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Generalization of Ostrowski's Inequality for Differentiable Functions and its Applications

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Abstract. We first establish weighted Ostrowski type inequalities for bounded differentiable functions. This inequality is also obtained for bounded above and bounded below differentiable functions. Some applications of the proposed results are presented to numerical standard and non standard quadrature rules. We recapture known results as well as obtain new results.

Keywords: Ostrowski's inequality, Numerical integration, differentiable functions.

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1. Introduction

Our work mostly deals with integral inequalities. To highlight its importance we quote here from [20], "Among the many types of inequalities, integral

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inequalities are of supreme importance because over the last few decades this field has proven to be an extensively applicable field. The integral inequalities of various types have been widely studied in most subjects involving mathematical analysis. These inequalities are particularly useful in approximation theory and in numerical analysis where estimates of approximation errors are involved." Ostrowski inequality [25] is one of the most famous inequalities, first presented by Alexander Markowich Ostrowski in 1938. It can be used to determine absolute deviation of functional values from its mean values. It is extremely important because of its wide range of applications in different areas of mathematics such as numerical integration, integral operator theory, probability theory and statistics. This inequality states that:

Proposition 1.1. *Let $\rho : I \to \mathbb{R}$ be a differentiable function on I^o such that $\rho \in L[j,k]$ where $j < k$ whose derivative ρ' is bounded on interior of I, i.e., $\|\rho'\|_\infty := \sup_{t\in(j,k)} |\rho'(t)| < \infty$. Then*

$$\left|\rho(\theta) - \frac{1}{k-j}\int_j^k \rho(t)dt\right| \leq (k-j)\left[\frac{1}{4} + \frac{\left(\theta - \frac{j+k}{2}\right)^2}{(k-j)^2}\right]\|\rho'(x)\|_\infty. \tag{1.1}$$

The constant $\frac{1}{4}$ is the best possible constant that it cannot be replaced by smaller one.

Ostrowski inequality for differentiable functions has been generalized many times, as stated in [6, 18, 19, 27]. For latest work related to Ostrowski inequality, we refer the reader to following articles [1, 2, 3, 5, 8, 12, 13, 14, 15, 16, 17, 21, 22, 23, 24, 26]

To prove our main results, we need the following two lemmas from [7] and [10].

Lemma 1.2. *Let ρ: $I \to \mathbb{R}$, be a differentiable function in interior I^0 of interval I and also let $[j,k] \subset I^0$. Then the following identity holds*

$$\begin{aligned}\int_j^k K(\theta,t)\rho'(t)dt &= \rho(k)\int_\beta^k \omega(t)dt - \rho(j)\int_\alpha^j \omega(t)dt + \rho(\theta)\int_\alpha^{\frac{\alpha+\beta}{2}} \omega(t)dt \\ &+ \rho(j+k-\theta)\int_{\frac{\alpha+\beta}{2}}^\beta \omega(t)dt - \int_j^k \rho(t)\omega(t)dt,\end{aligned} \tag{1.2}$$

where $K(\theta,t)$ is defined as:

$$K(\theta,t) = \begin{cases} \int_\alpha^t \omega(t)dt, & \text{if} \quad t \in [j,\theta]; \\ \int_{\frac{\alpha+\beta}{2}}^t \omega(t)dt, & \text{if} \quad t \in (\theta, j+k-\theta]; \\ \int_\beta^t \omega(t)dt, & \text{if} \quad t \in (j+k-\theta, k]; \end{cases} \tag{1.3}$$

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where $\forall\ \theta \in [j,k]$, $\alpha = j + \lambda\frac{k-j}{2}$, $\beta = k - \lambda\frac{k-j}{2}$ *and* $\lambda \in [0,1]$.

Lemma 1.3. *Let* $\rho : [j,k] \to \mathbb{R}$ *be a differentiable function such that* $\gamma(\theta) \leq \rho'(\theta) \leq \Gamma(\theta)$ *for any* $\gamma, \Gamma \in C[j,k]$ *and* $\theta \in [j,k]$. *Then we have*

$$\left|\rho'(t) - \frac{\gamma(t)+\Gamma(t)}{2}\right| \leq \frac{\Gamma(t)-\gamma(t)}{2}. \tag{1.4}$$

We are ready to present our main theorem, which will be generalized in two ways: first, by adding weights that are probability density functions, and second, by adding a parameter. In this way, we will capture variety of results from various articles as special cases.

2. Main results

Theorem 2.1. *Let* ρ: $I \to \mathbb{R}$, *be a differentiable function in* I^0 *and also let* $\gamma(\theta) \leq \rho'(\theta) \leq \Gamma(\theta)$ *for any* $\gamma, \Gamma \in C[j,k]$ *and* $\theta \in [j,k]$. *Then*

$$\begin{aligned} m(\theta,\lambda) \leq \rho(k)\int_\beta^k \omega(t)dt - \rho(j)\int_\alpha^j \omega(t)dt + \rho(\theta)\int_\alpha^{\frac{\alpha+\beta}{2}} \omega(t)dt \\ + \rho(j+k-\theta)\int_{\frac{\alpha+\beta}{2}}^\beta \omega(t)dt - \int_j^k \rho(t)\omega(t)dt \leq M(\theta,\lambda), \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} m(\theta,\lambda) &= \int_j^\theta \left(\left(\int_\alpha^t \omega(t)dt - \left|\int_\alpha^t \omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right. \\ &+ \left.\left(\int_\alpha^t \omega(t)dt + \left|\int_\alpha^t \omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt \\ &+ \int_\theta^{j+k-\theta}\left(\left(\int_{\frac{\alpha+\beta}{2}}^t \omega(t)dt - \left|\int_{\frac{\alpha+\beta}{2}}^t \omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right. \\ &+ \left.\left(\int_{\frac{\alpha+\beta}{2}}^t \omega(t)dt + \left|\int_{\frac{\alpha+\beta}{2}}^t \omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt \\ &+ \int_{j+k-\theta}^k \left(\left(\int_\beta^t \omega(t)dt - \left|\int_\beta^t \omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right. \\ &+ \left.\left(\int_\beta^t \omega(t)dt + \left|\int_\beta^t \omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt \end{aligned}$$

and

$$\begin{aligned} M(\theta,\lambda) &= \int_j^\theta \left(\left(\int_\alpha^t \omega(t)dt + \left|\int_\alpha^t \omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right. \\ &+ \left.\left(\int_\alpha^t \omega(t)dt - \left|\int_\alpha^t \omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt \end{aligned}$$

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$$+ \int_{\theta}^{j+k-\theta} \left(\left(\int_{\frac{\alpha+\beta}{2}}^{t} \omega(t)dt + \left| \int_{\frac{\alpha+\beta}{2}}^{t} \omega(t)dt \right| \right) \frac{\Gamma(t)}{2} \right.$$
$$\left. + \left(\int_{\frac{\alpha+\beta}{2}}^{t} \omega(t)dt - \left| \int_{\frac{\alpha+\beta}{2}}^{t} \omega(t)dt \right| \right) \frac{\gamma(t)}{2} \right) dt$$
$$+ \int_{j+k-\theta}^{k} \left(\left(\int_{\beta}^{t} \omega(t)dt + \left| \int_{\beta}^{t} \omega(t)dt \right| \right) \frac{\Gamma(t)}{2} \right.$$
$$\left. + \left(\int_{\beta}^{t} \omega(t)dt - \left| \int_{\beta}^{t} \omega(t)dt \right| \right) \frac{\gamma(t)}{2} \right) dt.$$

Proof. Using (1.2) and (1.3), we obtain

$$\int_{j}^{k} K(\theta,t) \left(\rho'(t) - \frac{\gamma(t)+\Gamma(t)}{2} \right) dt$$
$$= \int_{j}^{k} K(\theta,t)\rho'(t)dt - \frac{1}{2} \left(\int_{j}^{k} K(\theta,t) \left(\gamma(t)+\Gamma(t) \right) dt \right)$$
$$= \rho(k) \int_{\beta}^{k} \omega(t)dt - \rho(j) \int_{\alpha}^{j} \omega(t)dt + \rho(\theta) \int_{\alpha}^{\frac{\alpha+\beta}{2}} \omega(t)dt$$
$$+ \rho(j+k-\theta) \int_{\frac{\alpha+\beta}{2}}^{\beta} \omega(t)dt - \int_{j}^{k} \omega(t)\rho(t)dt$$
$$- \frac{1}{2} \left[\int_{j}^{\theta} \int_{\alpha}^{t} \omega(t)dt \left(\gamma(t)+\Gamma(t) \right) dt + \int_{\theta}^{j+k-\theta} \int_{\frac{\alpha+\beta}{2}}^{t} \omega(t)dt \left(\gamma(t)+\Gamma(t) \right) dt \right.$$
$$\left. + \int_{j+k-\theta}^{k} \int_{\beta}^{t} \omega(t)dt \left(\gamma(t)+\Gamma(t) \right) dt \right]. \tag{2.2}$$

Applying absolute value and using (1.4) we get

$$\left| \int_{\beta}^{k} \omega(t)dt\rho(k) - \rho(j) \int_{\alpha}^{j} \omega(t)dt + \rho(\theta) \int_{\alpha}^{\frac{\alpha+\beta}{2}} \omega(t)dt + \rho(j+k-\theta) \right.$$
$$\times \int_{\frac{\alpha+\beta}{2}}^{\beta} \omega(t)dt - \int_{j}^{k} \omega(t)\rho(t)dt - \frac{1}{2} \left[\int_{j}^{\theta} \int_{\alpha}^{t} \omega(t)dt \left(\gamma(t)+\Gamma(t) \right) dt \right.$$
$$+ \int_{\theta}^{j+k-\theta} \int_{\frac{\alpha+\beta}{2}}^{t} \omega(t)dt \left(\gamma(t)+\Gamma(t) \right) dt$$
$$\left. \left. + \int_{j+k-\theta}^{k} \int_{\beta}^{t} \omega(t)dt \left(\gamma(t)+\Gamma(t) \right) dt \right] \right|$$
$$= \left| \int_{j}^{k} K(\theta,t) \left(\rho'(t) - \frac{\gamma(t)+\Gamma(t)}{2} \right) dt \right|$$

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$$\leq \int_j^k |K(\theta,t)| \left| \left(\rho'(t) - \frac{\gamma(t)+\Gamma(t)}{2} \right) dt \right|$$
$$\leq \int_j^k |K(\theta,t)| \left(\frac{\Gamma(t)-\gamma(t)}{2} \right) dt$$
$$= \frac{1}{2} \left[\int_j^\theta \left| \int_\alpha^t \omega(t)dt \right| (\Gamma(t)-\gamma(t))\, dt + \int_\theta^{j+k-\theta} \left| \int_{\frac{\alpha+\beta}{2}}^t \omega(t)dt \right| \right.$$
$$\left. \times (\Gamma(t)-\gamma(t))\, dt + \int_{j+k-\theta}^k \left| \int_\beta^t \omega(t)dt \right| (\Gamma(t)-\gamma(t))\, dt \right]. \tag{2.3}$$

After rearranging (2.3), we get the required result. □

Remark 2.2. It is worth mentioning that if we put $\omega(t) = \frac{1}{k-j}$ in our main result we will get the following result.

