

## OD-characterization of Almost Simple Groups Related to $D_4(4)$

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**ABSTRACT.** Let  $G$  be a finite group and  $\pi_e(G)$  be the set of orders of all elements in  $G$ . The set  $\pi_e(G)$  determines the prime graph (or Grunberg-Kegel graph)  $\Gamma(G)$  whose vertex set is  $\pi(G)$ . The set of primes dividing the order of  $G$ , and two vertices  $p$  and  $q$  are adjacent if and only if  $pq \in \pi_e(G)$ . The degree  $\deg(p)$  of a vertex  $p \in \pi(G)$ , is the number of edges incident on  $p$ . Let  $\pi(G) = \{p_1, p_2, \dots, p_k\}$  with  $p_1 < p_2 < \dots < p_k$ . We define  $D(G) := (\deg(p_1), \deg(p_2), \dots, \deg(p_k))$ , which is called the degree pattern of  $G$ . The group  $G$  is called  $k$ -fold OD-characterizable if there exist exactly  $k$  non-isomorphic groups  $M$  satisfying conditions  $|G| = |M|$  and  $D(G) = D(M)$ . Usually a 1-fold OD-characterizable group is simply called OD-characterizable. In this paper, we classify all finite groups with the same order and degree pattern as an almost simple groups related to  $D_4(4)$ .

**Keywords:** Degree pattern,  $k$ -fold OD-characterizable, Almost simple group.

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## 1. INTRODUCTION

Let  $G$  be a finite group,  $\pi(G)$  the set of all prime divisors of  $|G|$  and  $\pi_e(G)$  be the set of orders of elements in  $G$ . The prime graph (or Grunberg-Kegel graph)  $\Gamma(G)$  of  $G$  is a simple graph with vertex set  $\pi(G)$  in which two vertices  $p$  and  $q$  are joined by an edge ( and we write  $p \sim q$  ) if and only if  $G$  contains an element of order  $pq$  (i.e.  $pq \in \pi_e(G)$  ).

The degree  $\deg(p)$  of a vertex  $p \in \pi(G)$  is the number of edges incident on  $p$ . If  $\pi(G) = \{p_1, p_2, \dots, p_k\}$  with  $p_1 < p_2 < \dots < p_k$ , then we define  $D(G) := (\deg(p_1), \deg(p_2), \dots, \deg(p_k))$ , which is called the degree pattern of  $G$ , and leads a following definition.

**Definition 1.1.** The finite group  $G$  is called  $k$ -fold OD-characterizable if there exist exactly  $k$  non-isomorphic groups  $H$  satisfying conditions  $|G| = |H|$  and  $D(G) = D(H)$ . In particular, a 1-fold OD-characterizable group is simply called OD-characterizable.

The interest in characterizing finite groups by their degree patterns started in [7] by M. R. Darafsheh and et. all, in which the authors proved that the following simple groups are uniquely determined by their order and degree patterns: All sporadic simple groups, the alternating groups  $A_p$  with  $p$  and  $p - 2$  primes and some simple groups of Lie type. Also in a series of articles (see [4, 6, 8, 9, 14, 17]), it was shown that many finite simple groups are OD-characterizable.

Let  $A$  and  $B$  be two groups then a split extension is denoted by  $A : B$ . If  $L$  is a finite simple group and  $Aut(L) \cong L : A$ , then if  $B$  is a cyclic subgroup of  $A$  of order  $n$  we will write  $L : n$  for the split extension  $L : B$ . Moreover if there are more than one subgroup of orders  $n$  in  $A$ , then we will denote them by  $L : n_1, L : n_2$ , etc.

**Definition 1.2.** A group  $G$  is said to be an almost simple group related to  $S$  if and only if  $S \leq G \lesssim Aut(S)$ , for some non-abelian simple group  $S$ .

In many papers (see [2, 3, 10, 13, 15, 16]), it has been proved, up to now, that many finite almost simple groups are OD-characterizable or  $k$ -fold OD-characterizable for certain  $k \geq 2$ .

We denote the socle of  $G$  by  $Soc(G)$ , which is the subgroup generated by the set of all minimal normal subgroups of  $G$ . For  $p \in \pi(G)$ , we denote by  $G_p$  and  $Syl_p(G)$  a Sylow  $p$ -subgroup of  $G$  and the set of all Sylow  $p$ -subgroups of  $G$  respectively, all further unexplained notation are standard and can be found in [11].

In this article our main aim is to show the recognizability of the almost simple groups related to  $L := D_4(4)$  by degree pattern in the prime graph and

order of the group. In fact, we will prove the following Theorem.

**Main Theorem** Let  $M$  be an almost simple group related to  $L := D_4(4)$ . If  $G$  is a finite group such that  $D(G) = D(M)$  and  $|G| = |M|$ , then the following assertions hold:

- (a) If  $M = L$ , then  $G \cong L$ .
- (b) If  $M = L : 2_1$ , then  $G \cong L : 2_1$  or  $L : 2_3$ .
- (c) If  $M = L : 2_2$ , then  $G \cong L : 2_2$  or  $\mathbb{Z}_2 \times L$ .
- (d) If  $M = L : 2_3$ , then  $G \cong L : 2_3$  or  $L : 2_1$ .
- (e) If  $M = L : 3$ , then  $G \cong L : 3$  or  $\mathbb{Z}_3 \times L$ .
- (f) If  $M = L : 2^2$ , then  $G \cong L : 2^2$ ,  $\mathbb{Z}_2 \times (L : 2_1)$ ,  $\mathbb{Z}_2 \times (L : 2_2)$ ,  $\mathbb{Z}_2 \times (L : 2_3)$ ,  $\mathbb{Z}_4 \times L$  or  $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times L$ .
- (g) If  $M = L : (D_6)_1$ , then  $G \cong L : (D_6)_1$ ,  $L : 6$ ,  $\mathbb{Z}_3 \times (L : 2_1)$ ,  $\mathbb{Z}_3 \times (L : 2_3)$  or  $(\mathbb{Z}_3 \times L) \cdot \mathbb{Z}_2$ .
- (h) If  $M = L : (D_6)_2$ , then  $G \cong L : (D_6)_2$ ,  $\mathbb{Z}_2 \times (L : 3)$ ,  $\mathbb{Z}_3 \times (L : 2_2)$ ,  $(\mathbb{Z}_3 \times L) \cdot \mathbb{Z}_2$ ,  $\mathbb{Z}_6 \times L$  or  $D_6 \times L$ .
- (i) If  $M = L : 6$ , then  $G \cong L : 6$ ,  $L : (D_6)_1$ ,  $\mathbb{Z}_3 \times (L : 2_1)$ ,  $\mathbb{Z}_3 \times (L : 2_3)$  or  $(\mathbb{Z}_3 \times L) \cdot \mathbb{Z}_2$ .
- (j) If  $M = L : D_{12}$ , then  $G \cong L : D_{12}$ ,  $\mathbb{Z}_2 \times (L : (D_6)_1)$ ,  $\mathbb{Z}_2 \times (L : (D_6)_2)$ ,  $\mathbb{Z}_2 \times (L : 6)$ ,  $\mathbb{Z}_3 \times (L : 2^2)$ ,  $(\mathbb{Z}_3 \times (L : 2_1)) \cdot \mathbb{Z}_2$ ,  $(\mathbb{Z}_3 \times (L : 2_2)) \cdot \mathbb{Z}_2$ ,  $(\mathbb{Z}_3 \times (L : 2_3)) \cdot \mathbb{Z}_2$ ,  $\mathbb{Z}_4 \times (L : 3)$ ,  $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times (L : 3)$ ,  $(\mathbb{Z}_4 \times L) \cdot \mathbb{Z}_3$ ,  $((\mathbb{Z}_2 \times \mathbb{Z}_2) \times L) \cdot \mathbb{Z}_3$ ,  $\mathbb{Z}_6 \times (L : 2_1)$ ,  $\mathbb{Z}_6 \times (L : 2_2)$ ,  $\mathbb{Z}_6 \times (L : 2_3)$ ,  $(\mathbb{Z}_6 \times L) \cdot \mathbb{Z}_2$ ,  $D_6 \times (L : 2_1)$ ,  $D_6 \times (L : 2_2)$ ,  $D_6 \times (L : 2_3)$ ,  $\mathbb{Z}_{12} \times L$ ,  $(\mathbb{Z}_2 \times \mathbb{Z}_6) \times L$ ,  $(\mathbb{Z}_2 \times L) \cdot D_6$ ,  $\mathbb{A}_4 \times L$ ,  $L \cdot \mathbb{A}_4$ ,  $D_{12} \times L$  or  $T \times L$ .

## 2. PRELIMINARY RESULTS

It is well-known that  $\text{Aut}(D_4(4)) \cong D_4(4) : D_{12}$  where  $D_{12}$  denotes the dihedral group of order 12. We remark that  $D_{12}$  has the following non-trivial proper subgroups up to conjugacy: three subgroups of order 2, one cyclic subgroup each of order 3 and 6, two subgroups isomorphic to  $D_6 \cong \mathbb{S}_3$  and one subgroup of order 4 isomorphic to the Klein's four group denoted by  $2^2$ . The field and the duality automorphisms of  $D_4(4)$  are denoted by  $2_1$  and  $2_2$  respectively, and we set  $2_3 = 2_1 \cdot 2_2$  (field\*duality which is called the diagonal automorphism). Therefore up to conjugacy we have the following almost simple groups related to  $D_4(4)$ .

