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Zero Divisors of Support Size 3 in Complex Group Algebras of Finite Groups

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Abstract. It is proved that if $1 + x + y$ or $1 + x - y$ cannot occur as a zero divisor of the complex group algebra of a finite group G for any two distinct $x, y \in G \setminus \{1\}$, then G is solvable. We also characterize all finite abelian groups with the latter property. The motivation of studying such property for finite groups is to settle the existence of zero divisors with support size 3 in the integral group algebra of torsion free residually finite groups.

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1. Introduction

Let $\mathbb{C}[G]$ be the complex group algebra of a finite group G. In this paper we study finite groups G for which $\mathbb{C}[G]$ does not contain zero divisors of the form $1 + x + y$ or $1 + x - y$ for some distinct non-trivial elements $x, y \in G$. The motivation of studying such finite groups G is the following. There is a

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famous conjecture due to Kaplansky [7] saying that group algebras of torsion-free groups have no zero divisor. It is known that there is no zero divisor of support size 2 in the latter group algebras (see [8, Theorem 2.1]). The conjecture is still open and one of the important open cases is the validity of the conjecture for the integral group algebras (see e.g., Problem 1.3 of [9]). By [2, Theorem 1.1], every possible zero divisor of support size 3 in an integral group algebra of a torsion-free group Γ is a scaler multiple of elements of the form $1+x+y$ or $1+x-y$ for some distinct and non-trivial elements $x, y \in \Gamma$. On the other hand, the conjecture is even open for the group algebras of torsion-free residually finite groups (see e.g. [1] and [6, Question 7.2.6 and (C.29) of p. 60]). Note that the image of a zero divisor $\mathbb{C}[\Gamma]$ is still a zero divisor with the same support size in $\mathbb{C}[G]$ for arbitrary "sufficiently large" finite quotients G of a torsion-free residually finite group Γ.

Our main results are as follows.

Theorem 1.1. *Let G be a finite non-solvable group. Then*

(a) *$\mathbb{C}[G]$ has a zero divisor of the form $1 + a + b$, for some non-trivial distinct elements $a, b \in G$.*
(b) *$\mathbb{C}[G]$ has a zero divisor of the form $1+x-y$ for some non-trivial distinct elements $x, y \in G$.*

Theorem 1.2. *Let G be a finite abelian group. Then*

(a) *$\mathbb{C}[G]$ has a zero divisor of the form $1+a+b$ for some non-trivial distinct elements $a, b \in G$ if and only if $3 \mid |G|$.*
(b) *$\mathbb{C}[G]$ has a zero divisor of the form $1+x-y$ for some non-trivial distinct elements $x, y \in G$ if and only if $6 \mid |G|$.*

In Section 2 we prove some preliminary results. In Section 3 we prove Theorem 1.1 and in Section 4 we prove Theorem 1.2.

2. Preliminaries

Lemma 2.1. *Let G be a finite group. An element $z \in \mathbb{C}[G]$ is a zero divisor if and only if there exists a $\mathbb{C}[G]$-module V and a non-zero element $v \in V$ such that $zv = 0$.*

Proof. Suppose that $z \in \mathbb{C}[G]$ is a zero divisor so that $zw = 0$ for some non-zero $w \in \mathbb{C}[G]$. Then as $\mathbb{C}[G]$ is a $\mathbb{C}[G]$-module, we have done.

Now suppose that there exists a $\mathbb{C}[G]$-module V and a non-zero element $v \in V$ such that $zv = 0$. Since V is completely reducible, V is a direct sum of some irreducible $\mathbb{C}[G]$-modules. It follows that there exists an irreducible $\mathbb{C}[G]$-module W and a non-zero element $w \in W$ such that $zw = 0$. Since G is finite, there exists an irreducible $\mathbb{C}[G]$-submodule M of $\mathbb{C}[G]$ such that W is

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isomorphic to M as $\mathbb{C}[G]$-modules. It follows that $zm = 0$ for some non-zero $m \in M$. This completes the proof. $\square$