Corollary 2.3. *Let all the assumptions of Theorem* 2.1 *be valid. Then*

$$m_0(\theta,\lambda) \leq \lambda \frac{\rho(j)+\rho(k)}{2} + (1-\lambda)\frac{\rho(\theta)+\rho(j+k-\theta)}{2} - \int_j^k \rho(t)d(t) \leq M_0(\theta,\lambda),$$

where

$$m_0(\theta,\lambda) = \frac{1}{k-j} \left[\int_{-\lambda\frac{k-j}{2}}^{\theta-(j+\lambda\frac{k-j}{2})} \left(\frac{\eta+|\eta|}{2} \Gamma\left(\eta + j + \lambda\frac{k-j}{2} \right) \right.\right.$$
$$\left. + \frac{\eta-|\eta|}{2} \gamma\left(\eta + j + \lambda\frac{k-j}{2} \right) \right) d\eta$$
$$+ \int_{\theta-\frac{j+k}{2}}^{\frac{j+k}{2}-\theta} \left(\frac{\eta+|\eta|}{2} \Gamma\left(\eta + \frac{j+k}{2} \right) + \frac{\eta-|\eta|}{2} \gamma\left(\eta + \frac{j+k}{2} \right) \right) d\eta$$
$$+ \int_{j+\lambda\frac{k-j}{2}-\theta}^{\lambda\frac{k-j}{2}} \left(\frac{\eta+|\eta|}{2} \Gamma\left(\eta + j - \lambda\frac{k-j}{2} \right) \right.$$
$$\left.\left. + \frac{\eta-|\eta|}{2} \gamma\left(\eta + j - \lambda\frac{k-j}{2} \right) \right) d\eta \right]$$

and

$$M_0(\theta,\lambda) = \frac{1}{k-j} \left[\int_{-\lambda\frac{k-j}{2}}^{\theta-(j+\lambda\frac{k-j}{2})} \left(\frac{\eta-|\eta|}{2} \Gamma\left(\eta + j + \lambda\frac{k-j}{2} \right) \right.\right.$$
$$\left. + \frac{\eta+|\eta|}{2} \gamma\left(\eta + j + \lambda\frac{k-j}{2} \right) \right) d\eta$$

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$$+\int_{\theta-\frac{j+k}{2}}^{\frac{j+k}{2}-\theta}\left(\frac{\eta-|\eta|}{2}\Gamma\left(\eta+\frac{j+k}{2}\right)+\frac{\eta+|\eta|}{2}\gamma\left(\eta+\frac{j+k}{2}\right)\right)d\eta$$
$$+\int_{j+\lambda\frac{k-j}{2}-\theta}^{\lambda\frac{k-j}{2}}\left(\frac{\eta-|\eta|}{2}\Gamma\left(\eta+j-\lambda\frac{k-j}{2}\right)\right.$$
$$\left.+\frac{\eta-|\eta|}{2}\gamma\left(\eta+j-\lambda\frac{k-j}{2}\right)\right)d\eta\Bigg],$$

which is Theorem 2.3 of [8] and hence all its Corollaries and Remarks and further consequences would become our special cases.

Throughout the section γ_0, γ_1, Γ_0, Γ_1 are real constants.

Remark 2.4. If we put $\lambda = 1$ in (2.1), we obtain following result.

Corollary 2.5. *Let all the assumptions of Theorem* 2.1 *be valid. Then*

$$m_1 \le \rho(k)\int_{\frac{j+k}{2}}^{k}\omega(t)dt+\rho(j)\int_{j}^{\frac{j+k}{2}}\omega(t)dt-\int_{j}^{k}\rho(t)\omega(t)dt\le M_1, \qquad (2.4)$$

where

$$\begin{aligned} m_1 &= \int_j^k\left[\left(\left(\int_{\frac{j+k}{2}}^{t}\omega(t)dt-\left|\int_{\frac{j+k}{2}}^{t}\omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right.\right.\\ &+ \left.\left.\left(\int_{\frac{j+k}{2}}^{t}\omega(t)dt+\left|\int_{\frac{j+k}{2}}^{t}\omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)\right]dt \end{aligned}$$

and

$$\begin{aligned} M_1 &= \int_j^k\left[\left(\left(\int_{\frac{j+k}{2}}^{t}\omega(t)dt+\left|\int_{\frac{j+k}{2}}^{t}\omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right.\right.\\ &+ \left.\left.\left(\int_{\frac{j+k}{2}}^{t}\omega(t)dt-\left|\int_{\frac{j+k}{2}}^{t}\omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)\right]dt. \end{aligned}$$

Special Case 2.5.(a) If we take, $\gamma(t)=\gamma_0\neq 0$, $\Gamma(t)=\Gamma_0\neq 0$ and $\omega(t)=\frac{1}{k-j}$ in (2.4), then

$$\frac{(k-j)}{8}(\gamma_0-\Gamma_0)\le\frac{\rho(j)+\rho(k)}{2}-\frac{1}{k-j}\int_j^k\rho(t)dt \le\frac{(k-j)}{8}(\Gamma_0-\gamma_0),$$

which is the Corollary 2 of [27] and Special Case 2.4.1 of [8].

Special Case 2.5.(b) If we take, $\gamma(t)=\gamma_1 t+\gamma_0\neq 0$, $\Gamma(t)=\Gamma_1 t+\Gamma_0\neq 0$ and $\omega(t)=\frac{1}{k-j}$ in (2.4), then

$$m_2\le\frac{\rho(j)+\rho(k)}{2}-\frac{1}{k-j}\int_j^k\rho(t)dt\le M_2,$$

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where

$$m_2 = \frac{(k-j)}{8}\left[\frac{(k-j)}{3}(\gamma_1+\Gamma_1)+\frac{(j+k)}{2}(\gamma_1-\Gamma_1)+\gamma_0-\Gamma_0\right]$$

and

$$M_2 = \frac{(k-j)}{8}\left[\frac{(k-j)}{3}(\gamma_1+\Gamma_1)+\frac{(j+k)}{2}(\Gamma_1-\gamma_1)+\Gamma_0-\gamma_0\right],$$

which is Special Case 2.4.2 of [8].

Remark 2.6. If we choose $\theta = \frac{j+k}{2}$ in (2.1), we obtain the following result.

Corollary 2.7. *Let all the assumptions of Theorem* 2.1 *be valid. Then*

$$m_3(\lambda) \le \rho(k)\int_\beta^k \omega(t)dt - \rho(j)\int_\alpha^j \omega(t)dt + \rho\left(\frac{k+j}{2}\right)\int_\alpha^\beta \omega(t)dt - \int_j^k \rho(t)\omega(t)dt \le M_3(\lambda), \quad (2.5)$$

where

$$\begin{aligned} m_3(\lambda) &= \int_j^{\frac{j+k}{2}} \left(\left(\int_\alpha^t \omega(t)dt - \left|\int_\alpha^t \omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right. \\ &+ \left.\left(\int_\alpha^t \omega(t)dt + \left|\int_\alpha^t \omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt \\ &+ \int_{\frac{j+k}{2}}^k \left(\left(\int_\beta^t \omega(t)dt - \left|\int_\beta^t \omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right. \\ &+ \left.\left(\int_\beta^t \omega(t)dt + \left|\int_\beta^t \omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt \end{aligned}$$

and

$$\begin{aligned} M_3(\lambda) &= \int_j^{\frac{j+k}{2}} \left(\left(\int_\alpha^t \omega(t)dt + \left|\int_\alpha^t \omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right. \\ &+ \left.\left(\int_\alpha^t \omega(t)dt - \left|\int_\alpha^t \omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt \\ &+ \int_{\frac{j+k}{2}}^k \left(\left(\int_\beta^t \omega(t)dt + \left|\int_\beta^t \omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right. \\ &+ \left.\left(\int_\beta^t \omega(t)dt - \left|\int_\beta^t \omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt. \end{aligned}$$

Remark 2.8. If we choose $\lambda = 0$ in (2.5), we obtain the following result.

Corollary 2.9. *Let all the assumptions of Theorem* 2.1 *be valid. Then*

$$m_4 \le \rho\left(\frac{k+j}{2}\right) - \int_j^k \rho(t)\omega(t)dt \le M_4(t) \quad (2.6)$$

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where

$$\begin{aligned} m_4 &= \int_j^{\frac{j+k}{2}} \left(\left(\int_j^t \omega(t)dt - \left| \int_j^t \omega(t)dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &+ \left. \left(\int_j^t \omega(t)dt + \left| \int_j^t \omega(t)dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\ &+ \int_{\frac{j+k}{2}}^k \left(\left(\int_k^t \omega(t)dt - \left| \int_k^t \omega(t)dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &+ \left. \left(\int_k^t \omega(t)dt + \left| \int_k^t \omega(t)dt \right| \right) \frac{\gamma(t)}{2} \right) dt \end{aligned}$$

and

$$\begin{aligned} M_4 &= \int_j^{\frac{j+k}{2}} \left(\left(\int_j^t \omega(t)dt + \left| \int_j^t \omega(t)dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &+ \left. \left(\int_j^t \omega(t)dt - \left| \int_j^t \omega(t)dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\ &+ \int_{\frac{j+k}{2}}^k \left(\left(\int_k^t \omega(t)dt + \left| \int_k^t \omega(t)dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &+ \left. \left(\int_k^t \omega(t)dt - \left| \int_k^t \omega(t)dt \right| \right) \frac{\gamma(t)}{2} \right) dt. \end{aligned}$$

Special Case 2.9.(a) If we take, $\gamma(\theta) = \gamma_0 \neq 0$, $\Gamma(\theta) = \Gamma_0 \neq 0$, $\theta = \frac{j+k}{2}$ and $\omega(t) = \frac{1}{k-j}$, in (2.5), then

$$\frac{(k-j)}{8}(\gamma_0 - \Gamma_0) \leq \rho\left(\frac{j+k}{2}\right) - \frac{1}{k-j}\int_j^k \rho(t)dt \leq \frac{(k-j)}{8}(\Gamma_0 - \gamma_0),$$

which is in fact the Special Case 1 of Theorem 1 presented in [18], Corollary 1 of [27] and Special Case 2.6.1 of [8].