**Lemma 2.1.** *If  $G$  is an almost simple group related to  $L := D_4(4)$ , then  $G$  is isomorphic to one of the following groups:  $L$ ,  $L : 2_1$ ,  $L : 2_2$ ,  $L : 2_3$ ,  $L : 3$ ,  $L : 2^2$ ,  $L : (D_6)_1$ ,  $L : (D_6)_2$ ,  $L : 6$ ,  $L : D_{12}$ .*

**Lemma 2.2** ([5]). *Let  $G$  be a Frobenius group with kernel  $K$  and complement  $H$ . Then:*

- (a)  $K$  is a nilpotent group.
- (b)  $|K| \equiv 1 \pmod{|H|}$ .

Let  $p \geq 5$  be a prime. We denote by  $\mathfrak{S}_p$  the set of all simple groups with prime divisors at most  $p$ . Clearly, if  $q \leq p$ , then  $\mathfrak{S}_q \subseteq \mathfrak{S}_p$ . We list all the simple groups in class  $\mathfrak{S}_{17}$  with their order and the order of their outer automorphisms in TABLE 1, taken from [12].

TABLE 1: Simple groups in  $\mathfrak{S}_p$ ,  $p \leq 17$ .

$S$	$ S $	$ \text{Out}(S) $	$S$	$ S $	$ \text{Out}(S) $
$A_5$	$2^2 \cdot 3 \cdot 5$	2	$G_2(3)$	$2^6 \cdot 3^6 \cdot 7 \cdot 13$	2
$A_6$	$2^3 \cdot 3^2 \cdot 5$	4	${}^3D_4(2)$	$2^{12} \cdot 3^4 \cdot 7^2 \cdot 13$	3
$S_4(3)$	$2^6 \cdot 3^4 \cdot 5$	2	$L_2(64)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$	6
$L_2(7)$	$2^3 \cdot 3 \cdot 7$	2	$U_4(5)$	$2^7 \cdot 3^4 \cdot 5^6 \cdot 7 \cdot 13$	4
$L_2(8)$	$2^3 \cdot 3^2 \cdot 7$	3	$L_3(9)$	$2^7 \cdot 3^6 \cdot 5 \cdot 7 \cdot 13$	4
$U_3(3)$	$2^5 \cdot 3^3 \cdot 7$	2	$S_6(3)$	$2^9 \cdot 3^9 \cdot 5 \cdot 7 \cdot 13$	2
$A_7$	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	2	$O_7(3)$	$2^9 \cdot 3^9 \cdot 5 \cdot 7 \cdot 13$	2
$L_2(49)$	$2^4 \cdot 3 \cdot 5^2 \cdot 7^2$	4	$G_2(4)$	$2^{12} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13$	2
$U_3(5)$	$2^4 \cdot 3^2 \cdot 5^3 \cdot 7$	6	$S_4(8)$	$2^{12} \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 13$	6
$L_3(4)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	12	$O_8^+(3)$	$2^{12} \cdot 3^{12} \cdot 5^2 \cdot 7 \cdot 13$	24
$A_8$	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	2	$L_5(3)$	$2^9 \cdot 3^{10} \cdot 5 \cdot 11^2 \cdot 13$	2
$A_9$	$2^6 \cdot 3^4 \cdot 5 \cdot 7$	2	$A_{13}$	$2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	2
$J_2$	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	2	$A_{14}$	$2^{10} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13$	2
$A_{10}$	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7$	2	$A_{15}$	$2^{10} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$	2
$U_4(3)$	$2^7 \cdot 3^6 \cdot 5 \cdot 7$	8	$L_6(3)$	$2^{11} \cdot 3^{15} \cdot 5 \cdot 7 \cdot 11^2 \cdot 13^2$	4
$S_4(7)$	$2^8 \cdot 3^2 \cdot 5^2 \cdot 7^4$	2	$Suz$	$2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	2
$S_6(2)$	$2^9 \cdot 3^4 \cdot 5 \cdot 7$	1	$A_{16}$	$2^{14} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$	2
$O_8^+(2)$	$2^{12} \cdot 3^5 \cdot 5^2 \cdot 7$	6	$Fi_{22}$	$2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	2
$L_2(11)$	$2^2 \cdot 3 \cdot 5 \cdot 11$	2	$L_2(17)$	$2^4 \cdot 3^2 \cdot 17$	2
$M_{11}$	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	1	$L_2(16)$	$2^4 \cdot 3 \cdot 5 \cdot 17$	4
$M_{12}$	$2^6 \cdot 3^3 \cdot 5 \cdot 11$	2	$S_4(4)$	$2^8 \cdot 3^2 \cdot 5^2 \cdot 17$	4
$U_5(2)$	$2^{10} \cdot 3^5 \cdot 5 \cdot 11$	2	$He$	$2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$	2
$M_{22}$	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	2	$O_8^-(2)$	$2^{12} \cdot 3^4 \cdot 5 \cdot 7 \cdot 17$	2
$A_{11}$	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$	2	$L_4(4)$	$2^{12} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 17$	4
$M^c L$	$2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$	2	$S_8(2)$	$2^{16} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17$	1
$HS$	$2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$	2	$U_4(4)$	$2^{12} \cdot 3^2 \cdot 5^3 \cdot 13 \cdot 17$	4
$A_{12}$	$2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$	2	$U_3(17)$	$2^6 \cdot 3^4 \cdot 7 \cdot 13 \cdot 17^3$	6
$U_6(2)$	$2^{15} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11$	6	$O_{10}^-(2)$	$2^{20} \cdot 3^6 \cdot 5^2 \cdot 7 \cdot 11 \cdot 17$	2
$L_3(3)$	$2^4 \cdot 3^3 \cdot 13$	2	$L_2(13^2)$	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 13^2 \cdot 17$	4
$L_2(25)$	$2^3 \cdot 3 \cdot 5^2 \cdot 13$	4	$S_4(13)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 13^4 \cdot 17$	2
$U_3(4)$	$2^6 \cdot 3 \cdot 5^2 \cdot 13$	4	$L_3(16)$	$2^{12} \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 13 \cdot 17$	24
$S_4(5)$	$2^6 \cdot 3^2 \cdot 5^4 \cdot 13$	2	$S_6(4)$	$2^{18} \cdot 3^4 \cdot 5^3 \cdot 7 \cdot 13 \cdot 17$	2
$L_4(3)$	$2^7 \cdot 3^6 \cdot 5 \cdot 13$	4	$O_8^+(4)$	$2^{24} \cdot 3^5 \cdot 5^4 \cdot 7 \cdot 13 \cdot 17^2$	12
${}^2F_4(2)'$	$2^{11} \cdot 3^3 \cdot 5^2 \cdot 13$	2	$F_4(2)$	$2^{24} \cdot 3^6 \cdot 5^2 \cdot 7^2 \cdot 13 \cdot 17$	2
$L_2(13)$	$2^2 \cdot 3 \cdot 7 \cdot 13$	2	$A_{17}$	$2^{14} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17$	2
$L_2(27)$	$2^2 \cdot 3^3 \cdot 7 \cdot 13$	6	$A_{18}$	$2^{15} \cdot 3^8 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17$	2

**Definition 2.3.** A completely reducible group will be called a *CR*-group. The center of a *CR*-group is a direct product of the abelian factor in the decomposition. Hence, a *CR*-group is centerless, that is, has trivial center, if and only if it is a direct product of non-abelian simple groups. The following Lemma determines the structure of the automorphism group of a centerless *CR*-group.

**Lemma 2.3** ([11]). *Let  $R$  be a finite centerless *CR*-group and write  $R = R_1 \times R_2 \times \dots \times R_k$ , where  $R_i$  is a direct product of  $n_i$  isomorphic copies of a simple group  $H_i$ , and  $H_i$  and  $H_j$  are not isomorphic if  $i \neq j$ . Then  $\text{Aut}(R) = \text{Aut}(R_1) \times \text{Aut}(R_2) \times \dots \times \text{Aut}(R_k)$  and  $\text{Aut}(R_i) \cong \text{Aut}(H_i) \wr \mathbb{S}_{n_i}$ , where in this wreath product  $\text{Aut}(H_i)$  appears in its right regular representation and the symmetric group  $\mathbb{S}_{n_i}$  in its natural permutation representation. Moreover, these isomorphisms induce isomorphisms  $\text{Out}(R) \cong \text{Out}(R_1) \times \text{Out}(R_2) \times \dots \times \text{Out}(R_k)$  and  $\text{Out}(R_i) \cong \text{Out}(H_i) \wr \mathbb{S}_{n_i}$ .*

### 3. OD-CHARACTERIZATION OF ALMOST SIMPLE GROUPS RELATED TO $D_4(4)$

In this section, we study the problem of characterizing almost simple groups by order and degree pattern. Especially we will focus our attention on almost simple groups related to  $L = D_4(4)$ , namely, we will prove the Main Theorem of Sec. 1. We break the proof into a number of separate propositions.

By assumption, we depict all possibilities for the prime graph associated with  $G$  by use of the variables for some vertices in each proposition. Also, we need to know the structure of  $\Gamma(M)$  to determine the possibilities for  $G$  in some proposition, therefore we depict the prime graph of all extension of  $L$  in pages 18 to 20. Note that the set of order elements in each of the following propositions is calculated using Magma.

**Proposition 3.1.** *If  $M = L$ , then  $G \cong L$ .*

*Proof.* By TABLE 1  $|L| = 2^{24} \cdot 3^5 \cdot 5^4 \cdot 7 \cdot 13 \cdot 17^2$ .  $\pi_e(L) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 15, 17, 20, 21, 30, 34, 51, 63, 65, 85, 255\}$ , so  $D(L) = (3, 4, 4, 1, 1, 3)$ . Since  $|G| = |L|$  and  $D(G) = D(L)$ , we conclude that the prime graph of  $G$  has following form:

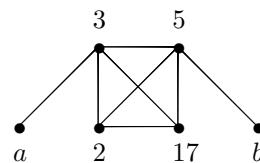


Figure 3.1

where  $\{a, b\} = \{7, 13\}$ .

