q. They characterized $PM_p(S)$ and $A_p(S)$ for Clifford semigroups, in the sense of p-pseudomeasures and Figa-Talamanca-Herz algebras of maximal semigroups of S, respectively[17].

Amini also worked on quantum version of Fourier transforms in [1].

In this paper we will combine what Medghalchi–Pourmahmood Aghababa and Runde have done. We will define the restricted representations on QSL_p -spaces and the p-analog of the Fourier-Stieltjes algebra on the restricted inverse semigroup.

Section 2 is a review of the theory of QSL_p -spaces. In Section 3 we define the restricted representations on QSL_p -spaces and study their tensor product. In Sections 4 and 5 we construct the p-analog of the restricted Fourier–Stieltjes algebra and study its order structure. The last section will be about Clifford semigroups and the p-analog of their restricted Fourier–Stieltjes algebra. Some new results which improves the results of [17] and [22] will be given in Section 6.

2. Review of the Theory of QSL_p -spaces

This section is a review of the paper of Runde [20].

Definition 2.1. A Banach space \mathcal{E} is called

Downloaded from ijmsi.com on 2025-11-16

- (ii) a of an L Section

 If E
- (i) an L_p -space if it is of the form $L_p(X)$, for some measure space X.
 - (ii) a QSL_p -space if it is isometrically isomorphic to a quotient of a subspace of an L_p -space (or equivalently, a subspace of a quotient of an L_p -space [20, Section 1, Remark 1]).

If E is a QSL_p -space and if $p' \in (1, \infty)$ is such that $\frac{1}{p} + \frac{1}{p'} = 1$, the dual space E^* is an $QSL_{p'}$ -spaces. In particular, every QSL_p -space is reflexive.

By [14, Theorem 2], the QSL_p -spaces are precisely the p-spaces in the sense of [11], i.e. those Banach spaces E such that for any two measure spaces X and Y the amplification map

$$B(L_p(X), L_p(Y)) \to B(L_p(X, E), L_p(Y, E)), T \to T \otimes id_E$$

is an isometry. In particular, an L_q -space is a QSL_p -space if and only if $2 \le q \le p$ or $p \le q \le 2$. Consequently, if $2 \le q \le p$ or $p \le q \le 2$, then every QSL_q -space is a QSL_p -space.

Runde equipped the algebraic tensor product of two QSL_p -spaces with a suitable norm, which comes in the following.

Theorem 2.2. [20, Theorem 3.1] Let E and F be QSL_p -spaces. Then there exists a norm $\|\cdot\|_p$ on the algebraic tensor product $E\otimes F$ such that:

- (i) $\|\cdot\|_p$ dominates the injective norm;
- (ii) $\|\cdot\|_p$ is a cross norm;
- (iii) the completion $E\tilde{\otimes}_p F$ of $E\otimes F$ with respect to $\|\cdot\|_p$ is a QSL_p -space. The Banach space $E\tilde{\otimes}_p F$ will be called p-projective tensor product of E and F.

3. RESTRICTED REPRESENTATION ON A BANACH SPACE

In this section we give an analog of the theory of group representations on a Hilbert space for the restricted representations for an inverse semigroup on a QSL_p -space.

Definition 3.1. A representation of a discrete inverse semigroup S on a Banach space E is a pair (π, E) consisting of a map $\pi: S \to B(E)$ satisfying $\pi(x)\pi(y) = \pi(xy)$, for $x, y \in S$ and $\|\pi(x)\| \leq 1$, for all $x \in S$.

Definition 3.2. A restricted representation of a discrete inverse semigroup S on a Banach space E is a pair (π, E) consisting of a map $\pi: S \to B(E)$ satisfying

$$\pi(x)\pi(y) = \begin{cases} \pi(xy) & \text{if } x^*x = yy^*, \\ 0 & \text{otherwise} \end{cases}$$

for $x, y \in S$, and $||\pi(x)|| < 1$, for all $x \in S$.

Definition 3.3. Let S be an inverse semigroup, and let (π, E) and (ρ, F) be restricted representations of S, then these restricted representations are said to be equivalent if there exists a surjective isometry $T: E \to F$ such that

$$T\pi(x)T^{-1} = \rho(x), \quad (x \in S).$$

For any inverse semigroup S and $p \in (1, \infty)$, we denote by $\Sigma_{p,r}(S)$ the collection of all (equivalence classes) of restricted representations of S on a QSL_p -space.