Special Case 2.9.(b) If we take, $\gamma(\theta) = \gamma_1\theta + \gamma_0 \neq 0$, $\Gamma(\theta) = \Gamma_1\theta + \Gamma_0 \neq 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.5), then

$$m_5 \leq \rho\left(\frac{j+k}{2}\right) - \frac{1}{k-j}\int_j^k \rho(t)dt \leq M_5,$$

where

$$m_5 = \frac{(k-j)}{8}\left(\frac{k-j}{3}(\gamma_1 + \Gamma_1) + j\gamma_1 - k\Gamma_1 + \gamma_0 - \Gamma_0\right)$$

and

$$M_5 = \frac{(k-j)}{8}\left(\frac{k-j}{3}(\gamma_1 + \Gamma_1) + j\Gamma_1 - k\gamma_1 + \Gamma_0 - \gamma_0\right),$$

which is example of Corollary 1 of [18] and Special Case 2.6.2 of [8].

Remark 2.10. By choosing $\lambda = \frac{1}{3}$ in (2.5), then we get the following corollary.

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Corollary 2.11. *Let all the assumptions of Theorem* 2.1 *be valid. Then*

$$m_6 \leq \rho(k)\int_{\frac{j+5k}{6}}^{k}\omega(t)dt-\rho(j)\int_{\frac{5j+k}{6}}^{j}\omega(t)dt+\rho\left(\frac{k+j}{2}\right)\int_{\frac{5j+k}{6}}^{\frac{j+k}{2}}\omega(t)dt$$
$$+\rho\left(\frac{k+j}{2}\right)\int_{\frac{j+k}{2}}^{\frac{j+5k}{6}}\omega(t)dt-\int_{j}^{k}\rho(t)\omega(t)dt\leq M_6, \quad (2.7)$$

where

$$\begin{aligned} m_6 &= \int_{-\frac{k-j}{6}}^{\frac{j+k}{2}}\left(\left(\int_{\frac{5j+k}{6}}^{t}\omega(t)dt-\left|\int_{\frac{5j+k}{6}}^{t}\omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right.\\ &+ \left.\left(\int_{\frac{5j+k}{6}}^{t}\omega(t)dt+\left|\int_{\frac{5j+k}{6}}^{t}\omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt\\ &+ \int_{\frac{j+k}{2}}^{\frac{k-j}{6}}\left(\left(\int_{\frac{j+5k}{6}}^{t}\omega(t)dt-\left|\int_{\frac{j+5k}{6}}^{t}\omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right.\\ &+ \left.\left(\int_{\frac{j+5k}{6}}^{t}\omega(t)dt+\left|\int_{\frac{j+5k}{6}}^{t}\omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt \end{aligned}$$

and

$$\begin{aligned} M_6 &= \int_{-\frac{k-j}{6}}^{\frac{j+k}{2}}\left(\left(\int_{\frac{5j+k}{6}}^{t}\omega(t)dt+\left|\int_{\frac{5j+k}{6}}^{t}\omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right.\\ &+ \left.\left(\int_{\frac{5j+k}{6}}^{t}\omega(t)dt-\left|\int_{\frac{5j+k}{6}}^{t}\omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt\\ &+ \int_{\frac{j+k}{2}}^{\frac{k-j}{6}}\left(\left(\int_{\frac{j+5k}{6}}^{t}\omega(t)dt+\left|\int_{\frac{j+5k}{6}}^{t}\omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right.\\ &+ \left.\left(\int_{\frac{j+5k}{6}}^{t}\omega(t)dt-\left|\int_{\frac{j+5k}{6}}^{t}\omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt\,. \end{aligned}$$

Special Case 2.11.(a) If we take, $\gamma(t)=\gamma_0\neq 0$, $\Gamma(t)=\Gamma_0\neq 0$ and $\omega(t)=\frac{1}{k-j}$ in (2.7) , then

$$\frac{(k-j)}{72}(\gamma_0-\Gamma_0)\leq\frac{1}{3}\left[\frac{\rho(j)+\rho(k)}{2}+2\rho\left(\frac{j+k}{2}\right)\right]-\frac{1}{k-j}\int_{j}^{k}\rho(t)dt$$
$$\leq\frac{(k-j)}{72}(\Gamma_0-\gamma_0),$$

which is $\frac{1}{3}$ Simpson's rule and Corollary 4 of [27] and Special Case 2.7.1 of [8].

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Special Case 2.11.(b) If we take, $\gamma(t) = \gamma_1 t + \gamma_0 \neq 0$, $\Gamma(t) = \Gamma_1 t + \Gamma_0 \neq 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.7), then

$$m_7 \leq \frac{1}{3}\left[\frac{\rho(j)+\rho(k)}{2} + 2\rho\left(\frac{j+k}{2}\right)\right] - \frac{1}{k-j}\int_j^k \rho(t)dt \leq M_7,$$

where

$$m_7 = \frac{(k-j)}{72}\left[(k-j)(\gamma_1+\Gamma_1) + \frac{j}{2}(7\gamma_1 - 3\Gamma_1) + \frac{k}{2}(3\gamma_1 - 7\Gamma_1) + 5(\gamma_0 - \Gamma_0)\right]$$

and

$$\begin{aligned} M_7 &= \frac{(k-j)}{72}\left[(k-j)(\gamma_1+\Gamma_1) + \frac{j}{2}(7\Gamma_1 - 3\gamma_1) + \frac{k}{2}(3\Gamma_1 - 7\gamma_1)\right. \\ &+ \left. 5(\Gamma_0 - \gamma_0)\right], \end{aligned}$$

which is $\frac{1}{3}$ Simpson's rule and Special Case 2.7.2 of [8].

Remark 2.12. If we choose $\lambda = \frac{1}{2}$ in (2.5), we get the following corollary.

Corollary 2.13. *Let all the assumptions of Theorem* 2.1 *be valid. Then*

$$\begin{aligned} m_8 \leq \rho(k)\int_{\frac{j+3k}{4}}^{k}\omega(t)dt - \rho(j)\int_{\frac{3j+k}{4}}^{j}\omega(t)dt + \rho\left(\frac{k+j}{2}\right)\int_{\frac{3j+k}{4}}^{\frac{j+k}{2}}\omega(t)dt \\ +\rho\left(\frac{k+j}{2}\right)\int_{\frac{j+k}{2}}^{\frac{j+3k}{4}}\omega(t)dt - \int_j^k \rho(t)\omega(t)dt \leq M_8, \quad (2.8) \end{aligned}$$

where

$$\begin{aligned} m_8 &= \int_{-\frac{k-j}{4}}^{\frac{j+k}{2}}\left(\left(\int_{\frac{3j+k}{4}}^{t}\omega(t)dt - \left|\int_{\frac{3j+k}{4}}^{t}\omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right. \\ &+ \left.\left(\int_{\frac{3j+k}{4}}^{t}\omega(t)dt + \left|\int_{\frac{3j+k}{4}}^{t}\omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt \\ &+ \int_{\frac{j+k}{2}}^{\frac{k-j}{4}}\left(\left(\int_{\frac{j+3k}{4}}^{t}\omega(t)dt - \left|\int_{\frac{j+3k}{4}}^{t}\omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right. \\ &+ \left.\left(\int_{\frac{j+3k}{4}}^{t}\omega(t)dt + \left|\int_{\frac{j+3k}{4}}^{t}\omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt \end{aligned}$$

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and

$$\begin{aligned} M_8 &= \int_{-\frac{k-j}{4}}^{\frac{j+k}{2}} \left(\left(\int_{\frac{3j+k}{4}}^{t} \omega(t)dt + \left| \int_{\frac{3j+k}{4}}^{t} \omega(t)dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &+ \left. \left(\int_{\frac{3j+k}{4}}^{t} \omega(t)dt - \left| \int_{\frac{3j+k}{4}}^{t} \omega(t)dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\ &+ \int_{\frac{j+k}{2}}^{\frac{k-j}{4}} \left(\left(\int_{\frac{j+3k}{4}}^{t} \omega(t)dt + \left| \int_{\frac{j+3k}{4}}^{t} \omega(t)dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &+ \left. \left(\int_{\frac{j+3k}{4}}^{t} \omega(t)dt - \left| \int_{\frac{j+3k}{4}}^{t} \omega(t)dt \right| \right) \frac{\gamma(t)}{2} \right) dt. \end{aligned}$$

Special Case 2.13.(a) If we take, $\gamma(t) = \gamma_0 \neq 0$, $\Gamma(t) = \Gamma_0 \neq 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.8), then

$$\frac{(k-j)}{16}(\gamma_0 - \Gamma_0) \leq \frac{1}{2}\left[\frac{\rho(j)+\rho(k)}{2} + \rho\left(\frac{j+k}{2}\right)\right] - \frac{1}{k-j}\int_j^k \rho(t)dt \leq \frac{(k-j)}{16}(\Gamma_0 - \gamma_0),$$

which is average midpoint and trapezoidal and Corollary 3 of [27] and Special Case 2.8.1 of [8].