We will show that  $G$  is isomorphic to  $L = D_4(4)$ . We break up the proof into a several steps.

**Step1.** Let  $K$  be the maximal normal solvable subgroup of  $G$ . Then  $K$  is a  $\{2, 3, 5\}$ -group. In particular,  $G$  is non-solvable.

First we show that  $K$  is a  $17'$ -group. Assume the contrary and let  $17 \in \pi(K)$ . Then 13 dose not divide the order of  $K$ . Otherwise, we may suppose that  $T$  is a Hall  $\{13, 17\}$ -subgroup of  $K$ . It is seen that  $T$  is a nilpotent subgroup of order  $13 \cdot 17^i$  for  $i = 1$  or  $2$ . Thus,  $13 \cdot 17 \in \pi_e(K) \subseteq \pi_e(G)$ , a contradiction. Thus  $\{17\} \subseteq \pi(K) \subseteq \pi(G) - \{13\}$ . Let  $K_{17} \in \text{Syl}_{17}(K)$ . By Frattini argument,  $G = KN_G(K_{17})$ . Therefore,  $N_G(K_{17})$  contains an element  $x$  of order 13. Since  $G$  has no element of order  $13 \cdot 17$ ,  $\langle x \rangle$  should act fixed point freely on  $K_{17}$ , that is implying  $\langle x \rangle K_{17}$  is a Frobenius group. By Lemma 2.2(b),  $|\langle x \rangle| \mid (|K_{17}| - 1)$ . It follows that  $13 \mid 17^i - 1$  for  $i = 1$  or  $2$ , which is a contradiction.

Next, we show that  $K$  is a  $p'$ -group for  $p \in \{a, b\}$ . Let  $p \mid |K|$  and  $K_p \in \text{Syl}_p(K)$ . Now by Frattini argument,  $G = KN_G(K_p)$ , so 17 must divide the order of  $N_G(K_p)$ . Therefore, the normalizer  $N_G(K_p)$  contains an element of order 17, say  $x$ . So  $\langle x \rangle K_p$  is a cyclic subgroup of  $G$  of order  $17 \cdot p$ , and so  $p \sim 17$  in  $\Gamma(G)$ , which is a contradiction. Therefore  $K$  is a  $\{2, 3, 5\}$ -group. In addition, since  $K$  is a proper subgroup of  $G$ , it follows that  $G$  is non-solvable.

**Step 2.** The quotient  $G/K$  is an almost simple group. In fact,  $S \leq G/K \lesssim \text{Aut}(S)$ , where  $S$  is a finite non-abelian simple group isomorphic to  $L := D_4(4)$ .

Let  $\overline{G} = G/K$ . Then  $S := \text{Soc}(\overline{G}) = P_1 \times P_2 \times \dots \times P_m$ , where  $P_i$ 's are finite non-abelian simple groups and  $S \leq \frac{G}{K} \lesssim \text{Aut}(S)$ . If we show that  $m = 1$ , the proof of Step 2 will be completed.

Suppose that  $m \geq 2$ . In this case, we claim that 13 does not divide  $|S|$ . Assume the contrary and let  $13 \mid |S|$ , on the other hand,  $\{2, 3\} \subset \pi(P_i)$  for every  $i$  (by TABLE 1), hence  $2 \sim 13$  and  $3 \sim 13$ , which is a contradiction. Now, by step 1, we observe that  $13 \in \pi(\overline{G}) \subseteq \pi(\text{Aut}(S))$ . But  $\text{Aut}(S) = \text{Aut}(S_1) \times \text{Aut}(S_2) \times \dots \times \text{Aut}(S_r)$ , where the groups  $S_j$  are direct products of isomorphic  $P_i$ 's such that  $S = S_1 \times S_2 \times \dots \times S_r$ . Therefore, for some  $j$ , 13 divides the order of an automorphism group of a direct product  $S_j$  of  $t$  isomorphic simple groups  $P_i$ . Since  $P_i \in \mathfrak{S}_{17}$ , it follows that  $|\text{Out}(P_i)|$  is not divisible by 13 (see TABLE 1). Now, by Lemma 2.3, we obtain  $|\text{Aut}(S_j)| = |\text{Aut}(P_i)|^{t!} \cdot t!$ . Therefore,  $t \geq 13$  and so  $2^{26}$  must divide the order of  $G$ , which is a contradiction. Therefore  $m = 1$  and  $S = P_1$ .

By TABLE 1 and Step 1, it is evident that  $|S| = 2^\alpha \cdot 3^\beta \cdot 5^\gamma \cdot 7 \cdot 13 \cdot 17^2$ , where  $2 \leq \alpha \leq 24$ ,  $1 \leq \beta \leq 5$  and  $0 \leq \gamma \leq 4$ . Now, using collected results contained in TABLE 1, we deduce that  $S \cong D_4(4)$  and by Step 2,  $L \trianglelefteq G/K \lesssim \text{Aut}(L)$  is completed. As  $|G| = |L|$ , we deduce  $K = 1$ , so  $G \cong L$  and the proof is completed.  $\square$

**Proposition 3.2.** *If  $M = L : 2_1$ , then  $G \cong L : 2_1$  or  $L : 2_3$ .*

*Proof.* As  $|L : 2_1| = 2^{25} \cdot 3^5 \cdot 5^4 \cdot 7 \cdot 13 \cdot 17^2$  and  $\pi_e(L : 2_1) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 24, 30, 34, 40, 42, 51, 60, 63, 65, 68, 85, 102, 126, 130, 170, 255\}$ , then  $D(L : 2_1) = (4, 4, 4, 2, 1, 3)$ . Since  $|G| = |L : 2_1|$  and  $D(G) = D(L : 2_1)$ , we conclude that there exist several possibilities for  $\Gamma(G)$ :

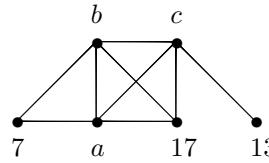


Figure 3.2

where  $\{a, b, c\} = \{2, 3, 5\}$ .

**Step1.** Let  $K$  be the maximal normal solvable subgroup of  $G$ . Then  $K$  is a  $\{2, 3, 5\}$ -group. In particular,  $G$  is non-solvable.

By a similar argument to that in Proposition 3.1, we can obtain this assertion.

**Step 2.** The quotient  $\frac{G}{K}$  is an almost simple group. In fact,  $S \leq \frac{G}{K} \lesssim \text{Aut}(S)$ , where  $S$  is a finite non-abelian simple group.

The proof is similar to Step 2 of Proposition 3.1.

By TABLE 1 and Step 1, it is evident that  $|S| = 2^\alpha \cdot 3^\beta \cdot 5^\gamma \cdot 7 \cdot 13 \cdot 17^2$ , where  $2 \leq \alpha \leq 25$ ,  $1 \leq \beta \leq 5$  and  $0 \leq \gamma \leq 4$ . Now, using collected results contained in TABLE 1, we conclude that  $S \cong D_4(4)$  and by Step 2,  $L \trianglelefteq \frac{G}{K} \lesssim \text{Aut}(L)$ . As  $|G| = |L : 2_1| = 2|L|$ , we deduce  $|K| = 1$  or  $2$ .

If  $|K| = 1$ , then  $G \cong L : 2_1, L : 2_2$  or  $L : 2_3$ . Obviously,  $G \cong L : 2_1$  or  $L : 2_3$  because  $\deg(2) = 5$  in  $\Gamma(L : 2_2)$  (see page 16).

If  $|K| = 2$ , then  $K \leq Z(G)$  and so  $\deg(2) = 5$ , which is a contradiction.  $\square$

**Proposition 3.3.** *If  $M = L : 2_2$ , then  $G \cong L : 2_2$  or  $\mathbb{Z}_2 \times L$ .*

*Proof.* As  $|L : 2_2| = 2^{25} \cdot 3^5 \cdot 5^4 \cdot 7 \cdot 13 \cdot 17^2$  and  $\pi_e(L : 2_2) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 17, 18, 20, 21, 24, 26, 30, 34, 40, 42, 51, 60, 63, 65, 68, 85, 102, 126, 130, 170, 255\}$ , then  $D(L : 2_2) = (5, 4, 4, 2, 2, 3)$ . By assumption  $|G| = |L : 2_2|$  and  $D(G) = D(L : 2_2)$ , so the prime graph of  $G$  has following form:

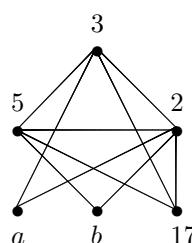


Figure 3.3

where  $\{a, b\} = \{7, 13\}$ .

**Step1.** Let  $K$  be the maximal normal solvable subgroup of  $G$ . Then  $K$  is a  $\{2, 3, 5\}$ -group. In particular,  $G$  is non-solvable.

By similar arguments as in the proof of Step 1 in Proposition 3.1, we conclude that  $K$  is a  $\{2, 3, 5\}$ -group and  $G$  is non-solvable.

**Step 2.** The quotient  $\frac{G}{K}$  is an almost simple group. In fact,  $S \leq \frac{G}{K} \lesssim \text{Aut}(S)$ , where  $S$  is a finite non-abelian simple group.