Remark 3.4. By [17] for $p \in (1, \infty)$ the restricted left regular representation $\lambda_p : S \longrightarrow B(\ell^p(S))$

$$\lambda_p(s)(\delta_t) = \begin{cases} \delta_{st} & \text{if } s^*s = tt^*, \\ 0 & \text{otherwise} \end{cases}$$

for $s, t \in S$ is a restricted representation so it belongs to $\Sigma_{p,r}(S)$.

The following propositions are easy to check, similar to [2].

Proposition 3.5. For an inverse semigroup S and its related restricted semigroup S_r , each restricted representation of S on a Banach space is a representation on S_r which is zero on $0 \in S_r$, i.e. it is multiplicative with respect to the restricted multiplication.

Proposition 3.6. For an inverse semigroup S, each restricted representation π of S on a Banach space lifts to a representation of $\ell_r^1(S)$, via

$$\tilde{\pi}(f) = \sum_{x \in S} f(x)\pi(x).$$

4. Banach Algebra
$$B_{p,r}(S)$$

In this section we define the p-analog of the Fourier–Stieltjes algebra on a inverse semigroup. We show that for p=2 we get the known algebra $B_{r,e}(S)$, defined in [2].

Theorem 4.1. Let $(\pi, E), (\rho, F) \in \Sigma_{p,r}(S)$ then $(\pi \otimes \rho, E \tilde{\otimes}_p F) \in \Sigma_{p,r}(S)$.

Proof. By the definition of $\pi \otimes \rho$ we have $\pi \otimes \rho(x)(\xi \otimes \eta) = \pi(x)\xi \otimes \rho(x)\eta$. For $x, y \in S$, $x^*x = yy^*$,

$$\pi \otimes \rho(xy)(\xi \otimes \eta) = \pi(xy)\xi \otimes \rho(xy)\eta$$

$$= \pi(x)\pi(y)\xi \otimes \rho(x)\rho(y)\eta$$

$$= \pi(x)(\pi(y)\xi) \otimes \rho(x)(\rho(y)\eta)$$

$$= \pi \otimes \rho(x)(\pi(y)\xi \otimes \rho(y)\eta)$$

$$= \pi \otimes \rho(x)\pi \otimes \rho(y)(\xi \otimes \eta)$$

when $x^*x \neq yy^*$

$$\pi \otimes \rho(x)\pi \otimes \rho(y)(\xi \otimes \eta) = \pi \otimes \rho(x)(\pi(y)\xi \otimes \rho(y)\eta)$$
$$= \pi(x) (\pi(y)\xi) \otimes \rho(x) (\rho(y)\eta)$$

which is equal to zero. Now it is enough to show that $\pi(x) \in B(E)$ and $\rho(y) \in B(F)$, $\pi(x) \otimes \rho(y)$ could be extend to $E \tilde{\otimes}_p F$. This is shown as in the group case [20, Therem 3.1].

Definition 4.2. Let S be an invesre semigroup, and let $(\pi, E) \in \Sigma_{p,r}(S)$. A coefficient function of (π, E) is a function $f: S \longrightarrow \mathbb{C}$ of the form

$$f(x) = \langle \pi(x)\xi, \phi \rangle \quad (x \in S),$$

where $\xi \in E$ and $\phi \in E^*$.

Definition 4.3. Let S be an inverse semigroup, let $p \in (1, \infty)$, and let $q \in (1, \infty)$ be the dual scalar to p, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. We define

 $B_{p,r}(S) := \{ f : S \to \mathbb{C} : f \text{ is a coefficient function of some } (\pi, E) \in \Sigma_{q,r}(S) \}.$

Proposition 4.4. Let S be an inverse semigroup, let $p \in (1, \infty)$, and let $q \in (1, \infty)$ be the dual scalar to p, i.e. $\frac{1}{p} + \frac{1}{q} = 1$, and let $f : S \longrightarrow \mathbb{C}$ defined by

$$f(x) = \sum_{n=1}^{\infty} \langle \pi_n(x)\xi_n, \phi_n \rangle, \quad (x \in S),$$

where $((\pi_n, E_n))_{n=1}^{\infty}$, $(\xi_n)_{n=1}^{\infty}$, and $(\phi_n)_{n=1}^{\infty}$ are sequences with $(\pi_n, E_n) \in \Sigma_{q,r}(S)$, $\xi_n \in E_n$, and $\phi_n \in E_n^*$, for $n \in \mathbb{N}$ such that

$$\sum_{n=1}^{\infty} \|\xi_n\| \|\phi_n\| < \infty.$$

Then f lies in $B_{p,r}(S)$.