Special Case 2.13.(b) If we take, $\gamma(t) = \gamma_1 t + \gamma_0 \neq 0$, $\Gamma(t) = \Gamma_1 t + \Gamma_0 \neq 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.8), then

$$m_9 \leq \frac{1}{2}\left[\frac{\rho(j)+\rho(k)}{2} + \rho\left(\frac{j+k}{2}\right)\right] - \frac{1}{k-j}\int_j^k \rho(t)dt \leq M_9,$$

where

$$m_9 = \frac{(k-j)}{16}\left[\frac{(k-j)}{6}(\gamma_1+\Gamma_1) + \frac{j}{2}(\gamma_1-\Gamma_1) + \frac{k}{2}(\gamma_1-\Gamma_1) + \gamma_0 - \Gamma_0\right]$$

and

$$M_9 = \frac{(k-j)}{16}\left[\frac{(k-j)}{6}(\gamma_1+\Gamma_1) + \frac{j}{2}(\Gamma_1-\gamma_1) + \frac{k}{2}(\Gamma_1-\gamma_1) + \Gamma_0 - \gamma_0\right],$$

which is midpoint and trapezoidal rule and Special Case 2.8.2 of [8].

Remark 2.14. If we choose $\theta = j$ in (2.1), for any value of $\lambda \in [0,1]$ we obtain the following result.

Corollary 2.15. *Let all the assumptions of Theorem* 2.1 *be valid. Then*

$$\begin{aligned} m_{10}(\lambda) \leq \rho(k)\int_\beta^k \omega(t)dt - \rho(j)\int_\alpha^j \omega(t)dt + \rho(p)\int_\alpha^{\frac{\alpha+\beta}{2}} \omega(t)dt \\ +\rho(k)\int_{\frac{\alpha+\beta}{2}}^{\beta} \omega(t)dt - \int_j^k \rho(t)\omega(t)dt \leq M_{10}(\lambda), \end{aligned} \tag{2.9}$$

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where

$$\begin{aligned} m_{10}(\lambda) &= \int_j^k \left(\left(\int_{\frac{\alpha+\beta}{2}}^t \omega(t)dt - \left| \int_{\frac{\alpha+\beta}{2}}^t \omega(t)dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &+ \left. \left(\int_{\frac{\alpha+\beta}{2}}^t \omega(t)dt + \left| \int_{\frac{\alpha+\beta}{2}}^t \omega(t)dt \right| \right) \frac{\gamma(t)}{2} \right) dt \end{aligned}$$

and

$$\begin{aligned} M_{10}(\lambda) &= \int_j^k \left(\left(\int_{\frac{\alpha+\beta}{2}}^t \omega(t)dt + \left| \int_{\frac{\alpha+\beta}{2}}^t \omega(t)dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &+ \left. \left(\int_{\frac{\alpha+\beta}{2}}^t \omega(t)dt - \left| \int_{\frac{\alpha+\beta}{2}}^t \omega(t)dt \right| \right) \frac{\gamma(t)}{2} \right) dt. \end{aligned}$$

Remark 2.16. (1) If we take $\Gamma(t) = \Gamma_0 \neq 0$, $\gamma(t) = \gamma_0 \neq 0$, and with $\omega(t) = \frac{1}{k-j}$ in (2.9), then we obtain results similar to **Special Case 2.5.(a)**.

(2) $\Gamma(t) = \Gamma_1 t + \Gamma_0 \neq 0$,$\gamma(t) = \gamma_1 t + \gamma_0 \neq 0$, and $\omega(t) = \frac{1}{k-j}$ in (2.9), then we obtain results similar to **Special Case 2.5.(b)**.

Remark 2.17. If we choose $\theta = k$, and $\lambda = 0$ in (2.1), we obtain the following result.

Corollary 2.18. *Let all the assumptions of Theorem* 2.1 *be valid. Then*

$$m_{11} \leq \frac{\rho(j) + \rho(k)}{2} - \frac{1}{k-j} \int_j^k \rho(t)dt \leq M_{11}, \tag{2.10}$$

where

$$\begin{aligned} m_{11} &= \int_j^k \left(\left(\int_j^t \omega(t)dt - \left| \int_j^t \omega(t)dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &+ \left. \left(\int_j^t \omega(t)dt + \left| \int_j^t \omega(t)dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\ &+ \int_k^j \left(\left(\int_{\frac{j+k}{2}}^t \omega(t)dt - \left| \int_{\frac{j+k}{2}}^t \omega(t)dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &+ \left. \left(\int_{\frac{j+k}{2}}^t \omega(t)dt + \left| \int_{\frac{j+k}{2}}^t \omega(t)dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\ &+ \int_j^k \left(\left(\int_k^t \omega(t)dt - \left| \int_k^t \omega(t)dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &+ \left. \left(\int_k^t \omega(t)dt + \left| \int_k^t \omega(t)dt \right| \right) \frac{\gamma(t)}{2} \right) dt. \end{aligned}$$

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and

$$\begin{aligned} M_{11} &= \int_j^k \left(\left(\int_j^t \omega(t)dt + \left| \int_j^t \omega(t)dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &+ \left. \left(\int_j^t \omega(t)dt - \left| \int_j^t \omega(t)dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\ &+ \int_k^j \left(\left(\int_{\frac{j+k}{2}}^t \omega(t)dt + \left| \int_{\frac{j+k}{2}}^t \omega(t)dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &+ \left. \left(\int_{\frac{j+k}{2}}^t \omega(t)dt - \left| \int_{\frac{j+k}{2}}^t \omega(t)dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\ &+ \int_j^k \left(\left(\int_k^t \omega(t)dt + \left| \int_k^t \omega(t)dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &+ \left. \left(\int_k^t \omega(t)dt - \left| \int_k^t \omega(t)dt \right| \right) \frac{\gamma(t)}{2} \right) dt. \end{aligned}$$

Special Case 2.18.(a) If we take, $\gamma(t) = \gamma_0 \neq 0$, $\Gamma(t) = \Gamma_0 \neq 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.10), then

$$\frac{3(k-j)}{8}(\gamma_0 - \Gamma_0) \leq \left[\frac{\rho(j) + \rho(k)}{2} \right] - \frac{1}{k-j} \int_j^k \rho(t)dt \leq \frac{3(k-j)}{8}(\Gamma_0 - \gamma_0),$$

which is Special Case 2.10.1 of [8].

Special Case 2.18.(b) If we take, $\gamma(t) = \gamma_1 t + \gamma_0 \neq 0$, $\Gamma(t) = \Gamma_1 t + \Gamma_0 \neq 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.10), then

$$m_{12} \leq \left[\frac{\rho(j) + \rho(k)}{2} \right] - \frac{1}{k-j} \int_j^k \rho(t)dt \leq M_{12},$$

where

$$m_{12} = \frac{(k-j)}{2} \left[\frac{7(k-j)}{12}(\gamma_1 + \Gamma_1) + \frac{j}{8}(7\gamma_1 + \Gamma_1) - \frac{k}{8}(\gamma_1 + 7\Gamma_1) + \frac{3}{4}(\gamma_0 - \Gamma_0) \right]$$

and

$$M_{12} = \frac{(k-j)}{2} \left[\frac{7(k-j)}{12}(\gamma_1 + \Gamma_1) + \frac{j}{8}(\gamma_1 + 7\Gamma_1) - \frac{k}{8}(7\gamma_1 + \Gamma_1) + \frac{3}{4}(\Gamma_0 - \gamma_0) \right],$$

which is Special Case 2.10.2 of [8].

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Remark 2.19. If we choose $\theta = k$ and $\lambda = \frac{1}{2}$ in (2.1), we obtain the following result.

Corollary 2.20. *Let all the assumptions of Theorem* 2.1 *be valid. Then*

$$m_{13} \leq \rho(k)\int_{\frac{j+3k}{4}}^{k}\omega(t)dt - \rho(j)\int_{\frac{3j+k}{4}}^{j}\omega(t)dt + \rho(k)\int_{\frac{3j+k}{4}}^{\frac{j+k}{2}}\omega(t)dt + \rho(j)\int_{\frac{j+k}{2}}^{\frac{j+3k}{4}}\omega(t)dt - \int_{j}^{k}\rho(t)\omega(t)dt \leq M_{13}, \tag{2.11}$$

where

$$\begin{aligned}
m_{13} &= \int_j^k \left(\left(\int_{\frac{3j+k}{4}}^t \omega(t)dt - \left|\int_{\frac{3j+k}{4}}^t \omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right.\\
&+ \left.\left(\int_{\frac{3j+k}{4}}^t \omega(t)dt + \left|\int_{\frac{3j+k}{4}}^t \omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt\\
&+ \int_k^j \left(\left(\int_{\frac{j+k}{2}}^t \omega(t)dt - \left|\int_{\frac{j+k}{2}}^t \omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right.\\
&+ \left.\left(\int_{\frac{j+k}{2}}^t \omega(t)dt + \left|\int_{\frac{j+k}{2}}^t \omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt\\
&+ \int_j^k \left(\left(\int_{\frac{j+3k}{4}}^t \omega(t)dt - \left|\int_{\frac{j+3k}{4}}^t \omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right.\\
&+ \left.\left(\int_{\frac{j+3k}{4}}^t \omega(t)dt + \left|\int_{\frac{j+3k}{4}}^t \omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt
\end{aligned}$$

and

$$\begin{aligned}
M_{13} &= \int_j^k \left(\left(\int_{\frac{3j+k}{4}}^t \omega(t)dt + \left|\int_{\frac{3j+k}{4}}^t \omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right.\\
&+ \left.\left(\int_{\frac{3j+k}{4}}^t \omega(t)dt - \left|\int_{\frac{3j+k}{4}}^t \omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt\\
&+ \int_k^j \left(\left(\int_{\frac{j+k}{2}}^t \omega(t)dt + \left|\int_{\frac{j+k}{2}}^t \omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right.\\
&+ \left.\left(\int_{\frac{j+k}{2}}^t \omega(t)dt - \left|\int_{\frac{j+k}{2}}^t \omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt\\
&+ \int_j^k \left(\left(\int_{\frac{j+3k}{4}}^t \omega(t)dt + \left|\int_{\frac{j+3k}{4}}^t \omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right.\\
&+ \left.\left(\int_{\frac{j+3k}{4}}^t \omega(t)dt - \left|\int_{\frac{j+3k}{4}}^t \omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt\,.
\end{aligned}$$

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Special Case 2.20.(a) If we take, $\gamma(t) = \gamma_0 \neq 0$, $\Gamma(t) = \Gamma_0 \neq 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.11), then

$$\frac{9(k-j)}{32}(\gamma_0 - \Gamma_0) \leq \left[\frac{\rho(j)+\rho(k)}{2}\right] - \frac{1}{k-j}\int_j^k \rho(t)dt \leq \frac{9(k-j)}{32}(\Gamma_0 - \gamma_0),$$

which is Special Case 2.11.1 of [8].