Let  $\overline{G} = \frac{G}{K}$ . Then  $S := \text{Soc}(\overline{G})$ ,  $S = P_1 \times P_2 \times \dots \times P_m$ , where  $P_i$ 's are finite non-abelian simple groups and  $S \leq \frac{G}{K} \lesssim \text{Aut}(S)$ . We are going to prove that  $m = 1$  and  $S = P_1$ . Suppose that  $m \geq 2$ . We claim  $a$  does not divide  $|S|$ . Assume the contrary and let  $a \mid |S|$ , we conclude that  $a$  just divide the order of one of the simple groups  $P_i$ 's. Without loss of generality, we assume that  $a \mid |P_1|$ . Then the rest of the  $P_i$ 's must be  $\{2, 3\}$ -group (because only 2 and 3 are adjacent to a in  $\Gamma(G)$ ), this is a contradiction because  $P_i$ 's are finite non-abelian simple groups. Now, by Step 1, we observe that  $a \in \pi(\overline{G}) \subseteq \pi(\text{Aut}(S))$ . But  $\text{Aut}(S) = \text{Aut}(S_1) \times \text{Aut}(S_2) \times \dots \times \text{Aut}(S_r)$ , where the groups  $S_j$  are direct products of isomorphic  $P_i$ 's such that  $S = S_1 \times S_2 \times \dots \times S_r$ . Therefore, for some  $j$ ,  $a$  divides the order of an automorphism group of a direct product  $S_j$  of  $t$  isomorphic simple groups  $P_i$ . Since  $P_i \in \mathfrak{S}_{17}$ , it follows that  $|\text{Out}(P_i)|$  is not divisible by  $a$  (see TABLE 1), so  $a$  does not divide the order of  $\text{Aut}(P_i)$ . Now, by Lemma 2.3, we obtain  $|\text{Aut}(S_j)| = |\text{Aut}(P_i)|^{t!} \cdot t!$ . Therefore,  $t \geq a$  and so  $3^a$  must divide the order of  $G$ , which is a contradiction. Therefore  $m = 1$  and  $S = P_1$ .

By TABLE 1 and Step 1, it is evident that  $|S| = 2^\alpha \cdot 3^\beta \cdot 5^\gamma \cdot 7 \cdot 13 \cdot 17^2$ , where  $2 \leq \alpha \leq 25$ ,  $1 \leq \beta \leq 5$  and  $0 \leq \gamma \leq 4$ . Now, using collected results contained in TABLE 1, we conclude that  $S \cong D_4(4)$  and by Step 2,  $L \trianglelefteq \frac{G}{K} \lesssim \text{Aut}(L)$ . As  $|G| = |L : 2_2| = 2|L|$ , we deduce  $|K| = 1$  or 2.

If  $|K| = 1$ , then  $G \cong L : 2_1, L : 2_2$  or  $L : 2_3$  because  $|G| = 2|L|$ . It is obvious that  $G \cong L : 2_2$ , because  $\deg(13) = 1$  in  $\Gamma(L : 2_1)$  and  $\Gamma(L : 2_3)$  (see page 17).

If  $|K| = 2$ , then  $G/K \cong L$  and  $K \leq Z(G)$ . It follows that  $G$  is a central extension of  $K$  by  $L$ . If  $G$  is a non-split extension of  $K$  by  $L$ , then  $|K|$  must divide the Schur multiplier of  $L$ , which is 1. But this is a contradiction, so we obtain that  $G$  split over  $|K|$ . Hence  $G \cong \mathbb{Z}_2 \times L$ .  $\square$

**Proposition 3.4.** *If  $M = L : 2_3$ , then  $G \cong L : 2_3$  or  $L : 2_1$ .*

*Proof.* As  $|L : 2_3| = 2^{25} \cdot 3^5 \cdot 5^4 \cdot 7 \cdot 13 \cdot 17^2$  and  $\pi_e(L : 2_3) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 24, 30, 34, 51, 63, 65, 85, 255\}$ , then  $D(L : 2_3) = (4, 4, 4, 2, 1, 3)$ . Since  $|G| = |L : 2_3|$  and  $D(G) = D(L : 2_3)$ , we conclude that  $\Gamma(G)$  has the following form similarly to Proposition 3.2:

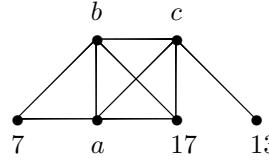


Figure 3.4

where  $\{a, b, c\} = \{2, 3, 5\}$ .

**Step1.** Let  $K$  be the maximal normal solvable subgroup of  $G$ . Then  $K$  is a  $\{2, 3, 5\}$ -group. In particular,  $G$  is non-solvable.

We can prove this by the similar way to that in Proposition 3.2.

**Step 2.** The quotient  $\frac{G}{K}$  is an almost simple group. In fact,  $S \leq \frac{G}{K} \lesssim \text{Aut}(S)$ , where  $S$  is a finite non-abelian simple group.

By using a similar argument, as in the proof of Proposition 3.2, we can verify that  $\frac{G}{K}$  is an almost simple group.

By TABLE 1 and Step 1, it is evident that  $|S| = 2^\alpha \cdot 3^\beta \cdot 5^\gamma \cdot 7 \cdot 13 \cdot 17^2$ , where  $2 \leq \alpha \leq 25$ ,  $1 \leq \beta \leq 5$  and  $0 \leq \gamma \leq 4$ . Now, using collected results contained in TABLE 1, we conclude that  $S \cong D_4(4)$  and by Step 2,  $L \trianglelefteq \frac{G}{K} \lesssim \text{Aut}(L)$ . As  $|G| = |L : 2_3| = 2|L|$ , we deduce  $|K| = 1$  or  $2$ .

If  $|K| = 1$ , then  $G \cong L : 2_1, L : 2_2$  or  $L : 2_3$  because  $|G| = 2|L|$ . Obviously,  $G \cong L : 2_3$  or  $L : 2_1$ , because  $\text{deg}(2) = 5$  in  $\Gamma(L : 2_2)$  (see page 16).

If  $|K| = 2$ , then  $K \leq Z(G)$  and so  $\text{deg}(2) = 5$ , which is a contradiction.  $\square$

**Proposition 3.5.** *If  $M = L : 3$ , then  $G \cong L : 3$  or  $\mathbb{Z}_3 \times L$ .*

*Proof.* As  $|L : 3| = 2^{24} \cdot 3^6 \cdot 5^4 \cdot 7 \cdot 13 \cdot 17^2$  and  $\pi_e(L : 3) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 15, 17, 18, 20, 21, 24, 30, 34, 39, 45, 51, 63, 65, 85, 255\}$ , then  $D(L : 3) = (3, 5, 4, 1, 2, 3)$ . since  $|G| = |L : 3|$  and  $D(G) = D(L : 3)$ , we conclude that  $\Gamma(G)$  has the following form (like  $\Gamma(L : 3)$ ):

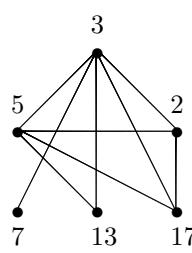


Figure 3.5

**Step1.** Let  $K$  be the maximal normal solvable subgroup of  $G$ . Then  $K$  is a  $\{2, 3\}$ -group. In particular,  $G$  is non-solvable.

First, we show that  $K$  is a  $p'$ -group for  $p = 7, 13$  and  $17$ . Since the proof is quite similar to the proof of Step 1 in Proposition 3.1, so we avoid here full explanation of all details.

Next we consider  $K$  is a  $5'$ -group. Assume the contrary,  $5 \in \pi_e(K)$ . Let  $K_5 \in \text{Syl}_5(K)$ . By Frattini argument,  $G = KN_G(K_5)$ . Therefore,  $N_G(K_5)$  has an element  $x$  of order 7. Since  $G$  has no element of order 5.7,  $\langle x \rangle$  should act fixed point freely on  $K_5$ , implying  $\langle x \rangle K_5$  is a Frobenius group. By Lemma 2.2(b),  $|\langle x \rangle| \mid (|K_5| - 1)$ , which is impossible. Therefore  $K$  is a  $\{2, 3\}$ -group. In addition since  $K$  is a proper subgroup of  $G$ , then  $G$  is non-solvable and the proof of this step is completed.

**Step 2.** The quotient  $\frac{G}{K}$  is an almost simple group. In fact,  $S \leq \frac{G}{K} \lesssim \text{Aut}(S)$ , where  $S$  is a finite non-abelian simple group.

In a similar way as in the proof of Step 2 in Proposition 3.1, we conclude that  $\frac{G}{K}$  is an almost simple group.

By TABLE 1 and Step 1, it is evident that  $|S| = 2^\alpha \cdot 3^\beta \cdot 5^4 \cdot 7 \cdot 13 \cdot 17^2$ , where  $2 \leq \alpha \leq 24$  and  $1 \leq \beta \leq 6$ . Now, using collected results contained in TABLE 1, we conclude that  $S \cong D_4(4)$  and by Step 2,  $L \trianglelefteq \frac{G}{K} \lesssim \text{Aut}(L)$ . As  $|G| = |L| : 3| = 3|L|$ , we deduce  $|K| = 1$  or 3.

If  $|K| = 1$ , then  $G \cong L : 3$ .

If  $|K| = 3$ , then  $G/K \cong L$ . In this case we have  $G/C_G(K) \lesssim \text{Aut}(K) \cong \mathbb{Z}_2$ . Thus  $|G/C_G(K)| = 1$  or 2. If  $|G/C_G(K)| = 1$ , then  $K \leq Z(G)$ , that is,  $G$  is a central extension of  $K$  by  $L$ . If  $G$  is a non-split extension of  $K$  by  $L$ , then  $|K|$  must divide the Schur multiplier of  $L$ , which is 1. But this is a contradiction, so we obtain that  $G$  split over  $K$ . Hence  $G \cong \mathbb{Z}_3 \times L$ . If  $|G/C_G(K)| = 2$ , then  $K < C_G(K)$  and  $1 \neq C_G(K)/K \trianglelefteq G/K \cong L$ , which is a contradiction since  $L$  is simple.  $\square$

**Proposition 3.6.** *If  $M = L : 2^2$ , then  $G \cong L : 2^2$ ,  $\mathbb{Z}_2 \times (L : 2_1)$ ,  $\mathbb{Z}_2 \times (L : 2_2)$ ,  $\mathbb{Z}_2 \times (L : 2_3)$ ,  $\mathbb{Z}_4 \times L$  or  $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times L$ .*

*Proof.* As  $|L : 2^2| = 2^{26} \cdot 3^5 \cdot 5^4 \cdot 7 \cdot 13 \cdot 17^2$  and  $\pi_e(L : 2^2) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 24, 26, 30, 34, 42, 51, 60, 63, 65, 68, 85, 102, 126, 130, 170, 255\}$ , then  $D(L : 2^2) = (5, 4, 4, 2, 2, 3)$ . Since  $|G| = |L : 2^2|$  and  $D(G) = D(L : 2^2)$ , so the prime graph of  $G$  has following form similarly to Proposition 3.3:

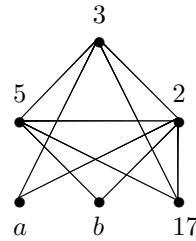


Figure 3.6

where  $\{a, b\} = \{7, 13\}$ .