Proof. The proof is similar to [20]. Without loss of generality, we may suppose that

$$\sum_{i=1}^{\infty} \|\xi_n\|_q < \infty, \quad \text{and } \sum_{i=1}^{\infty} \|\phi_n\|_p < \infty.$$

Then $E := \ell_q - \bigoplus_{n=1}^{\infty} E_n$ is a QSL_q -space and for $\xi := (\xi_1, \xi_2, ...)$ and $\phi := (\phi_1, \phi_2, ...)$, we have $\xi \in E$ and $\phi \in E^*$. Now the map $\pi : S \longrightarrow B(E)$ with $\pi(x)\eta = (\pi_1(x)\eta, \pi_2(x)\eta, ...)$ is a restricted representation of S on E, and f is the coefficient function of π .

Definition 4.5. [17, Definition 3.1]. Let S be an inverse semigroup and let $p,q \in (1,\infty)$ be dual pairs. The space $A_q(S)$ consists of those $u \in c_0(S)$ such that there exist sequences $(f_n)_{n=1}^{\infty} \subseteq \ell_q(S)$ and $(g_n)_{n=1}^{\infty} \subseteq \ell_p(S)$ with $\sum_{n=1}^{\infty} \|f_n\|_q \|g_n\|_q \le \infty$ and $u = \sum_{n=1}^{\infty} f_n \bullet \check{g_n}$. For $u \in A_q(S)$, let

$$||u|| = \inf \left\{ \sum_{n=1}^{\infty} ||f_n||_q ||g_n||_p : u = \sum_{n=1}^{\infty} f_n \bullet \check{g_n} \right\}$$

Proposition 4.6. [17, Proposition 3.2]. Let S be an inverse semigroup and let $p \in (1, \infty)$, then $A_p(S)$ is a Banach space and is the closure of finite support functions on S.

Proposition 4.7. Let S be an inverse semigroup, let $p \in (1, \infty)$. Then $B_{p,r}(S)$ is a linear subspace of $c_b(S)$ containing $A_p(S)$. Moreover, if $2 \le q \le p$ or $p \le q \le 2$, we have $B_{q,r}(S) \subseteq B_{p,r}(S)$.

Proof. Every thing is easy to check, and is similar to [20].

Definition 4.8. Let S be an inverse semigroup, and let (π, E) be a restricted representation of S on the Banach space E. Then (π, E) is called cyclic if there exists $x \in E$ such that $\pi(\ell_r^1(S))x$ is dense in E. For $p \in (1, \infty)$, we set $Cyc_{p,r}(S) := \{(\pi, E) : (\pi, E) \text{ is a cyclic restricted representation on a } QSL_p\text{- space } E\}.$

Definition 4.9. Let S be an inverse semigroup, let $p,q\in(1,\infty)$ be the dual scalars, and let $f\in B_{p,r}(S)$. We define $\|f\|_{B_{p,r}(S)}$ as the infimum over all expressions $\sum_{n=1}^{\infty}\|\xi_n\|\|\phi_n\|$, where, for each $n\in\mathbb{N}$, there is $(\pi_n,E_n)\in Cyc_{q,r}(S)$ with $\xi_n\in E_n$ and $\phi_n\in E_n^*$ such that $\sum_{n=1}^{\infty}\|\xi_n\|\|\phi_n\|<\infty$ and

$$f(x) = \sum_{n=1}^{\infty} \langle \pi_n(x)\xi_n, \phi_n \rangle, \quad (x \in S).$$

The proof of the following theorem is similar to the group case.

Theorem 4.10. Let S be an inverse semigroup, let $p \in (1, \infty)$, and let $f, g : S \longrightarrow \mathbb{C}$ be coefficient functions of (π, E) and (ρ, F) in $\Sigma_{p,r}(S)$, respectively. Then the pointwise product of f and g is a coefficient function of $(\pi \otimes \rho, E \tilde{\otimes}_p F)$.

In the next theorem we give some result about our new constructed space and also the relation between semigroup restricted Herz algebra and our new space.