Lemma 2.2. *Let G be a finite group, H a subgroup of G and $z \in \mathbb{C}[H]$. Then z is a zero divisor of $\mathbb{C}[G]$ if and only if z is a zero divisor of $\mathbb{C}[H]$.*

Proof. Let z be a zero divisor of $\mathbb{C}[G]$ so that $zw = 0$ for some non-zero $w \in \mathbb{C}[G]$. Suppose that $w' = \sum_{x \in G} w'_x x \in \mathbb{C}[G]$ is an element with minimum support size with respect to the properties: $zw' = 0$ and $1 \in supp(w') := \{x \in G \mid w'_x \neq 0\}$. It follows from [3, Lemma 2.5] that $supp(w') \subseteq \langle supp(z) \rangle \leq H$. Hence $w' \in \mathbb{C}[H]$ and so z is a zero divisor of $\mathbb{C}[H]$. $\square$

Lemma 2.3. *Let G be a finite group, N a normal subgroup of G and $z \in \mathbb{C}[G]$. Suppose that $^-\colon \mathbb{C}[G] \to \mathbb{C}[\frac{G}{N}]$ is the induced ring homomorphism by the natural group homomorphism from G to $\frac{G}{N}$ sending x to xN. If $\bar{z} \in \mathbb{C}[\frac{G}{N}]$ is a zero divisor, then z is a zero divisor of $\mathbb{C}[G]$.*

Proof. Note that $\mathbb{C}[\frac{G}{N}]$ is a $\mathbb{C}[G]$-module with the action $\alpha \cdot v = \bar{\alpha} v$ for all $\alpha \in \mathbb{C}[G]$ and $v \in \mathbb{C}[\frac{G}{N}]$. Now Lemma 2.1 completes the proof. $\square$

Theorem 2.4. *Let $p \geq 3$ be a prime and $G := \langle a, b \mid a^p = b^{p-1} = 1,\ b^{-1}ab = a^k \rangle$, where k is a primitive root of p. Then $1 + a \pm b$ is a zero divisor of G.*

Proof. Let $H := \langle a \rangle = G'$ and $\alpha = e^{\frac{2\pi \mathrm{i}}{p}}$. Now $\mathbb{C}[G]$ acts on $\mathbb{C}[\frac{G}{H}]$ as follows

$$a \star b^m H = \alpha^{k^m} b^m H, b \star b^m H = b^{m+1} H,$$

where $0 \leq m \leq p-2$ is an integer. It is easy to see tath $(\mathbb{C}[\frac{G}{H}], \star)$ is a $\mathbb{C}[G]$-module. Now it follows from Lemma 2.1 that $1 + a \pm b$ is a zero divisor of $\mathbb{C}[G]$ if and only if there exist $c_0, c_1, \ldots, c_{p-2} \in \mathbb{C}$ not all of them zero such that $(1 + a \pm b) \star (\sum_{m=0}^{p-2} c_m b^m H) = 0$. The latter is equivalent to $(1+\alpha)(1+\alpha^k)(1+\alpha^{k^2}) \cdots (1+\alpha^{k^{p-2}}) = 1$. Since k is a primitive root of p, $\{1, \alpha, \alpha^k, \ldots, \alpha^{k^{p-2}}\}$ is the set of all roots of $x^p - 1 = 0$. Hence $\{1+1, 1+\alpha, 1+\alpha^k, \ldots, 1+\alpha^{k^{p-2}}\}$ is the set of roots of $(x-1)^p - 1 = 0$. Thus $(1+1)(1+\alpha) \cdots (1+\alpha^{k^{p-2}})$ is the product of all roots of $(x-1)^p - 1 = 0$. Therefore $(1+1)(1+\alpha) \cdots (1+\alpha^{k^{p-2}}) = 2$. This completes the proof. $\square$

Lemma 2.5. *The complex group algebra of the alternating group A_4 of degree 4 has a zero divisor of the form $1 + a - b$ for some non-trivial distinct elements $a, b \in A_4$.*