**Step1.** Let  $K$  be the maximal normal solvable subgroup of  $G$ . Then  $K$  is a  $\{2, 3, 5\}$ -group. In particular,  $G$  is non-solvable.

According to Step 1 in Proposition 3.3, we have  $K$  is a  $\{2, 3, 5\}$ -group and  $G$  is non-solvable.

**Step 2.** The quotient  $\frac{G}{K}$  is an almost simple group. In fact,  $S \leq \frac{G}{K} \lesssim \text{Aut}(S)$ , where  $S$  is a finite non-abelian simple group.

We can prove this by the similar argument in Step 2 in Proposition 3.3.

By TABLE 1 and Step 1, it is evident that  $|S| = 2^\alpha \cdot 3^\beta \cdot 5^\gamma \cdot 7 \cdot 13 \cdot 17^2$ , where  $2 \leq \alpha \leq 26$ ,  $1 \leq \beta \leq 5$  and  $0 \leq \gamma \leq 4$ . Now, using collected results contained in TABLE 1, we conclude that  $S \cong D_4(4)$  and by Step 2,  $L \trianglelefteq \frac{G}{K} \lesssim \text{Aut}(L)$ . As  $|G| = |L : 2^2| = 4|L|$ , we deduce  $|K| = 1, 2$  or  $4$ .

If  $|K| = 1$ , then  $G \cong L : 2^2$ .

If  $|K| = 2$ , then  $K \leq Z(G)$ . In this case  $G$  is a central extension of  $\mathbb{Z}_2$  by  $L : 2_1$ ,  $L : 2_2$  or  $L : 2_3$ . If  $G$  splits over  $K$  then  $G \cong \mathbb{Z}_2 \times (L : 2_1)$ ,  $\mathbb{Z}_2 \times (L : 2_2)$  or  $\mathbb{Z}_2 \times (L : 2_3)$ , otherwise we get a contradiction because  $|K|$  must divide the Schure multiplier of  $L : 2_1$ ,  $L : 2_2$  and  $L : 2_3$ , which is impossible.

If  $|K| = 4$ , then  $G/K \cong L$ . In this case we have  $G/C_G(K) \lesssim \text{Aut}(K) \cong \mathbb{Z}_2$  or  $S_3$ . Thus  $|G/C_G(K)| = 1, 2, 3$  or  $6$ . If  $|G/C_G(K)| = 1$ , then  $K \leq Z(G)$ , that is,  $G$  is a central extension of  $K$  by  $L$ . If  $G$  is a non-split extension of  $K$  by  $L$ , then  $|K|$  must divide the Schur multiplier of  $L$ , which is 1, but this is a contradiction. Therefore  $G$  splits over  $K$ . Hence  $G \cong K \times L$ . So we have  $G \cong \mathbb{Z}_4 \times L$  or  $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times L$  because  $K \cong \mathbb{Z}_4$  or  $\mathbb{Z}_2 \times \mathbb{Z}_2$ . If  $|G/C_G(K)| = 2, 3$  or  $6$ , then  $K < C_G(K)$  and  $1 \neq C_G(K)/K \trianglelefteq G/K \cong L$ . Which is a contradiction, since  $L$  is simple.  $\square$

**Proposition 3.7.** If  $M = L : (D_6)_1$ , then  $G \cong L : (D_6)_1$ ,  $L : 6$ ,  $\mathbb{Z}_3 \times (L : 2_1)$ ,  $\mathbb{Z}_3 \times (L : 2_3)$  or  $(\mathbb{Z}_3 \times L).\mathbb{Z}_2$ .

*Proof.* As  $|L : (D_6)_1| = 2^{25} \cdot 3^6 \cdot 5^4 \cdot 7 \cdot 13 \cdot 17^2$  and  $\pi_e(L : (D_6)_1) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 24, 30, 34, 39, 42, 45, 51, 60, 63, 65, 85, 255\}$ , then  $D(L : (D_6)_1) = (4, 5, 4, 2, 2, 3)$ . Since  $|G| = |L : (D_6)_1|$  and  $D(G) = D(L : (D_6)_1)$ , we conclude that there exist several possibilities for  $\Gamma(G)$ :

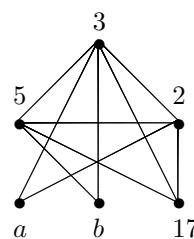


Figure 3.7

where  $\{a, b\} = \{7, 13\}$ .

**Step1.** Let  $K$  be the maximal normal solvable subgroup of  $G$ . Then  $K$  is a  $\{2, 3, 5\}$ -group. In particular,  $G$  is non-solvable.

By the similar argument to that in Step 1 in Proposition 3.1, we can obtain this assertion.

**Step 2.** The quotient  $\frac{G}{K}$  is an almost simple group. In fact,  $S \leq \frac{G}{K} \lesssim \text{Aut}(S)$ , where  $S$  is a finite non-abelian simple group.

The proof is similar to Step 2 in Proposition 3.3.

By TABLE 1 and Step 1, it is evident that  $|S| = 2^\alpha \cdot 3^\beta \cdot 5^\gamma \cdot 7 \cdot 13 \cdot 17^2$ , where  $2 \leq \alpha \leq 25$ ,  $1 \leq \beta \leq 6$  and  $0 \leq \gamma \leq 4$ . Now, using collected results contained in TABLE 1, we conclude that  $S \cong D_4(4)$  and by Step 2,  $L \trianglelefteq \frac{G}{K} \lesssim \text{Aut}(L)$ . As  $|G| = |L : D_6|_1 = 6|L|$ , we deduce  $|K| = 1, 2, 3$  or  $6$ .

If  $|K| = 1$ , then  $G \cong L : (D_6)_1, L : (D_6)_2$  or  $L : 6$  because  $|G| = 6|L|$ . Obviously,  $G \cong L : (D_6)_1$  or  $L : 6$  because  $\deg(2) = 5$  in  $\Gamma(L : (D_6)_2)$ .

If  $|K| = 2$ , then  $K \leq Z(G)$  and so  $\deg(2) = 5$ , which is a contradiction (see page 18).

If  $|K| = 3$ , then  $G/K \cong L : 2_1, L : 2_2$  or  $L : 2_3$ . But  $G/C_G(K) \lesssim \text{Aut}(K) \cong \mathbb{Z}_2$ . Thus  $|G/C_G(K)| = 1$  or  $2$ . If  $|G/C_G(K)| = 1$ , then  $K \leq Z(G)$ , that is,  $G$  is a central extension of  $K$  by  $L : 2_1, L : 2_2$  or  $L : 2_3$ . If  $G$  splits over  $K$ , then  $G \cong \mathbb{Z}_3 \times (L : 2_1)$  or  $\mathbb{Z}_3 \times (L : 2_3)$  because in  $\Gamma(\mathbb{Z}_3 \times (L : 2_2))$  the degree of 2 is 5. Otherwise we get a contradiction because  $|K|$  must divide the Schure multiplier of  $L : 2_1, L : 2_2$  and  $L : 2_3$ , which is impossible. If  $|G/C_G(K)| = 2$ , then  $K < C_G(K)$  and  $1 \neq C_G(K)/K \trianglelefteq G/K \cong L : 2_1, L : 2_2$  or  $L : 2_3$ , we obtain  $C_G(K)/K \cong L$ . Since  $K \leq Z(C_G(K))$ ,  $C_G(K)$  is a central extension of  $K$  by  $L$ . If  $C_G(K)$  splits over  $K$ , then  $C_G(K) \cong \mathbb{Z}_3 \times L$ , otherwise we get a contradiction because  $|K|$  must divide the Schure multiplier of  $L$ , which is impossible. Therefore,  $G \cong (\mathbb{Z}_3 \times L).\mathbb{Z}_2$ .

If  $|K| = 6$ , then  $G/K \cong L$  and  $K \cong \mathbb{Z}_6$  or  $D_6$ .

If  $K \cong \mathbb{Z}_6$ , then  $G/C_G(K) \lesssim \mathbb{Z}_2$  and so  $|G/C_G(K)| = 1$  or  $2$ . If  $|G/C_G(K)| = 1$ , then  $K \leq Z(G)$ . It follows that  $\deg(2) = 5$ , a contradiction. If  $|G/C_G(K)| = 2$ , then  $K < C_G(K)$  and  $1 \neq C_G(K)/K \trianglelefteq G/K \cong L$ , which is a contradiction because  $L$  is simple.