Theorem 4.11. Let S be an inverse semigroup, let $p \in (1, \infty)$. Then:

- (i) $B_{p,r}(S)$ is a commutative Banach algebra.
- (ii) the inclusion $A_p(S) \subseteq B_{p,r}(S)$ is a contraction.
- (iii) for $2 \le p' \le p$ or $p \le p' \le 2$, the inclusion $B_{p',r}(S) \subseteq B_{p,r}(S)$ is a contraction.
- (iv) for p = 2, $B_{r,e}(S)$ is isometrically isomorphic to $B_{p,r}(S)$ as Banach algebras.
- Proof. (i) Let $\frac{1}{p} + \frac{1}{q} = 1$. The space $B_{p,r}(S)$ is the quotient space of complete q-projective tensor product of $E\tilde{\otimes}_q E^*$, for the universal restricted representation (π, E) , on QSL_q -space E. Also Theorem 4.10 shows it is an algebra. The submultiplicative property for norm of $B_{p,r}(S)$ is similar to the group case in [20] and it is only based on characteristic property of infimum.
- (ii) By the definition of semigroup Herz algebra in [17] for conjugate numbers p,q, each $f \in A_p(S)$ is a coefficient function of the restricted left regular representation on the ℓ_q -space, $\ell_q(S)$. So $A_p(S) \subseteq B_{p,r}(S)$. By the definition of the norm of $f \in B_{p,r}(S)$, the infimum is taken on all expressions of f as the coefficient function of some restricted representation on a QSL_q -space, and

the norm on the $A_p(S)$ is the infimum only on expressions of f as the coefficient function of restricted left regular representation, so the inclusion map is a contraction.

- (iii) For $2 \leq p' \leq p$ or $p \leq p' \leq 2$ and q,q' conjugate scalars to p and p' respectively. Then each restricted representation on a $QSL_{q'}$ -space is a restricted representation on a QSL_q -space.
- (iv) By the definition, each element of $B_{r,e}(S)$ is a coefficient function of a 2-restricted representation [3].

Remark 4.12. A very natural question is that when $A_p(S)$ is an ideal in $B_{p,r}(S)$. Even in p=2 this question is not studied. If we want to go along the proof of the group case, a difficulty to prove this is that in general for $p \in (1, \infty)$, and $(\pi, E) \in \Sigma_{p,r}(S)$, the representations $(\lambda_p \otimes \pi, \ell_p(S, E))$ and $(\lambda_p \otimes id_E, \ell_p(S, E))$ are not equivalent. In fact we can not find a suitable substitution for representation $id: S \to B(E), \quad id(s) = id_E$ in the class of restricted representations. But in a special case, such as Clifford semigroups, we can give a better result.

5. Order Structure of the p-Analog of the Semigroup Fourier-Steiltjes Algebras $B_{p,r}(S)$

Studying the ordered spaces and order structures has a long history. The natural order structure of the Fourier-Stieltjes algebras was favorite in 80s. In [21] the authors studied the order structure of Figà-Talamanca- Herz algebra and generalized results on Fourier algebras. In this section, we consider the p-analog of the restricted Fourier-Stieltjes algebra, $B_{p,r}(S)$, introduced in Section 4, and study its order structure given by the p-analog of positive definite continuous functions.

A compatible couple of Banach spaces in the sense of interpolation theory (see [3]) is a pair $(\mathcal{E}_0, \mathcal{E}_1)$ of Banach spaces such that both \mathcal{E}_0 and \mathcal{E}_1 are embedded continuously in some (Hausdorff) topological vector space. In this case, the intersection $\mathcal{E}_0 \cap \mathcal{E}_1$ is again a Banach space under the norm $\|\cdot\|_{(\mathcal{E}_0, \mathcal{E}_1)} = \max\{\|\cdot\|_{\mathcal{E}_0}, \|\cdot\|_{\mathcal{E}_1}\}$. For example, for a locally compact group G, the pairs $(A_p(G), A_q(G))$ and $(L_p(G), L_q(G))$ are compatible couples.

Definition 5.1. Let (π, E) be a restricted representation of S on a Banach space E, such that $(\mathcal{E}, \mathcal{E}^*)$ is a compatible couple. We mean by a π_r -positive definite function on S, a function which has a representation as $f(x) = \langle \pi(x)\xi, \xi \rangle$, $(x \in S)$, where $\xi \in \mathcal{E} \cap \mathcal{E}^*$. For dual scalars $p, q \in (1, \infty)$, we call each element in the closure of the set of all π_r -positive definite functions on S in $B_{p,r}(S)$, where π is a restricted representation of S on an L_q -space, a restricted p-positive definite function on S and the set of all restricted p-positive definite functions on S, will be denoted by $P_{p,r}(S)$.