Proof. Let $G := A_4$, $H := G'$, $a := (3,2,1)$ and $b := (4,2,1)$. Then $G := \langle a, b \rangle$ and $\mathbb{C}[G]$ acts on $\mathbb{C}[\frac{G}{H}]$ as follows

$$a \star H := aH, a \star aH := -bH, a \star bH := -H,$$

$$b \star H := bH, b \star aH := -H, b \star bH := -aH.$$

It is easy to see that $(\mathbb{C}[\frac{G}{H}], \star)$ is a $\mathbb{C}[G]$-module. Since $(1+a-b) \star (H - aH) = 0$, Lemma 2.1 completes the proof. $\square$

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3. Solvablility of finite groups whose complex group algebras not containing zero divisors of the form $1+x+y$ or $1+x-y$

Proof of Theorem 1.1 (a). Since G is non-solvable, there exist normal subgroups M and N of G such that $M/N \cong \underbrace{S \times \cdots \times S}_{k-\text{times}}$, where $k \geq 1$ is an integer and S is a non-abelian finite simple group. From Lemmas 2.2 and 2.3, it is enough to show that S has a zero divisor of the form $1+a+b$. If $3 \mid |S|$, we have done. Thus we assume that $3 \nmid |S|$. Hence by [5, Remarks 3.7, p. 188] $S \cong \mathrm{Sz}(q)$ is the Suzuki simple group, where $q = 2^{2m+1}$ for some $m \in \mathbb{N}$. Since $\mathrm{Sz}(2)$ is the Frobenious group of order 20 and $\mathrm{Sz}(q)$ has $\mathrm{Sz}(2)$ as a subgroup, Theorem 2.4 and Lemma 2.2 imply that S has a zero divisor of the form $1+a+b$. This completes the proof. $\square$

Proof of Theorem 1.1 (b). Similar to Theorem 1.1, it is enough to show that a simple non-abelian group S has a zero divisor of the form $1+a-b$. If there exists a section H of S such that $H \cong S_3$, then by Theorem 2.4 and Lemmas 2.2 and 2.3, we have done. Hence we may assume that S is non-abelian finite simple S_3-free group. It follows from [4, Corollary 3] one of the following cases occurs:
Case 1. S is a simple Suzuki group. Then similar to Theorem 1.1, there exists a subgroup K of S which is a Frobenious group of order 20, and so Theorem 2.4 completes the proof.
Case 2. $S \cong \mathrm{PSL}_2(3^{2m+1})$ the projective special linear group of dimension 2 over field with 3^{2m+1} for some $m \in \mathbb{N}$. Since $A_4 \cong \mathrm{PSL}_2(3)$, S contains a subgroup isomorphic to A_4 and so Lemma 2.5 completes the proof. $\square$

4. Finite abelian groups whose complex group algebras containing zero divisors of the form $1+x+y$ or $1+x-y$

Lemma 4.1. *Let G be a finite group, $z \in \mathbb{C}[G]$, λ a linear complex character of G and $e_\lambda = \frac{1}{|G|}\sum_{g\in G}\lambda(g^{-1})g$ be the idempotent element of $\mathbb{C}[G]$ defined by λ. Then $ze_\lambda = 0$ if and only if $\lambda(z) = 0$.*

Proof. Now let h be an arbitrary element of G. Then

$$he_\lambda = h\Big(\frac{1}{|G|}\sum_{g\in G}\lambda(g^{-1})g\Big) =$$
$$\frac{1}{|G|}\sum_{g\in G}\lambda(g^{-1})hg = \frac{\lambda(h)}{|G|}\sum_{g\in G}\lambda(h^{-1}g^{-1})hg =$$
$$\frac{\lambda(h)}{|G|}\sum_{g\in G}\lambda((hg)^{-1})hg = \lambda(h)e_\lambda.$$