If  $K \cong D_6$ , then  $K \cap C_G(K) = 1$  and  $G/C_G(K) \lesssim D_6$ . Thus  $C_G(K) \neq 1$ . Hence,  $1 \neq C_G(K) \cong C_G(K)K/K \trianglelefteq G/K \cong L$ . It follows that  $L \cong G/K \cong C_G(K)$  because  $L$  is simple. Therefore,  $G \cong D_6 \times L$ , which implies that  $\deg(2) = 5$ , a contradiction.  $\square$

**Proposition 3.8.** *If  $M = L : (D_6)_2$ , then  $G \cong L : (D_6)_2, \mathbb{Z}_2 \times (L : 3), \mathbb{Z}_3 \times (L : 2_2), (\mathbb{Z}_3 \times L).\mathbb{Z}_2, \mathbb{Z}_6 \times L$  or  $S_3 \times L$ .*

*Proof.* As  $|L : (D_6)_2| = 2^{25} \cdot 3^6 \cdot 5^4 \cdot 7 \cdot 13 \cdot 17^2$  and  $\pi_e(L : (D_6)_2) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 17, 18, 20, 21, 24, 26, 30, 34, 39, 40, 42, 45, 51, 60, 63, 65, 68, 85, 102, 126, 130, 170, 255\}$ , then  $D(L : (D_6)_2) = (5, 5, 4, 2, 3, 3)$ . Since  $|G| = |L : (D_6)_2|$  and  $D(G) = D(L : (D_6)_2)$ , we conclude that  $\Gamma(G)$  has the following form (like  $\Gamma(L : (D_6)_2)$ ):

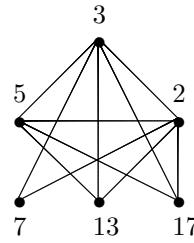


Figure 3.8

**Step1.** Let  $K$  be the maximal normal solvable subgroup of  $G$ . Then  $K$  is a  $\{2, 3\}$ -group. In particular,  $G$  is non-solvable.

The proof is similar to Step 1 in Proposition 3.5.

**Step 2.** The quotient  $\frac{G}{K}$  is an almost simple group. In fact,  $S \leq \frac{G}{K} \lesssim \text{Aut}(S)$ , where  $S$  is a finite non-abelian simple group.

Let  $\overline{G} = \frac{G}{K}$ . Then  $S := \text{Soc}(\overline{G})$ ,  $S = P_1 \times P_2 \times \dots \times P_m$ , where  $P_i$ 's are finite non-abelian simple groups and  $S \leq \frac{G}{K} \lesssim \text{Aut}(S)$ . We are going to prove that  $m = 1$  and  $S = P_1$ . Suppose that  $m \geq 2$ . By the same argument in Step 2 of Proposition 3.3 and considering 7 instead of a, we get a contradiction. Therefore  $m = 1$  and  $S = P_1$ .

By TABLE 1 and Step 1, it is evident that  $|S| = 2^\alpha \cdot 3^\beta \cdot 5^4 \cdot 7 \cdot 13 \cdot 17^2$ , where  $2 \leq \alpha \leq 25$  and  $1 \leq \beta \leq 6$ . Now, using collected results contained in TABLE 1, we conclude that  $S \cong D_4(4)$  and by Step 2,  $L \trianglelefteq \frac{G}{K} \lesssim \text{Aut}(L)$ . As  $|G| = |L : (D_6)_2| = 6|L|$ , we deduce  $|K| = 1, 2, 3$  or 6.

If  $|K| = 1$ , then  $G \cong L : (D_6)_1, L : (D_6)_2$  or  $L : 6$  because  $|G| = 6|L|$ . Obviously  $G \cong L : (D_6)_2$  because in  $\Gamma(L : (D_6)_1)$  and  $\Gamma(L : 6)$ , we have  $\deg(13) = 2$  (see page 17).

If  $|K| = 2$ , then  $K \leq Z(G)$  and  $G/K \cong L : 3$ . Hence  $G$  is a central extension of  $K$  by  $L : 3$ . If  $G$  splits over  $K$ , then  $G \cong \mathbb{Z}_2 \times (L : 3)$ . Otherwise we get a contradiction because  $|K|$  must divide the Schure multiplier of  $L : 3$ , which is impossible.

If  $|K| = 3$ , then  $G/K \cong L : 2_1, L : 2_2$  or  $L : 2_3$ . But  $G/C_G(K) \lesssim \text{Aut}(K) \cong \mathbb{Z}_2$ . Thus  $|G/C_G(K)| = 1$  or 2. If  $|G/C_G(K)| = 1$ , then  $K \leq Z(G)$ , that is,  $G$  is a central extension of  $K$  by  $L : 2_1, L : 2_2$  or  $L : 2_3$ . If  $G$  splits over  $K$ , then only  $G \cong \mathbb{Z}_3 \times (L : 2_2)$  because  $2 \not\sim 13$  in  $\Gamma(\mathbb{Z}_3 \times (L : 2_1))$  and  $\Gamma(\mathbb{Z}_3 \times (L : 2_3))$ . Otherwise we get a contradiction because  $|K|$  must divide the Schure multiplier of  $L : 2_1, L : 2_2$  and  $L : 2_3$ , which is impossible. If

$|G/C_G(K)| = 2$ , then  $K < C_G(K)$  and  $1 \neq C_G(K)/K \trianglelefteq G/K \cong L : 2_1, L : 2_2$  or  $L : 2_3$ , we obtain  $C_G(K)/K \cong L$ . Since  $K \leq Z(C_G(K))$ ,  $C_G(K)$  is a central extension of  $K$  by  $L$ . If  $C_G(K)$  splits over  $K$ , then  $C_G(K) \cong \mathbb{Z}_3 \times L$ , otherwise we get a contradiction because  $|K|$  must divide the Schure multiplier of  $L$ , which is impossible. Therefore,  $G \cong (\mathbb{Z}_3 \times L).\mathbb{Z}_2$ .

If  $|K| = 6$ , then  $G/K \cong L$  and  $K \cong \mathbb{Z}_6$  or  $D_6$ . If  $K \cong \mathbb{Z}_6$ , then  $G/C_G(K) \lesssim \mathbb{Z}_2$  and so  $|G/C_G(K)| = 1$  or  $2$ . If  $|G/C_G(K)| = 1$ , then  $K \leq Z(G)$  and  $G/K \cong L$ . Therefore  $G$  is a central extension of  $K$  by  $L$ . If  $G$  is a non-split extension of  $K$  by  $L$ , then  $|K|$  must divide the Schure multiplier of  $L$ , which is  $1$ . But this is a contradiction. So we obtain that  $G$  splits over  $K$ . Hence  $G \cong \mathbb{Z}_6 \times L$ . If  $|G/C_G(K)| = 2$ , then  $K < C_G(K)$  and  $1 \neq C_G(K)/K \trianglelefteq G/K \cong L$ , which is a contradiction because  $L$  is simple. If  $K \cong D_6$ , then  $K \cap C_G(K) = 1$  and  $G/C_G(K) \lesssim D_6$ . Thus  $C_G(K) \neq 1$ . Hence,  $1 \neq C_G(K) \cong C_G(K)K/K \trianglelefteq G/K \cong L$ . It follows that  $L \cong G/K \cong C_G(K)$  because  $L$  is simple. Therefore  $G \cong D_6 \times L$ .  $\square$

**Proposition 3.9.** *If  $M = L : 6$ , then  $G \cong L : 6$ ,  $L : (D_6)_1$ ,  $\mathbb{Z}_3 \times (L : 2_1)$ ,  $\mathbb{Z}_3 \times (L : 2_3)$  or  $(\mathbb{Z}_3 \times L).\mathbb{Z}_2$ .*

*Proof.* As  $|L : 6| = 2^{25} \cdot 3^6 \cdot 5^4 \cdot 7 \cdot 13 \cdot 17^2$  and  $\pi_e(L : 6) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 24, 30, 34, 36, 39, 42, 45, 48, 51, 63, 65, 85, 255\}$ , then  $D(L : 6) = (4, 5, 4, 2, 2, 3)$ . Since  $|G| = |L : 6|$  and  $D(G) = D(L : 6)$ , there exist several possibilities for  $\Gamma(G)$  similarly to Proposition 3.7:

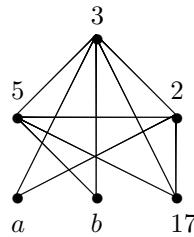


Figure 3.9

where  $\{a, b\} = \{7, 13\}$ .

**Step1.** Let  $K$  be the maximal normal solvable subgroup of  $G$ . Then  $K$  is a  $\{2, 3, 5\}$ -group. In particular,  $G$  is non-solvable.

The proof is similar to that in Proposition 3.3.

**Step 2.** The quotient  $\frac{G}{K}$  is an almost simple group. In fact,  $S \leq \frac{G}{K} \lesssim \text{Aut}(S)$ , where  $S$  is a finite non-abelian simple group.

Again we refer to Step 2 of proposition 3.3 to get the proof.

By TABLE 1 and Step 1, it is evident that  $|S| = 2^\alpha \cdot 3^\beta \cdot 5^\gamma \cdot 7 \cdot 13 \cdot 17^2$ , where  $2 \leq \alpha \leq 25$ ,  $1 \leq \beta \leq 6$  and  $0 \leq \gamma \leq 4$ . Now, using collected results contained

in TABLE 1, we conclude that  $S \cong D_4(4)$  and by Step 2,  $L \trianglelefteq \frac{G}{K} \lesssim \text{Aut}(L)$ . As  $|G| = |L : 6| = 6|L|$ , we deduce  $|K| = 1, 2, 3$  or 6.

If  $|K| = 1$ , then  $G \cong L : 6$ ,  $L : (D_6)_1$  or  $L : (D_6)_2$  because  $|G| = 6|L|$ . Obviously,  $G \cong L : 6$  or  $L : (D_6)_1$  because  $\deg(2) = 5$  in  $\Gamma(L : (D_6)_2)$  (see page 18).

If  $|K| = 2$ , then  $K \leq Z(G)$  and so  $\deg(2) = 5$ , which is a contradiction.