Special Case 2.20.(b) If we take, $\gamma(t) = \gamma_1 t + \gamma_0 \neq 0$, $\Gamma(t) = \Gamma_1 t + \Gamma_0 \neq 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.11), then

$$m_{14} \leq \left[\frac{\rho(j)+\rho(k)}{2}\right] - \frac{1}{k-j}\int_j^k \rho(t)dt \leq M_{14},$$

where

$$\begin{aligned} m_{14} &= \frac{k-j}{16}\left[\frac{5(k-j)}{3}(\gamma_1+\Gamma_1) + \frac{j}{2}(5\gamma_1 - \Gamma_1) - \frac{k}{2}(\gamma_1 - 5\Gamma_1)\right. \\ &+ \left. 3(\gamma_0 - \Gamma_0)\right] \end{aligned}$$

and

$$\begin{aligned} M_{14} &= \frac{k-j}{16}\left[\frac{5(k-j)}{3}(\gamma_1+\Gamma_1) + \frac{j}{2}(5\Gamma_1 - \gamma_1) - \frac{k}{2}(\Gamma_1 - 5\gamma_1)\right. \\ &+ \left. 3(\Gamma_0 - \gamma_0)\right], \end{aligned}$$

which is Special Case 2.11.2 of [8].

Remark 2.21. If we choose $\theta = k$ and $\lambda = \frac{1}{3}$ in (2.1), then we get the following result.

Corollary 2.22. *Let all the assumptions of Theorem* 2.1 *be valid. Then*

$$\begin{aligned} m_{15} \leq \rho(k)\int_{\frac{j+5k}{6}}^{k} \omega(t)dt - \rho(j)\int_{\frac{5j+k}{6}}^{j} \omega(t)dt + \rho(k)\int_{\frac{5j+k}{6}}^{\frac{j+k}{2}} \omega(t)dt \\ +\rho(j)\int_{\frac{j+k}{2}}^{\frac{j+5k}{6}} \omega(t)dt - \int_j^k \rho(t)\omega(t)dt \leq M_{15}, \end{aligned} \tag{2.12}$$

where

$$m_{15} = \int_j^k \left(\left(\int_{\frac{5j+k}{6}}^{t} \omega(t)dt - \left|\int_{\frac{5j+k}{6}}^{t} \omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right.$$

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$$
\begin{aligned}
&+ \quad \left(\int_{\frac{5j+k}{6}}^{t} \omega(t)dt + \left| \int_{\frac{5j+k}{6}}^{t} \omega(t)dt \right| \right) \frac{\gamma(t)}{2} \Bigg) dt \\
&+ \quad \int_{k}^{j} \left(\left(\int_{\frac{j+k}{2}}^{t} \omega(t)dt - \left| \int_{\frac{j+k}{2}}^{t} \omega(t)dt \right| \right) \frac{\Gamma(t)}{2} \right. \\
&+ \quad \left(\int_{\frac{j+k}{2}}^{t} \omega(t)dt + \left| \int_{\frac{j+k}{2}}^{t} \omega(t)dt \right| \right) \frac{\gamma(t)}{2} \Bigg) dt \\
&+ \quad \int_{j}^{k} \left(\left(\int_{\frac{j+5k}{6}}^{t} \omega(t)dt - \left| \int_{\frac{j+5k}{6}}^{t} \omega(t)dt \right| \right) \frac{\Gamma(t)}{2} \right. \\
&+ \quad \left(\int_{\frac{j+5k}{6}}^{t} \omega(t)dt + \left| \int_{\frac{j+5k}{6}}^{t} \omega(t)dt \right| \right) \frac{\gamma(t)}{2} \Bigg) dt
\end{aligned}
$$

and

$$
\begin{aligned}
M_{15} \quad = \quad & \int_{j}^{k} \left(\left(\int_{\frac{5j+k}{6}}^{t} \omega(t)dt + \left| \int_{\frac{5j+k}{6}}^{t} \omega(t)dt \right| \right) \frac{\Gamma(t)}{2} \right. \\
& + \quad \left(\int_{\frac{5j+k}{6}}^{t} \omega(t)dt - \left| \int_{\frac{5j+k}{6}}^{t} \omega(t)dt \right| \right) \frac{\gamma(t)}{2} \Bigg) dt \\
& + \quad \int_{k}^{j} \left(\left(\int_{\frac{j+k}{2}}^{t} \omega(t)dt + \left| \int_{\frac{j+k}{2}}^{t} \omega(t)dt \right| \right) \frac{\Gamma(t)}{2} \right. \\
& + \quad \left(\int_{\frac{j+k}{2}}^{t} \omega(t)dt - \left| \int_{\frac{j+k}{2}}^{t} \omega(t)dt \right| \right) \frac{\gamma(t)}{2} \Bigg) dt \\
& + \quad \int_{j}^{k} \left(\left(\int_{\frac{j+5k}{6}}^{t} \omega(t)dt + \left| \int_{\frac{j+5k}{6}}^{t} \omega(t)dt \right| \right) \frac{\Gamma(t)}{2} \right. \\
& + \quad \left(\int_{\frac{j+5k}{6}}^{t} \omega(t)dt - \left| \int_{\frac{j+5k}{6}}^{t} \omega(t)dt \right| \right) \frac{\gamma(t)}{2} \Bigg) dt.
\end{aligned}
$$

Special Case 2.22.(a) If we take, $\gamma(t) = \gamma_0 \neq 0$, $\Gamma(t) = \Gamma_0 \neq 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.12), then

$$
\begin{aligned}
\frac{17(k-j)}{72}(\gamma_0 - \Gamma_0) \leq \left[\frac{\rho(j) + \rho(k)}{2} \right] - \frac{1}{k-j} \int_{j}^{k} \rho(t)dt \\
\leq \frac{17(k-j)}{72}(\Gamma_0 - \gamma_0),
\end{aligned}
$$

which is Special Case 2.12.1 of [8].

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Special Case 2.22.(b) If we take, $\gamma(t) = \gamma_1 t + \gamma_0 \neq 0$, $\Gamma(t) = \Gamma_1 t + \Gamma_0 \neq 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.12), then

$$m_{16} \leq \left[\frac{\rho(j)+\rho(k)}{2}\right] - \frac{1}{k-j}\int_j^k \rho(t)dt \leq M_{16},$$

where

$$m_{16} = \frac{(k-j)}{72}\left[11(k-j)(\gamma_1+\Gamma_1) + \frac{3j}{2}(33\gamma_1-\Gamma_1) + \frac{3k}{2}(\gamma_1-33\Gamma_1) + 17(\gamma_0-\Gamma_0)\right]$$

and

$$M_{16} = \frac{(k-j)}{72}\left[(k-j)(\gamma_1+\Gamma_1) + \frac{3j}{2}(33\Gamma_1-\gamma_1) + \frac{3k}{2}(\Gamma_1-33\gamma_1) + 17(\Gamma_0-\gamma_0)\right],$$

which is Special Case 2.12.2 of [8].

Remark 2.23. If we choose $\theta = k$, and $\lambda = \frac{1}{4}$ in (2.1), we obtain a bound for trapezoidal rule in the following result.

Corollary 2.24. *Let all the assumptions of Theorem* 2.1 *be valid. Then*

$$m_{17} \leq \rho(k)\int_{\frac{7j+k}{8}}^{k}\omega(t)dt - \rho(j)\int_{\frac{7j+k}{8}}^{j}\omega(t)dt + \rho(k)\int_{\frac{7j+k}{8}}^{\frac{j+k}{2}}\omega(t)dt + \rho(j)\int_{\frac{j+k}{2}}^{\beta}\omega(t)dt - \int_j^k \rho(t)\omega(t)dt \leq M_{17}, \qquad (2.13)$$

where

$$\begin{aligned} m_{17} &= \int_j^k \left(\left(\int_{\frac{7j+k}{8}}^t \omega(t)dt - \left|\int_{\frac{7j+k}{8}}^t \omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right. \\ &+ \left.\left(\int_{\frac{7j+k}{8}}^t \omega(t)dt + \left|\int_{\frac{7j+k}{8}}^t \omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt \\ &+ \int_k^j \left(\left(\int_{\frac{j+k}{2}}^t \omega(t)dt - \left|\int_{\frac{j+k}{2}}^t \omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right. \\ &+ \left.\left(\int_{\frac{j+k}{2}}^t \omega(t)dt + \left|\int_{\frac{j+k}{2}}^t \omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt \\ &+ \int_j^k \left(\left(\int_{\beta}^t \omega(t)dt - \left|\int_{\beta}^t \omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right. \\ &+ \left.\left(\int_{\beta}^t \omega(t)dt + \left|\int_{\beta}^t \omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt. \end{aligned}$$

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and

$$\begin{aligned} M_{17} &= \int_j^k \left(\left(\int_{\frac{7j+k}{8}}^t \omega(t)dt + \left| \int_{\frac{7j+k}{8}}^t \omega(t)dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &+ \left. \left(\int_{\frac{7j+k}{8}}^t \omega(t)dt - \left| \int_{\frac{7j+k}{8}}^t \omega(t)dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\ &+ \int_k^j \left(\left(\int_{\frac{j+k}{2}}^t \omega(t)dt + \left| \int_{\frac{j+k}{2}}^t \omega(t)dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &+ \left. \left(\int_{\frac{j+k}{2}}^t \omega(t)dt - \left| \int_{\frac{j+k}{2}}^t \omega(t)dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\ &+ \int_j^k \left(\left(\int_\beta^t \omega(t)dt + \left| \int_\beta^t \omega(t)dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &+ \left. \left(\int_\beta^t \omega(t)dt - \left| \int_\beta^t \omega(t)dt \right| \right) \frac{\gamma(t)}{2} \right) dt. \end{aligned}$$

Special Case 2.24.(a) If we take, $\gamma(t) = \gamma_0 \neq 0$, $\Gamma(t) = \Gamma_0 \neq 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.13), then

$$\frac{17(k-j)}{64}(\gamma_0 - \Gamma_0) \leq \left[\frac{\rho(j) + \rho(k)}{2} \right] - \frac{1}{k-j} \int_j^k \rho(t)dt \leq \frac{17(k-j)}{64}(\Gamma_0 - \gamma_0),$$

which is Special Case 2.13.1 of [8].