It follows that $ze_\lambda = 0$ if and only if $\lambda(z) = 0$. $\square$

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Proof of Theorem 1.2 (a). Suppose that $3 \mid |G|$. Then there exists $a \in G$ of order 3. Hence $(1+a+a^2)(1-a) = 0$. Thus $1+a+a^2$ is a zero divisor of G. Now suppose that $a, b \in G \setminus \{1\}$, $a \neq b$ and $1+a+b$ is a zero divisor of G. Thus using Lemma 2.1, there exist an irreducible $\mathbb{C}[G]$-module V and $0 \neq v \in V$ suth that $(1+a+b)v = 0$. Therefore for some $\lambda \in \mathrm{Irr}(G) = \mathrm{Lin}(G)$, $V = V_\lambda = e_\lambda \mathbb{C}[G]$ where V_λ is an irreducible $\mathbb{C}[G]$-module and $\dim_{\mathbb{C}} V_\lambda = \lambda(1) = 1$. Hence $V_\lambda = \mathbb{C}e_\lambda$. Thus by Lemma 4.1, $1+\lambda(a)+\lambda(b) = 0$. Now assume that $o(a) = m$ and $o(b) = n$. Thus for some $s \in \{1, \ldots, m-1\}$ and some $t \in \{1, \ldots, n-1\}$, $1 + e^{\frac{2\pi s \mathbf{i}}{m}} + e^{\frac{2\pi t \mathbf{i}}{n}} = 0$, where $\mathbf{i}^2 = -1$. Hence $\begin{cases} \sin \frac{2\pi s}{m} = -\sin \frac{2\pi t}{n} \\ \cos \frac{2\pi s}{m} + \cos \frac{2\pi t}{n} = -1 \end{cases}$. It follows that $\frac{s}{m} - \frac{t}{n} \in \{2k \pm \frac{1}{3}, 2k \pm \frac{2}{3} \mid k \in \mathbb{Z}\}$. Thus $3 \mid mn$ and so $3 \mid |G|$. $\square$

Proof of Theorem 1.2 (b). We first assume that $6 \mid |G|$. Since G is abelian, there exists $a \in G$ of order 6. Thus $(1-a+a^2)(1+a)(1-a^3) = 0$. Hence $1-a+a^2$ is a zero divisor of $\mathbb{C}[G]$.
Now suppose that there exist $a, b \in G \setminus \{1\}$, $a \neq b$ and $1+a-b$ is a zero divisor of $\mathbb{C}[G]$. Thus using Lemma 2.1, there exist an irreducible $\mathbb{C}[G]$-module V and $0 \neq v \in V$ suth that $(1+a-b)v = 0$. Therefore for some $\lambda \in \mathrm{Irr}(G) = \mathrm{Lin}(G)$, $V = V_\lambda = e_\lambda \mathbb{C}[G]$ where V_λ is an irreducible $\mathbb{C}[G]$-module and $\dim_{\mathbb{C}} V_\lambda = \lambda(1) = 1$. Hence $V_\lambda = \mathbb{C}e_\lambda$. Thus by Lemma 4.1, $1+\lambda(a)-\lambda(b) = 0$. Now let $o(a) = m$ and $o(b) = n$. Thus there exists $s \in \{1, \ldots, m-1\}$ and $t \in \{1, \ldots, n-1\}$ such that $1 + e^{\frac{2\pi s \mathbf{i}}{m}} - e^{\frac{2\pi t \mathbf{i}}{n}} = 0$, where $\mathbf{i}^2 = -1$. Therefore

$$\begin{cases} \sin \frac{2\pi s}{m} = \sin \frac{2\pi t}{n} \\ \cos \frac{2\pi s}{m} - \cos \frac{2\pi t}{n} = -1 \end{cases}.$$

It follows that $\frac{s}{m} + \frac{t}{n} \in \{2k \pm \frac{1}{6}, 2k+1 \pm \frac{1}{6} \mid k \in \mathbb{Z}\}$. Therefore $6 \mid mn$ and so $6 \mid |G|$. $\square$

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