It follows from [21] and the definition of $P_{p,r}(S)$, that for each $f \in P_{p,r}(S)$, associated to a representation (π, E) , for a QSL_q -space E, there exists a sequence $(\pi_n, \mathcal{E}_n)_{n=1}^{\infty}$ of cyclic restricted representations of S on closed subspaces E_n of $E \cap E^*$, and $\{\xi_n\}$ in \mathcal{E}_n , such that

$$f(x) = \sum_{n=1}^{\infty} \langle \pi_n(x)\xi_n, \xi_n \rangle \quad (x \in S).$$

Proposition 5.2. The linear span of all finite support elements in $P_{p,r}(S)$ is dense in $A_p(S)$, and $A_p(S)$ is an ordered space.

Proof. From [17, Proposition 3.2] $A_p(S)$ is a norm closure of the set of elements of the form $\sum_{i=1}^n f_i \bullet \check{g_i}$ where f_i, g_i are finite support functions on S, i = 1, ...n. Also $f_i \bullet \check{g_i}(x) = \langle \lambda_r(x^*)f_i, g_i \rangle$. Now by Polarization identity, we have the statement.

Since $A_p(S)$ is the set of coefficient functions of the restricted left regular representation of S on $\ell_p(S)$, we define the positive cone of $A_p(S)$ as the closure in $A_p(S)$, of the set of all function of the form $f = \sum_{i=1}^n \xi_i \bullet \tilde{\xi}_i$, for a sequence (ξ_i) in $\ell_p(S) \cap \ell_q(S)$, and denote it by $A_p(S)_+$.

This order structure, in the case where p=2, is the same as the order structure of $A_{r,e}(S)$, induced by the set $P_{r,e}(S) \cap A_{r,e}(S)$, as a positive cone. Because in the case p=2, the extensible restricted positive definitive functions are exactly the closed linear span of $h \bullet \tilde{h}$, for $h \in \ell^2(S)$.

6. p-Analog of the Fourier-Stiletjes Algebras on Clifford Semigroups

Let S be a semigroup. Then, by [13, Chapter 2], there is an equivalence relation \mathcal{D} on S by $s\mathcal{D}t$ if and only if there exists $x \in S$ such that

$$Ss \cup \{s\} = Sx \cup \{x\} \text{ and } tS \cup \{t\} = xS \cup \{x\}.$$

If S is an inverse semigroup, then by [13, Proposition 5.1.2(4)], $s\mathcal{D}t$ if and only if there exists $x \in S$ such that $s^*s = xx^*$ and $t^*t = x^*x$.

Proposition 6.1. [17, Proposition 4.1]. Let S be an inverse semigroup,

- (i) and let D be a D-class of S. Then $\ell_p(D)$ is a closed $\ell_r^1(S)$ -submodule of $\ell_p(S)$.
- (ii) and let $\{D_{\lambda}; \lambda \in \Lambda\}$ be the family of \mathcal{D} -classes of S indexed by some set Λ . Then there is an isometric isomorphism of Banach $\ell_r^1(S)$ -bimodules

$$\ell^p(S) \cong \ell^p - \bigoplus_{\lambda \in \Lambda} \ell_p(D_\lambda).$$
 (6.1)

Corollary 6.2. Let S be an inverse semigroup, and let $\{D_{\lambda}; \lambda \in \Lambda\}$ be the family of \mathcal{D} -classes of S indexed by some set Λ . Then for a QSL_p -space E of functions on S, there is a family of QSL_p -spaces $\{E_{\lambda}\}_{{\lambda}\in\Lambda}$, where for each ${\lambda}\in\Lambda$, E_{λ} consists of functions on D_{λ} , and $E\cong\ell^p-\oplus_{{\lambda}\in\Lambda}E_{\lambda}$.

Proof. This is clear by the definition of a QSL_p -space, and the fact that the isomorphism 6.1 is compatible with taking quotients and subspaces of $\ell_p(D_\lambda)$ s.

An inverse semigroup S is called a Clifford semigroup if $s^*s = ss^*$ for all $s \in S$. For $e \in E(S)$ define $G_e := \{s \in S | s^*s = ss^* = e\}$. Then G_e is a group with identity e. Here each \mathcal{D} -class D contains a single idempotent (say e) and we have $D = G_e$

We modified the isometrical isomorphism derived in [17, Section 5.3] in the following theorem.