Special Case 2.24.(b) If we take, $\gamma(t) = \gamma_1 t + \gamma_0 \neq 0$, $\Gamma(t) = \Gamma_1 t + \Gamma_0 \neq 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.13), then

$$m_{18} \leq \left[\frac{\rho(j) + \rho(k)}{2} \right] - \frac{1}{k-j} \int_j^k \rho(t)dt \leq M_{18},$$

where

$$m_{18} = \frac{(k-j)}{64} \left[\frac{35}{3}(k-j)(\gamma_1 + \Gamma_1) + \frac{j}{2}(35\gamma_1 + \Gamma_1) - \frac{k}{2}(\gamma_1 + 35\Gamma_1) + 17(\gamma_0 - \Gamma_0) \right]$$

and

$$M_{18} = \frac{(k-j)}{64} \left[\frac{35}{3}(k-j)(\gamma_1 + \Gamma_1) + \frac{j}{2}(35\Gamma_1 + \gamma_1 +) - \frac{k}{2}(\Gamma_1 + 35\gamma_1) + 17(\Gamma_0 - \gamma_0) \right],$$

which is Special Case 2.13.2 of [8].

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Remark 2.25. if we choose $\theta = \frac{2j+k}{3}$ and $\lambda = \frac{1}{4}$ in (2.1), then we get the following result.

Corollary 2.26. *Let all the assumptions of Theorem* 2.1 *be valid. Then*

$$m_{19} \leq \rho(k)\int_{\frac{j+7k}{8}}^{k}\omega(t)dt - \rho(j)\int_{\frac{7j+k}{8}}^{j}\omega(t)dt + \rho\left(\frac{2j+k}{3}\right)\int_{\frac{7j+k}{8}}^{\frac{j+k}{2}}\omega(t)dt$$
$$+\rho\left(\frac{j+2k}{3}\right)\int_{\frac{j+k}{2}}^{\frac{j+7k}{8}}\omega(t)dt - \int_{j}^{k}\rho(t)\omega(t)dt \leq M_{19}, \tag{2.14}$$

where

$$\begin{aligned}
m_{19} &= \int_{j}^{\frac{2j+k}{3}}\left(\left(\int_{\frac{7j+k}{8}}^{t}\omega(t)dt - \left|\int_{\frac{7j+k}{8}}^{t}\omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right.\\
&+ \left.\left(\int_{\frac{7j+k}{8}}^{t}\omega(t)dt + \left|\int_{\frac{7j+k}{8}}^{t}\omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt\\
&+ \int_{\frac{2j+k}{3}}^{\frac{j+2k}{3}}\left(\left(\int_{\frac{j+k}{2}}^{t}\omega(t)dt - \left|\int_{\frac{j+k}{2}}^{t}\omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right.\\
&+ \left.\left(\int_{\frac{j+k}{2}}^{t}\omega(t)dt + \left|\int_{\frac{j+k}{2}}^{t}\omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt\\
&+ \int_{\frac{j+2k}{3}}^{k}\left(\left(\int_{\frac{j+7k}{8}}^{t}\omega(t)dt - \left|\int_{\frac{j+7k}{8}}^{t}\omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right.\\
&+ \left.\left(\int_{\frac{j+7k}{8}}^{t}\omega(t)dt + \left|\int_{\frac{j+7k}{8}}^{t}\omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt.
\end{aligned}$$

and

$$\begin{aligned}
M_{19} &= \int_{j}^{\frac{2j+k}{3}}\left(\left(\int_{\frac{7j+k}{8}}^{t}\omega(t)dt + \left|\int_{\frac{7j+k}{8}}^{t}\omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right.\\
&+ \left.\left(\int_{\frac{7j+k}{8}}^{t}\omega(t)dt - \left|\int_{\frac{7j+k}{8}}^{t}\omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt\\
&+ \int_{\frac{2j+k}{3}}^{\frac{j+2k}{3}}\left(\left(\int_{\frac{j+k}{2}}^{t}\omega(t)dt + \left|\int_{\frac{j+k}{2}}^{t}\omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right.\\
&+ \left.\left(\int_{\frac{j+k}{2}}^{t}\omega(t)dt - \left|\int_{\frac{j+k}{2}}^{t}\omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt\\
&+ \int_{\frac{j+2k}{3}}^{k}\left(\left(\int_{\frac{j+7k}{8}}^{t}\omega(t)dt + \left|\int_{\frac{j+7k}{8}}^{t}\omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right.\\
&+ \left.\left(\int_{\frac{j+7k}{8}}^{t}\omega(t)dt - \left|\int_{\frac{j+7k}{8}}^{t}\omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt.
\end{aligned}$$

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Special Case 2.26.(a) If we take, $\gamma(t) = \gamma_0 \neq 0$, $\Gamma(t) = \Gamma_0 \neq 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.14), then

$$\frac{25(k-j)}{576}(\Gamma_0 - \gamma_0) \leq \frac{3}{8}\left[\frac{\rho(j)+\rho(k)}{3} + \rho\left(\frac{2j+k}{3}\right) + \rho\left(\frac{j+2k}{3}\right)\right] - \frac{1}{k-j}\int_j^k \rho(t)dt \leq \frac{25(k-j)}{576}(\gamma_0 - \Gamma_0),$$

which is Special Case 2.15.1 of [8].

Special Case 2.26.(b) If we take, $\gamma(t) = \gamma_1 t + \gamma_0 \neq 0$, $\Gamma(t) = \Gamma_1 t + \Gamma_0 \neq 0$ and $\omega(t) = \frac{1}{k-j}$, in (2.14) then

$$m_{20} \leq \frac{3}{8}\left[\frac{\rho(j)+\rho(k)}{3} + \rho\left(\frac{2j+k}{3}\right) + \rho\left(\frac{j+2k}{3}\right)\right] - \frac{1}{k-j}\int_j^k \rho(t)dt \leq M_{20},$$

where

$$\begin{aligned} m_{20} &= \frac{(k-j)}{192}\left[(k-j)(\gamma_1+\Gamma_1) + \frac{31}{6}(j\gamma_1 - k\Gamma_1) + \frac{19}{6}(k\gamma_1 - j\Gamma_1)\right. \\ &+ \left.\frac{25}{3}(\gamma_0 - \Gamma_0)\right] \end{aligned}$$

and

$$\begin{aligned} M_{20} &= \frac{(k-j)}{192}\left[(k-j)(\gamma_1+\Gamma_1) + \frac{31}{6}(k\Gamma_1 - j\gamma_1) + \frac{19}{6}(j\Gamma_1 - k\gamma_1)\right. \\ &+ \left.\frac{25}{3}(\Gamma_0 - \gamma_0)\right], \end{aligned}$$

which is Special Case 2.15.2 of [8].

Now we state two results with their consequences for function ρ whose first derivative is bounded below only and bounded above only respectively.

Theorem 2.27. *Let $\rho : I \to \mathbb{R}$, be a differentiable function on I^0 of I, and let $[j,k] \subset I^0$. If ρ' is bounded below then $\gamma(\theta) \leq \rho'(\theta)$ for any $\gamma \in C[j,k]$, $\theta \in [j,k]$, then for all $\lambda \in [0,1]$ we have*

$$m_{21}(\theta,\lambda) \leq \rho(k)\int_\beta^k \omega(t)dt - \rho(j)\int_\alpha^j \omega(t)dt + \rho(\theta)\int_\alpha^{\frac{\alpha+\beta}{2}} \omega(t)dt + \rho(j+k-\theta)\int_{\frac{\alpha+\beta}{2}}^\beta \omega(t)dt - \int_j^k \rho(t)\omega(t)dt \leq M_{21}(\theta,\lambda), \quad (2.15)$$

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where

$$m_{21}(\theta,\lambda)=\int_j^k\left(\int_{\frac{\alpha+\beta}{2}}^t\omega(t)dt\right)\gamma(t)dt+\int_j^\theta\left(\int_\alpha^{\frac{\alpha+\beta}{2}}\omega(t)dt\right)\gamma(t)dt$$
$$-\int_{j+k-\theta}^k\left(\int_{\frac{\alpha+\beta}{2}}^\beta\omega(t)dt\right)\gamma(t)dt$$
$$-\max\left\{\int_\alpha^\theta\omega(t)dt,\int_{\frac{\alpha+\beta}{2}}^{j+k-\theta}\omega(t)dt,\int_\beta^k\omega(t)dt\right\}$$
$$\times\left(\rho(k)-\rho(j)-\int_j^k\gamma(t)dt\right)$$

and

$$M_{21}(\theta,\lambda)=\int_j^k\left(\int_{\frac{\alpha+\beta}{2}}^t\omega(t)dt\right)\gamma(t)dt+\int_j^\theta\left(\int_\alpha^{\frac{\alpha+\beta}{2}}\omega(t)dt\right)\gamma(t)dt$$
$$-\int_{j+k-\theta}^k\left(\int_{\frac{\alpha+\beta}{2}}^\beta\omega(t)dt\right)\gamma(t)dt$$

$$+\max\left\{\int_\alpha^\theta\omega(t)dt,\int_{\frac{\alpha+\beta}{2}}^{j+k-\theta}\omega(t)dt,\int_\beta^k\omega(t)dt\right\}$$
$$\times\left(\rho(k)-\rho(j)-\int_j^k\gamma(t)dt\right).$$