If  $|K| = 3$ , then  $G/K \cong L : 2_1, L : 2_2$  or  $L : 2_3$ . But  $G/C_G(K) \lesssim \text{Aut}(K) \cong \mathbb{Z}_2$ . Thus  $|G/C_G(K)| = 1$  or 2. If  $|G/C_G(K)| = 1$ , then  $K \leq Z(G)$ , that is,  $G$  is a central extension of  $K$  by  $L : 2_1, L : 2_2$  or  $L : 2_3$ . If  $G$  splits over  $K$ , then  $G \cong \mathbb{Z}_3 \times (L : 2_1)$  or  $\mathbb{Z}_3 \times (L : 2_3)$  because in  $\Gamma(\mathbb{Z}_3 \times (L : 2_2))$  the degree of 2 is 5. Otherwise we get a contradiction because  $|K|$  must divide the Schure multiplier of  $L : 2_1, L : 2_2$  and  $L : 2_3$ , which is impossible. If  $|G/C_G(K)| = 2$ , then  $K < C_G(K)$  and  $1 \neq C_G(K)/K \trianglelefteq G/K \cong L : 2_1, L : 2_2$  or  $L : 2_3$ , we obtain  $C_G(K)/K \cong L$ . Since  $K \leq Z(C_G(K))$ ,  $C_G(K)$  is a central extension of  $K$  by  $L$ . If  $C_G(K)$  splits over  $K$ , then  $C_G(K) \cong \mathbb{Z}_3 \times L$ , otherwise we get a contradiction because  $|K|$  must divide the Schure multiplier of  $L$ , which is impossible. Therefore,  $G \cong (\mathbb{Z}_3 \times L).\mathbb{Z}_2$ .

If  $|K| = 6$ , then  $G/K \cong L$  and  $K \cong \mathbb{Z}_6$  or  $D_6$ . If  $K \cong \mathbb{Z}_6$ , then  $G/C_G(K) \lesssim \mathbb{Z}_2$  and so  $|G/C_G(K)| = 1$  or 2. If  $|G/C_G(K)| = 1$ , then  $K \leq Z(G)$ . It follows that  $\deg(2) = 5$ , a contradiction. If  $|G/C_G(K)| = 2$ , then  $K < C_G(K)$  and  $1 \neq C_G(K)/K \trianglelefteq G/K \cong L$ , which is a contradiction because  $L$  is simple. If  $K \cong D_6$ , then  $K \cap C_G(K) = 1$  and  $G/C_G(K) \lesssim D_6$ . Thus  $C_G(K) \neq 1$ . Hence,  $1 \neq C_G(K) \cong C_G(K)K/K \trianglelefteq G/K \cong L$ . It follows that  $L \cong G/K \cong C_G(K)$  because  $L$  is simple. Therefore,  $G \cong D_6 \times L$ , which implies that  $\deg(2) = 5$ , a contradiction.  $\square$

**Proposition 3.10.** *If  $M = L : D_{12}$ , then  $G \cong L : D_{12}, \mathbb{Z}_2 \times (L : (D_6)_1), \mathbb{Z}_2 \times (L : (D_6)_2), \mathbb{Z}_2 \times (L : 6), \mathbb{Z}_3 \times (L : 2^2), (\mathbb{Z}_3 \times (L : 2_1)).\mathbb{Z}_2, (\mathbb{Z}_3 \times (L : 2_2)).\mathbb{Z}_2, (\mathbb{Z}_3 \times (L : 2_3)).\mathbb{Z}_2, \mathbb{Z}_4 \times (L : 3), (\mathbb{Z}_2 \times \mathbb{Z}_2) \times (L : 3), (\mathbb{Z}_4 \times L).\mathbb{Z}_3, ((\mathbb{Z}_2 \times \mathbb{Z}_2) \times L).\mathbb{Z}_3, \mathbb{Z}_6 \times (L : 2_1), \mathbb{Z}_6 \times (L : 2_2), \mathbb{Z}_6 \times (L : 2_3), (\mathbb{Z}_6 \times L).\mathbb{Z}_2, S_3 \times (L : 2_1), S_3 \times (L : 2_2), S_3 \times (L : 2_3), \mathbb{Z}_{12} \times L, (\mathbb{Z}_2 \times \mathbb{Z}_6) \times L, D_{12} \times L, (\mathbb{Z}_2 \times L).D_6, \mathbb{A}_4 \times L, L.\mathbb{A}_4$  or  $T \times L$ .*

*Proof.* As  $|L : D_{12}| = 2^{26} \cdot 3^6 \cdot 5^4 \cdot 7 \cdot 13 \cdot 17^2$  and  $\pi_e(L : (D_{12})) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 24, 26, 30, 34, 39, 40, 42, 45, 48, 51, 60, 63, 65, 68, 85, 102, 126, 130, 170, 255\}$ , then  $D(L : D_{12}) = (5, 5, 4, 2, 3, 3)$ . Since  $|G| = |L : D_{12}|$  and  $D(G) = D(L : D_{12})$ , we conclude that  $\Gamma(G)$  has the following form (like  $\Gamma(L : D_{12})$ ):

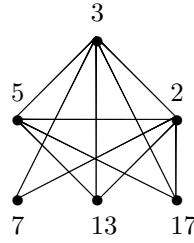


Figure 3.10

**Step1.** Let  $K$  be the maximal normal solvable subgroup of  $G$ . Then  $K$  is a  $\{2, 3\}$ -group. In particular,  $G$  is non-solvable.

The proof is similar to Step 1 in Proposition 3.5.

**Step 2.** The quotient  $\frac{G}{K}$  is an almost simple group. In fact,  $S \leq \frac{G}{K} \lesssim \text{Aut}(S)$ , where  $S$  is a finite non-abelian simple group.

To get the proof, follow the way in the proof of Step 2 in proposition 3.5.

By TABLE 1 and Step 1, it is evident that  $|S| = 2^\alpha \cdot 3^\beta \cdot 5^4 \cdot 7 \cdot 13 \cdot 17^2$ , where  $2 \leq \alpha \leq 26$  and  $1 \leq \beta \leq 6$ . Now, using collected results contained in TABLE 1, we conclude that  $S \cong D_4(4)$  and by Step 2,  $L \trianglelefteq \frac{G}{K} \lesssim \text{Aut}(L)$ . As  $|G| = |L : D_{12}| = 12|L|$ , we deduce  $|K| = 1, 2, 3, 4, 6$  or  $12$ .

If  $|K| = 1$ , then  $G \cong L : D_{12}$ .

If  $|K| = 2$ , then  $G/K \cong L : (D_6)_1, L : (D_6)_2$  or  $L : 6$  and  $K \leq Z(G)$ . It follows that  $G$  is a central extension of  $K$  by  $L : (D_6)_1, L : (D_6)_2$  or  $L : 6$ . If  $G$  splits over  $K$ , then  $G \cong \mathbb{Z}_2 \times (L : (D_6)_1), \mathbb{Z}_2 \times (L : (D_6)_2)$  or  $\mathbb{Z}_2 \times (L : 6)$ . Otherwise  $G \cong \mathbb{Z}_2 \cdot (L : (D_6)_1)$  or  $\mathbb{Z}_2 \cdot (L : (D_6)_2)$ .

If  $|K| = 3$ , then  $G/K \cong L : 2^2$ . But  $G/C_G(K) \lesssim \text{Aut}(K) \cong \mathbb{Z}_2$ . Thus  $|G/C_G(K)| = 1$  or  $2$ . If  $|G/C_G(K)| = 1$ , then  $K \leq Z(G)$ , that is,  $G$  is a central extension of  $K$  by  $L : 2^2$ . If  $G$  splits over  $K$ , then  $G \cong \mathbb{Z}_3 \times (L : 2^2)$ , Otherwise we get a contradiction because  $|K|$  must divide the Schure multiplier of  $L : 2^2$ , which is impossible. If  $|G/C_G(K)| = 2$ , then  $K < C_G(K)$  and  $1 \neq C_G(K)/K \trianglelefteq G/K \cong L : 2^2$ , and we obtain  $C_G(K)/K \cong L : 2_1, L : 2_2$  or  $L : 2_3$ . Since  $K \leq Z(C_G(K))$ ,  $C_G(K)$  is a central extension of  $K$  by  $L : 2_1, L : 2_2$  or  $L : 2_3$ . Thus  $C_G(K) \cong \mathbb{Z}_3 \times (L : 2_1), \mathbb{Z}_3 \times (L : 2_2)$  or  $\mathbb{Z}_3 \times (L : 2_3)$ , otherwise we get a contradiction because  $3$  must divide the Schure multiplier of  $L : 2_1, L : 2_2$  or  $L : 2_3$ , which is impossible. Therefore,  $G \cong (\mathbb{Z}_3 \times (L : 2_1)) \cdot \mathbb{Z}_2, (\mathbb{Z}_3 \times (L : 2_2)) \cdot \mathbb{Z}_2$  or  $(\mathbb{Z}_3 \times (L : 2_3)) \cdot \mathbb{Z}_2$ .