Theorem 6.3. Let S be a Clifford semigroup with the family of \mathcal{D} -classes $\{G_e\}_{e\in E(S)}$, and let $p\in (1,\infty)$. Then

$$B_{p,r}(S) \cong \ell^1 - \bigoplus_{e \in E(S)} B_p(G_e)$$

Proof. Let p,q are conjugate scalars. Fix $e \in E(S)$, assume that $G_e = \{x \in S; x^*x = e\}$. Define $\pi: S \to B(\ell_q(G_e))$

$$\pi(s)(\delta_t) = \begin{cases} \delta_t & \text{if } s^*s = e, \\ 0 & \text{otherwise} \end{cases}$$

for $s \in S$. Then π is a restricted representation and $\chi_{G_e}(s) = \langle \pi(s)\delta_t, \delta_t \rangle$. Hence χ_{G_e} is in $B_{p,r}(S)$, and indeed χ_{G_e} is a restricted positive definite function. Now for each $u \in B_{p,r}(S)$, $u \cdot \chi_{G_e}$ is in $B_{p,r}(S)$. In fact the set $\{u \in B_{p,r}(S); u(s) = 0 \text{ for all } s \in S \setminus G_e\}$ is a closed subspace of $B_{p,r}(S)$ and it is isometrically isomorphic to $B_p(G_e)$. This follows from the fact that, each coefficient function of a restricted representation of S on a QSL_q -space that is zero on G_e^c , is a coefficient function of a representation on a QSL_q -space of G_e , using Corollary 6.2.

Let $u \in B_{p,r}(S)$, then we could decompose u to $(u_e)_{e \in E(S)}$, for some $u_e \in B_p(G_e)$, by the above paragraph. Now for all $e \in E(S)$ and all explanations of u_e as $u_e = \langle \pi_e(\cdot)\xi_e, \eta_e \rangle$, where $\pi_e \in \Sigma_q(G_e)$, ξ_e in some QSL_q and η_e in some QSL_p -space for dual scalars p,q we have $\|u_e\| \leq \|\xi_e\| \|\eta_e\|$ and also $u = (u_e)_{e \in E(S)} = \langle \oplus \pi_e(\cdot) \oplus \xi_e, \oplus \eta_e \rangle$ and then $\oplus \pi_e$ is a restricted representation of S on a QSL_q -space and $\sum_{e \in E(S)} \|u_e\| \leq \sum_{e \in E(S)} \|\xi_e\| \|\eta_e\| \leq (\sum_{e \in E(S)} \|\xi_e\|^q)^{\frac{1}{q}} (\sum_{e \in E(S)} \|\eta_e\|^p)^{\frac{1}{p}} = \|(\xi_e)\| \|(\eta_e)\|$, the last equality comes from Proposition 6.1. Now we have:

$$\sum_{e \in E(S)} \|u_e\| \le \|(u_e)_{e \in E(S)}\| = \|\sum_{e \in E(S)} u_e\|$$

by the definition of the norm in the Fourier–Stieltjes algebras. So we have an isometric isomorphism of Banach algebras. \Box

The following corollary improves [22, Proposition 2.6].

Corollary 6.4. Let S be a Clifford semigroup with the family of \mathcal{D} -classes $\{G_e\}_{e\in E(S)}$, and let $p\in (1,\infty)$. Then $A_p(S)$ is an ideal of $B_{p,r}(S)$, and $B_{p,r}(S)$ is amenable if and only if $B_p(G_e)$ is amenable for all $e\in E(S)$.

Theorem 6.5. Let S be a Clifford semigroup with the family of amenable \mathcal{D} -classes $\{G_e\}_{e\in E(S)}$ and let $p\in (1,\infty)$. Then $A_p(S)$ is equal to the closure of $B_{p,r}(S)\cap F(S)$ in the norm of $A_p(S)$.