Proof. Since

$$\int_j^k K(\theta,t)\left(\rho'(t)-\gamma(t)\right)dt$$
$$=\rho(k)\int_\beta^k\omega(t)dt-\rho(j)\int_\alpha^j\omega(t)dt+\rho(\theta)\int_\alpha^{\frac{\alpha+\beta}{2}}\omega(t)dt$$
$$+\rho(j+k-\theta)\int_{\frac{\alpha+\beta}{2}}^\beta\omega(t)dt-\int_j^k\rho(t)\omega(t)dt-\int_j^k\omega(t)\rho(t)dt$$
$$-\int_j^k K(\theta,t)\gamma(t)dt$$
$$=\rho(k)\int_\beta^k\omega(t)dt-\rho(j)\int_\alpha^j\omega(t)dt+\rho(\theta)\int_\alpha^{\frac{\alpha+\beta}{2}}\omega(t)dt$$

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$$+\rho(j+k-\theta)\int_{\frac{\alpha+\beta}{2}}^{\beta}\omega(t)dt-\int_{j}^{k}\rho(t)\omega(t)dt$$
$$-\left[\int_{j}^{\theta}\left(\int_{\alpha}^{t}\omega(t)dt\right)\gamma(t)dt+\int_{\theta}^{j+k-\theta}\left(\int_{\frac{\alpha+\beta}{2}}^{t}\omega(t)dt\right)\gamma(t)dt\right.$$
$$\left.+\int_{j+k-\theta}^{k}\left(\int_{\beta}^{t}\omega(t)dt\right)\gamma(t)dt\right].$$

Using modulus property, we have

$$\left|\rho(k)\int_{\beta}^{k}\omega(t)dt-\rho(j)\int_{\alpha}^{j}\omega(t)dt+\rho(\theta)\int_{\alpha}^{\frac{\alpha+\beta}{2}}\omega(t)dt\right.$$
$$+\rho(j+k-\theta)\int_{\frac{\alpha+\beta}{2}}^{\beta}\omega(t)dt-\int_{j}^{k}\rho(t)\omega(t)dt$$
$$-\left[\int_{j}^{\theta}\left(\int_{\alpha}^{t}\omega(t)dt\right)\gamma(t)dt+\int_{\theta}^{j+k-\theta}\left(\int_{\frac{\alpha+\beta}{2}}^{t}\omega(t)dt\right)\gamma(t)dt\right.$$
$$\left.\left.+\int_{j+k-\theta}^{k}\left(\int_{\beta}^{t}\omega(t)dt\right)\gamma(t)dt\right]\right|$$
$$=\left|\int_{j}^{k}K(\theta,t)\left(\rho'(t)-\gamma(t)\right)dt\right|\leq\int_{j}^{k}|K(\theta,t)|\left(\rho'(t)-\gamma(t)\right)dt$$
$$\leq\max_{t\in[j,k]}|K(\theta,t)|\int_{j}^{k}\left(\rho'(t)-\gamma(t)\right)dt$$
$$=\max\left\{\int_{\alpha}^{\theta}\omega(t)dt,\int_{\frac{\alpha+\beta}{2}}^{j+k-\theta}\omega(t)dt,\int_{\beta}^{k}\omega(t)dt\right\}$$
$$\times\left(\rho(k)-\rho(j)-\int_{j}^{k}\gamma(t)dt\right).\tag{2.16}$$

After rearrangement of (2.16) we get required result. □

Remark 2.28. If we put $\omega(t)=\frac{1}{k-j}$ in inequality (2.15), then we will get the following result.

Corollary 2.29. *Let all the assumptions of Theorem* 2.27 *be valid. Then*

$$m_{22}(\theta,\lambda)\leq\lambda\frac{\rho(j)+\rho(k)}{2}+(1-\lambda)\frac{\rho(\theta)+\rho(j+k-\theta)}{2}-\int_{j}^{k}\rho(t)d(t)$$
$$\leq M_{22}(\theta,\lambda)\tag{2.17}$$

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where

$$m_{22}(\lambda) = \frac{1}{k-j}\left[\int_j^k \left(t - \frac{j+k}{2}\right)\gamma(t)dt + \frac{k-j}{2}\left(\int_j^\theta (1-\lambda)\,\gamma(t)dt - \int_{j+k-\theta}^k (1-\lambda)\,\gamma(t)dt\right) - \max\left\{\lambda\frac{k-j}{2}, \left(\theta - \frac{(2-\lambda)j+\lambda k}{2}\right), \left(\frac{j+k}{2} - \theta\right)\right\} \times \left(\rho(k) - \rho(j) - \int_j^k \gamma(t)dt\right)\right]$$

and

$$M_{22}(\lambda) = \frac{1}{k-j}\left[\int_j^k \left(t - \frac{j+k}{2}\right)\gamma(t)dt + \frac{k-j}{2}\left(\int_j^{\frac{j+k}{2}} (1-\lambda)\,\gamma(t)dt - \int_{\frac{j+k}{2}}^k (1-\lambda)\,\gamma(t)dt\right) + \max\left\{\lambda\frac{k-j}{2}, \left(\theta - \frac{(2-\lambda)j+\lambda k}{2}\right), \left(\frac{j+k}{2} - \theta\right)\right\} \times \left(\rho(k) - \rho(j) - \int_j^k \gamma(t)dt\right)\right],$$

which is Theorem 2.8 of [8].

Remark 2.30. If we choose $\theta = \frac{j+k}{2}$ in (2.15), we get the following result.

Corollary 2.31. *Let all the assumptions of Theorem* 2.1 *be valid. Then*

$$m_{23}(\lambda) \le \rho(k)\int_\beta^k \omega(t)dt - \rho(j)\int_\alpha^j \omega(t)dt + \rho\left(\frac{j+k}{2}\right)\int_\alpha^\beta \omega(t)dt - \int_j^k \rho(t)\omega(t)dt \le M_{23}(\lambda), \tag{2.18}$$

where

$$m_{23}(\lambda) = \int_j^{\frac{j+k}{2}} \left(\int_\alpha^t \omega(t)dt\right)\gamma(t)dt + \int_{\frac{j+k}{2}}^k \left(\int_\beta^t \omega(t)dt\right)\gamma(t)dt - \max\left\{\int_\alpha^{\frac{j+k}{2}} \omega(t)dt, \int_{\frac{\alpha+\beta}{2}}^{\frac{j+k}{2}} \omega(t)dt, \int_\beta^k \omega(t)dt\right\} \times \left(\rho(k) - \rho(j) - \int_j^k \gamma(t)dt\right)$$

and

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$$M_{23}(\lambda) = \int_j^{\frac{j+k}{2}} \int_\alpha^t \omega(t)dt\gamma(t)dt + \int_{\frac{j+k}{2}}^k \int_\beta^t \omega(t)dt\gamma(t)dt + \max\left\{\int_\alpha^{\frac{j+k}{2}} \omega(t)dt, \int_{\frac{\alpha+\beta}{2}}^{\frac{j+k}{2}} \omega(t)dt, \int_\beta^k \omega(t)dt\right\} \times \left(\rho(k) - \rho(j) - \int_j^k \gamma(t)dt\right).$$

Special Case 2.31. If we choose $\omega(t) = \frac{1}{k-j}$ in (2.18), we obtain the following result.

$$m_{24}(\lambda) \leq \left[\lambda\frac{\rho(j)+\rho(k)}{2} + (1-\lambda)\rho\left(\frac{j+k}{2}\right)\right] - \frac{1}{k-j}\int_j^k \rho(t)dt \leq M_{24}(\lambda), \quad (2.19)$$

where

$$m_{24}(\lambda) = \frac{1}{k-j}\left[\int_j^k \left(t - \frac{j+k}{2}\right)\gamma(t)dt + \frac{k-j}{2}\left(\int_j^{\frac{j+k}{2}} (1-\lambda)\,\gamma(t)dt - \int_{\frac{j+k}{2}}^k (1-\lambda)\,\gamma(t)dt\right) - \max\left\{\lambda\frac{k-j}{2}, (1-\lambda)\frac{k-j}{2}\right\} \times \left(\rho(k) - \rho(j) - \int_j^k \gamma(t)dt\right)\right]$$

and

$$M_{24}(\lambda) = \frac{1}{k-j}\left[\int_j^k \left(t - \frac{j+k}{2}\right)\gamma(t)dt + \frac{k-j}{2}\left(\int_j^{\frac{j+k}{2}} (1-\lambda)\,\gamma(t)dt - \int_{\frac{j+k}{2}}^k (1-\lambda)\,\gamma(t)dt\right) + \max\left\{\lambda\frac{k-j}{2}, (1-\lambda)\frac{k-j}{2}\right\} \times \left(\rho(k) - \rho(j) - \int_j^k \gamma(t)dt\right)\right],$$

which is Corollary 2.9 of [8].

Theorem 2.32. *Let $\rho : I \to \mathbb{R}$, be a differentiable function on I^0 of I, and let $[j,k] \subset I^0$. If ρ' is bounded above then $\rho'(\theta) \leq \Gamma(\theta)$ for any $\Gamma \in C[j,k]$, $\theta \in [j,k]$, then for all $\lambda \in [0,1]$ we have*

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$$m_{25}(\theta,\lambda) \le \rho(k)\int_{\beta}^{k}\omega(t)dt - \rho(j)\int_{\alpha}^{j}\omega(t)dt + \rho(\theta)\int_{\alpha}^{\frac{\alpha+\beta}{2}}\omega(t)dt + \rho(j+k-\theta)\int_{\frac{\alpha+\beta}{2}}^{\beta}\omega(t)dt - \int_{j}^{k}\rho(t)\omega(t)dt \le M_{25}(\theta,\lambda), \quad (2.20)$$

where

$$m_{25}(\theta,\lambda) = \int_{j}^{k}\left(\int_{\frac{\alpha+\beta}{2}}^{t}\omega(t)dt\right)\Gamma(t)dt + \int_{j}^{\theta}\left(\int_{\alpha}^{\frac{\alpha+\beta}{2}}\omega(t)dt\right)\Gamma(t)dt - \int_{j+k-\theta}^{k}\left(\int_{\frac{\alpha+\beta}{2}}^{t}\omega(t)dt\right)\Gamma(t)dt - \max\left\{\int_{\alpha}^{\theta}\omega(t)_0dt, \int_{\frac{\alpha+\beta}{2}}^{j+k-\theta}\omega(t)dt, \int_{\beta}^{k}\omega(t)dt\right\} \times \left(\int_{j}^{k}\Gamma(t)dt - \rho(k) + \rho(j)\right)$$

and

$$M_{25}(\theta,\lambda) = \int_{j}^{k}\left(\int_{\frac{\alpha+\beta}{2}}^{t}\omega(t)dt\right)\Gamma(t)dt + \int_{j}^{\theta}\left(\int_{\alpha}^{\frac{\alpha+\beta}{2}}\omega(t)dt\right)\Gamma(t)dt - \int_{j+k-\theta}^{k}\left(\int_{\frac{\alpha+\beta}{2}}^{t}\omega(t)dt\right)\Gamma(t)dt + \max\left\{\int_{\alpha}^{\theta}\omega(t)dt, \int_{\frac{\alpha+\beta}{2}}^{j+k-\theta}\omega(t)dt, \int_{\beta}^{k}\omega(t)dt\right\} \times \left(\int_{j}^{k}\Gamma(t)dt - \rho(k) + \rho(j)\right).$$