If  $|K| = 4$ , then  $G/K \cong L : 3$  and  $K \cong \mathbb{Z}_4$  or  $\mathbb{Z}_2 \times \mathbb{Z}_2$ . In this case we have  $G/C_G(K) \lesssim \text{Aut}(K) \cong \mathbb{Z}_2$  or  $S_3$ . Thus  $|G/C_G(K)| = 1, 2, 3$  or  $6$ . If  $|G/C_G(K)| = 1$ , then  $K \leq Z(G)$ , that is,  $G$  is a central extension of  $K$  by  $L : 3$ . If  $G$  split over  $K$  by  $L : 3$ , then  $G \cong \mathbb{Z}_4 \times (L : 3)$  or  $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times (L : 3)$ . Otherwise we get a contradiction because  $|K|$  must divide the Schure multiplier of  $L : 3$ , which is impossible. If  $|G/C_G(K)| \neq 1$ , since  $|G/C_G(K)| = 2, 3$  or  $6$ , it follows that  $K < C_G(K)$ . As  $L$  is simple, we conclude that  $1 \neq C_G(K)/K$  must

be an extension of  $L$ . Hence  $|G/C_G(K)| = 3$  and therefore  $C_G(K)/K \cong L$ . Now, since  $K \leq Z(C_G(K))$ , we conclude that  $C_G(K)$  is a central extension of  $K$  by  $L$ . Thus  $C_G(K) \cong \mathbb{Z}_4 \times L$ , or  $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times L$ , otherwise  $|K|$  must divide the Schure multiplier of  $L$ , which is 1 and it is impossible. Therefore,  $G \cong (\mathbb{Z}_4 \times L).\mathbb{Z}_3$  or  $((\mathbb{Z}_2 \times \mathbb{Z}_2) \times L).\mathbb{Z}_3$ .

If  $|K| = 6$ , then  $G/K \cong L : 2_1$ ,  $L : 2_2$  or  $L : 2_3$  and  $K \cong \mathbb{Z}_6$  or  $D_6$ . If  $K \cong \mathbb{Z}_6$ , then  $G/C_G(K) \lesssim \mathbb{Z}_2$  and so  $|G/C_G(K)| = 1$  or 2. If  $|G/C_G(K)| = 1$ , then  $K \leq Z(G)$ , that is  $G$  is a central extension of  $\mathbb{Z}_6$  by  $L : 2_1$ ,  $L : 2_2$  or  $L : 2_3$ . If  $G$  splits over  $K$ , we obtain  $G \cong \mathbb{Z}_6 \times (L : 2_1)$ ,  $\mathbb{Z}_6 \times (L : 2_2)$  or  $\mathbb{Z}_6 \times (L : 2_3)$ , otherwise we get a contradiction because  $|K|$  must divide the Schure multiplier of  $L : 2_1$ ,  $L : 2_2$  or  $L : 2_3$ , which is impossible. If  $|G/C_G(K)| = 2$ , then  $K < C_G(K)$  and  $1 \neq C_G(K)/K \trianglelefteq G/K \cong L : 2_1$ ,  $L : 2_2$  or  $L : 2_3$ , and we obtain  $C_G(K)/K \cong L$ . Since  $K \leq Z(C_G(K))$ ,  $C_G(K)$  is a central extension of  $K$  by  $L$ . Thus  $C_G(K) \cong \mathbb{Z}_6 \times L$ , otherwise we get a contradiction because  $|K|$  must divide the Schure multiplier of  $L$ . Therefore  $G \cong (\mathbb{Z}_6 \times L).\mathbb{Z}_2$ . If  $K \cong D_6$ , then  $G/C_G(K) \lesssim D_6$  and so  $|G/C_G(K)| = 1, 2, 3$  or 6. If  $|G/C_G(K)| = 1$ , then  $K \leq Z(G)$ , that is a contradiction. If  $|G/C_G(K)| = 2$ , then we have  $|KC_G(K)| = 6|G|/2 = 3|G|$  because  $K \cap C_G(K) = 1$ , which is a contradiction. If  $|G/C_G(K)| = 3$ , then we have  $|KC_G(K)| = 6|G|/3 = 2|G|$  because  $K \cap C_G(K) = 1$ , which is a contradiction. If  $|G/C_G(K)| = 6$ , then  $G/C_G(K) \cong D_6$  and  $C_G(K) \neq 1$ . Hence,  $1 \neq C_G(K) \cong C_G(K)K/K \trianglelefteq G/K \cong L : 2_1$ ,  $L : 2_2$  or  $L : 2_3$ . It follows that  $C_G(K) \cong L : 2_1$ ,  $L : 2_2$  or  $L : 2_3$  because  $L$  is simple. Therefore,  $G \cong D_6 \times (L : 2_1)$ ,  $D_6 \times (L : 2_2)$  or  $D_6 \times (L : 2_3)$ .

Before processing the last case, we recall the following facts.

There exist five non-isomorphic groups of order 12. Two of them are abelian and three are non-abelian. The non-abelian groups are: alternating group  $A_4$ , dihedral group  $D_{12}$  and the dicyclic group  $T$  with generators  $a$  and  $b$ , subject to the relations  $a^6 = 1$ ,  $a^3 = b^2$  and  $b^{-1}ab = a^{-1}$ .

If  $|K| = 12$ , then  $G/K \cong L$  and  $K \cong \mathbb{Z}_{12}$ ,  $\mathbb{Z}_2 \times \mathbb{Z}_6$ ,  $D_{12}$ ,  $\mathbb{A}_4$  or  $T$ . But  $C_G(K)K/K \trianglelefteq G/K \cong L$ . If  $C_G(K)K/K = 1$ , then  $C_G(K) \leq K$  and hence  $|L| = |G/K||G/C_G(K)||\text{Aut}(K)|$ . Thus  $|L||\text{Aut}(K)|$ , a contradiction. Therefore,  $C_G(K)K/K \neq 1$  and since  $L$  is simple group, we conclude that  $G = C_G(K)K$  and hence,  $G/C_G(K) \cong K/Z(K)$ . Now, we should consider the following cases:

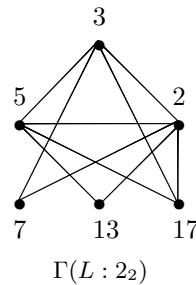
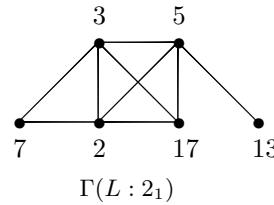
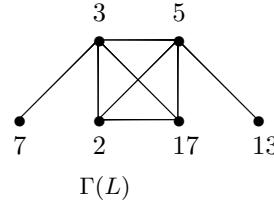
If  $K \cong \mathbb{Z}_{12}$  or  $\mathbb{Z}_2 \times \mathbb{Z}_6$ , then  $G/C_G(K) = 1$ . Therefore  $K \leq Z(G)$ , that is  $G$  is a central extension of  $\mathbb{Z}_{12}$  or  $\mathbb{Z}_2 \times \mathbb{Z}_6$  by  $L$ . If  $G$  splits over  $K$ , we obtain  $G \cong \mathbb{Z}_{12} \times L$  or  $(\mathbb{Z}_2 \times \mathbb{Z}_6) \times L$ , otherwise we get a contradiction because  $|K|$  must divide the Schure multiplier of  $L$ , which is 1 and it is impossible.

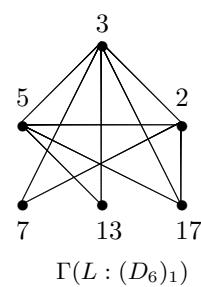
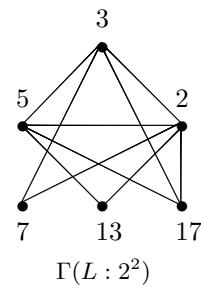
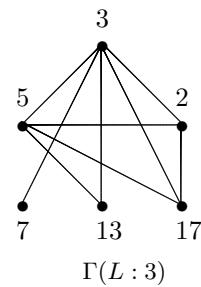
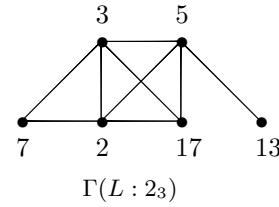
If  $K \cong D_{12}$ , then  $G = K \cdot L$  and  $G/C_G(K) \cong D_6$ . Since  $C_G(K)/Z(K) \cong G/K \cong L$  and  $Z(K) \leq Z(C_G(K))$ , we conclude that  $C_G(K)$  is a central extension of  $Z(K) \cong \mathbb{Z}_2$  by  $L$ . If  $C_G(K)$  is a non-split extension, then 2 must divide the Schure multiplier of  $L$ , which is 1 and it is impossible. Thus  $C_G(K) \cong \mathbb{Z}_2 \times L$  and hence,  $G$  is a split extension of  $K$  by  $L$ . Now, since  $\text{Hom}(L, \text{Aut}(D_{12}))$  is trivial, we have  $G \cong D_{12} \times L$ .

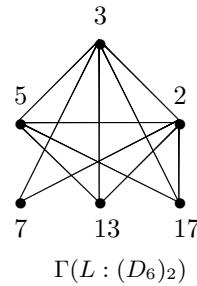
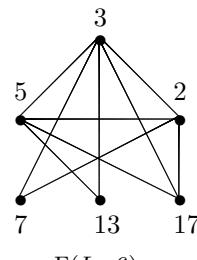
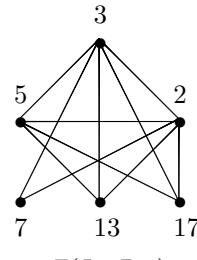
If  $K \cong \mathbb{A}_4$ , then  $G/C_G(K) \cong \mathbb{A}_4$ . As  $G = C_G(K)K$ , It follows that  $C_G(K) \cong L$ . Therefore  $G \cong L \times \mathbb{A}_4$  or  $L \cdot \mathbb{A}_4$ .

If  $K \cong T$ , then By the similar way in case  $K \cong D_{12}$ , we can conclude that  $G$  is a split extension of  $K$  by  $L$ . Also, since  $\text{Hom}(L, \text{Aut}(T))$  is trivial, we have  $G \cong T \times L$ .  $\square$

According to what we said before the proof, here we depict  $\Gamma(M)$  by  $|M|$  and  $\pi_e(M)$ , where  $M$  is an almost simple group related to  $L = D_4(4)$ .





 $\Gamma(L : (D_6)_2)$  $\Gamma(L : 6)$  $\Gamma(L : D_{12})$ 

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