Proof. Since for each $e \in E(S)$ the group G_e is amenable, the natural embedding $i_e: A_p(G_e) \to B_p(G_e)$ is an isometry by [20, Corollary 5.3]. Now let $f \in B_{p,r}(S) \cap F(S)$. Then by Theorem 6.3, $f = \sum_{e \in E(S)} f_e$, where $f_e \in B_p(G_e) \cap F(G_e)$, so f_e belongs to $A_p(G_e)$ by [20]. Now since

$$A_p(S) \cong \ell_1 - \bigoplus_{e \in E(S)} A_p(G_e),$$

[17, Equation 5.1], we conclude that $f \in A_p(S)$. On the other hand let $f \in A_p(S) \cap F(S)$. Then for each $e \in E(S)$ the function f_e , which has been defined in Theorem 6.3, belongs to $A_p(G_e) \cap F(G_e)$ and so to $B_p(G_e) \cap F(G_e)$ with the same norm because of amenability of G_e . Now, by Theorem 6.3 we have $f \in B_{p,r}(S) \cap F(S)$. Since F(S) is dense in $A_p(S)$ with norm of $A_p(S)$, [17, Proposition 3.2 vi], result follows.

ACKNOWLEDGMENT

The author is indebted to the anonymous referee for his/her suggestions which improved the paper.

REFERENCES

- M. Amini, Quantum Error-correction Codes on Abelian Groups, Iranian Journal of Mathematical Sciences and Informatics, 5(1), (2010), 55-67.
- 2. M. Amini, A. R. Medghalchi, Restricted Algebras on Inverse Semigroups I, *Math. Nachr.*, **279**(16), (2006), 1-10.
- 3. M. Amini, A. R. Medghalchi, Restricted Algebras on Inverse Semigroups II, Positive Definite Functions, *Int. J. Math. Math. Sci.*, (2011).
- M. Amini, A. R. Medghalchi, Spectrum of the Fourier-Stieltjes algebra of a Semigroup, Iranian Journal of Mathematical Sciences and Informatics, 1(2), (2006), 1–8.
- M. Cowling, An Application of Littlewood-Paley Theory in Harmonic Analysis. Math. Ann., 241, (1979), 83-96.
- 6. H. G. Dales, Banach Algebras and Automatic Continuity, Clarendon Press, Oxford, 2000.
- J. Duncan, A. L. T. Paterson, Amenability for Discrete Convolution Semigroup Algebras, Math. Scand., 66, (1990), 141-146.
- 8. C. F. Dunkl, D. E. Rumirez, L^{∞} -Representations of Commutative Semitopological semigroups, Semigroup Forum, 7, (1974), 180–199.
- 9. P. Eymard, Lalgebra de Fourier dun Groupe Localment Compact, Bull. Soc. Math. France, 92, (1964), 181–236.

- A. Figà-Talamanca, Translation Invariant Operators in L^p, Duke Math. J., 32, (1965), 495–501.
- C. Herz, The Theory of p-Spaces with Applications to Convolution Operators, Trans. Amer. Math. Soc., 154, (1971), 69-82.
- C. Herz, Harmonic Synthesis for Subgroups, Ann. Inst. Fourier (Grenoble), 23, (1973), 91-123.
- J. M. Howie, Fundamentals of Semigroup Theory, in: London Mathematical Society Monographs, vol. 12, Clarendon Press, Oxford, 1995.
- S. Kwapien, On Operators Factoring Through Lp-space, Bull. Soc. Math. France, Mem., 3132, (1972), 215-225.
- A. T. M. Lau, The Fourier Stieltjes algebra of a Topological Semigroup with Involution, Pac. J. Math., 77, (1978), 165–181.
- M. V. Lawson, Inverse Semigroups, the Theory of Partial Symmetries, World Scientific, Singapore, 1998.
- A. R. Medghalchi, H. Pourmahmood Aghababa, Figa-Talamanca-Herz Algebras for Restricted Inverse Semigroups and Clifford Semigroups, J. Math. Anal. Appl., 395, (2012), 473-485.
- A. T. Paterson, Groupoids, Inverse Semigroups, and their Operator Algebras, Progr. Math. vol. 170, Birkhuser, Boston, 1999.
- 19. J. P. Pier, Amenable Locally Compact Groups, Wiley-Interscience, 1984.
- V. Runde, Representations of Locally Compact Groups on QSL_p-spaces and a p-Analog of the Fourier-Stieltjes Algebra, Pacific J. Math., 221, (2005), 379–397.
- M. Shams Yousefi, M. Amini, F. Sady, Order structure of the Figà-Talamanca- Herz algebra, ISRN Geometry, (2011), Article ID 531023, 13 pages.
- M. Shams Yousefi, M. Amini, Amenability of the Restricted Fourier algebras, ISRN Algebra, (2012), Article ID 782953, 8 pages.