Proof. Since

$$\int_{j}^{k}K(\theta,t)\left(\rho'(t)-\Gamma(t)\right)dt = \rho(k)\int_{\beta}^{k}\omega(t)dt - \rho(j)\int_{\alpha}^{j}\omega(t)dt + \rho(\theta)\int_{\alpha}^{\frac{\alpha+\beta}{2}}\omega(t)dt + \rho(j+k-\theta)\int_{\frac{\alpha+\beta}{2}}^{\beta}\omega(t)dt - \int_{j}^{k}\rho(t)\omega(t)dt - \int_{j}^{k}K(\theta,t)\Gamma(t)dt$$

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$$
\begin{aligned}
&= \rho(k)\int_{\beta}^{k}\omega(t)dt - \rho(j)\int_{\alpha}^{j}\omega(t)dt + \rho(\theta)\int_{\alpha}^{\frac{\alpha+\beta}{2}}\omega(t)dt \\
&+\rho(j+k-\theta)\int_{\frac{\alpha+\beta}{2}}^{\beta}\omega(t)dt - \int_{j}^{k}\rho(t)\omega(t)dt \\
&-\left[\int_{j}^{\theta}\int_{\alpha}^{t}\omega(t)dt\Gamma(t)dt + \int_{\theta}^{j+k-\theta}\int_{\frac{\alpha+\beta}{2}}^{t}\omega(t)dt\Gamma(t)dt\right. \\
&\left.+\int_{j+k-\theta}^{k}\int_{\beta}^{t}\omega(t)dt\Gamma(t)dt.\right.
\end{aligned}
$$

so we have

$$
\begin{aligned}
&\left|\rho(k)\int_{\beta}^{k}\omega(t)dt - \rho(j)\int_{\alpha}^{j}\omega(t)dt + \rho(\theta)\int_{\alpha}^{\frac{\alpha+\beta}{2}}\omega(t)dt\right. \\
&+\rho(j+k-\theta)\int_{\frac{\alpha+\beta}{2}}^{\beta}\omega(t)dt - \int_{j}^{k}\rho(t)\omega(t)dt \\
&-\left[\int_{j}^{\theta}\int_{\alpha}^{t}\omega(t)dt\Gamma(t)dt + \int_{\theta}^{j+k-\theta}\int_{\frac{\alpha+\beta}{2}}^{t}\omega(t)dt\Gamma(t)dt\right. \\
&\left.\left.+\int_{j+k-\theta}^{k}\int_{\beta}^{t}\omega(t)dt\Gamma(t)dt\right]\right| \\
&= \left|\int_{j}^{k}K(\theta,t)\left(\rho'(t)-\Gamma(t)\right)dt\right| \\
&\leq \int_{j}^{k}|K(\theta,t)|\left(\Gamma(t)-\rho'(t)\right)dt \\
&\leq \max_{t\in[j,k]}|K(\theta,t)|\int_{j}^{k}\left(\Gamma(t)-\rho'(t)\right)dt \\
&= \max\left\{\int_{\alpha}^{\theta}\omega(t)dt, \int_{\frac{\alpha+\beta}{2}}^{j+k-\theta}\omega(t)dt, \int_{\beta}^{k}\omega(t)dt\right\} \\
&\times\left(\int_{j}^{k}\Gamma(t)dt - \rho(k)+\rho(j)\right), \qquad (2.21)
\end{aligned}
$$

we get required result after some rearrangement. □

Remark 2.33. If we put $\omega(t) = \frac{1}{k-j}$ in inequality (2.20) then we get Theorem 3 of [18] and Theorem 3 of [19].

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Corollary 2.34. *Let all the assumptions of Theorem* 2.32 *be valid and if we choose* $\theta = \frac{j+k}{2}$ *in* (2.20)*, then we get*

$$m_{26}(\lambda) \leq \rho(k)\int_{\beta}^{k}\omega(t)dt - \rho(j)\int_{\alpha}^{j}\omega(t)dt + \rho\left(\frac{j+k}{2}\right)\int_{\alpha}^{\beta}\omega(t)dt - \int_{j}^{k}\rho(t)\omega(t)dt \leq M_{26}(\lambda), \quad (2.22)$$

where

$$m_{26}(\lambda) = \int_{j}^{\frac{j+k}{2}}\int_{\alpha}^{t}\omega(t)dt\Gamma(t)dt + \int_{\frac{j+k}{2}}^{k}\int_{\beta}^{t}\omega(t)dt\Gamma(t)dt - \max\left\{\int_{\alpha}^{\frac{j+k}{2}}\omega(t)dt, \int_{\frac{\alpha+\beta}{2}}^{\frac{j+k}{2}}\omega(t)dt, \int_{\beta}^{k}\omega(t)dt\right\}\left(\int_{j}^{k}\Gamma(t)dt - \rho(k) + \rho(j)\right)$$

and

$$M_{26}(\lambda) = \int_{j}^{\frac{j+k}{2}}\int_{\alpha}^{t}\omega(t)dt\Gamma(t)dt + \int_{\frac{j+k}{2}}^{k}\int_{\beta}^{t}\omega(t)dt\Gamma(t)dt + \max\left\{\int_{\alpha}^{\frac{j+k}{2}}\omega(t)dt, \int_{\frac{\alpha+\beta}{2}}^{\frac{j+k}{2}}\omega(t)dt, \int_{\beta}^{k}\omega(t)dt\right\}\left(\int_{j}^{k}\Gamma(t)dt - \rho(k) + \rho(j)\right).$$

Special Case 2.34. If we choose $\omega(t) = \frac{1}{k-j}$ in (2.22), we obtain the following result.

$$m_{27}(\lambda) \leq \left[\lambda\frac{\rho(j)+\rho(k)}{2} + (1-\lambda)\rho\left(\frac{j+k}{2}\right)\right] - \frac{1}{k-j}\int_{j}^{k}\rho(t)dt \leq M_{27}(\lambda),$$

where

$$m_{27}(\lambda) = \frac{1}{k-j}\left[\int_{j}^{k}\left(t - \frac{j+k}{2}\right)\Gamma(t)dt + \frac{k-j}{2}\left(\int_{j}^{\frac{j+k}{2}}(1-\lambda)\,\Gamma(t)dt - \int_{\frac{j+k}{2}}^{k}(1-\lambda)\,\Gamma(t)dt\right) - \max\left\{\lambda\frac{k-j}{2}, (1-\lambda)\frac{k-j}{2}\right\}\left(\int_{j}^{k}\Gamma(t)dt - \rho(k) + \rho(j)\right)\right]$$

and

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$$M_{27}(\lambda) = \frac{1}{k-j}\left[\int_j^k \left(t - \frac{j+k}{2}\right)\Gamma(t)dt \right.$$
$$+ \frac{k-j}{2}\left(\int_j^{\frac{j+k}{2}} (1-\lambda)\,\Gamma(t)dt - \int_{\frac{j+k}{2}}^k (1-\lambda)\,\Gamma(t)dt\right)$$
$$\left. + \max\left\{\lambda\frac{k-j}{2}, (1-\lambda)\frac{k-j}{2}\right\}\left(\int_j^k \Gamma(t)dt - \rho(k) + \rho(j)\right)\right],$$

which is Corollary 2.11 of [8].

Remark 2.35. If $\gamma(\theta) \le \rho'(\theta) \le \Gamma(\theta)$ for any $\theta \in [j,k]$ and γ,$\Gamma \in C[j,k]$, and if one put $\lambda = 0$, then error bounds of non-standard quadrature are given as:

$$m_{29} \le \frac{1}{2}\left[-\rho(j) + 2\rho\left(\frac{j+k}{2}\right) + \rho(k)\right] - \frac{1}{k-j}\int_j^k \rho(t)dt \le M_{29}, \quad (2.23)$$

where

$$\begin{aligned} m_{29} &= \frac{1}{k-j}\left[\int_j^{\frac{j+k}{2}} (t-j)\,\gamma(t)dt + \int_{\frac{j+k}{2}}^k (t-k)\gamma(t)dt \right. \\ &+ \left. \frac{(k-j)}{2}\int_j^k \gamma(t)dt\right] \end{aligned}$$

and

$$\begin{aligned} M_{29} &= \frac{1}{k-j}\left[\int_j^{\frac{j+k}{2}} (t-j)\,\Gamma(t)dt + \int_{\frac{j+k}{2}}^k (t-k)\Gamma(t)dt \right. \\ &+ \left. \frac{(k-j)}{2}\int_j^k \Gamma(t)dt\right], \end{aligned}$$

which is Corollary 3 of [18], Corollary 4 of [19] and Corollary 2.11 of [8].

Proof. To prove (2.23), we use Corollaries 2.31 and 2.34. First by putting $\lambda = 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.22), we get

$$\frac{1}{k-j}\left[\int_j^k (t-j)\,\gamma(t)dt + \frac{k-j}{2}\left(\int_j^{\frac{j+k}{2}} \gamma(t)dt - \int_{\frac{j+k}{2}}^k \gamma(t)dt\right)\right]$$
$$\le \frac{1}{2}\left[-\rho(j) + 2\rho\left(\frac{j+k}{2}\right) + \rho(k)\right] - \frac{1}{k-j}\int_j^k \rho(t)dt \quad (2.24)$$

provided that $\gamma(t) \le \rho'(t)\ \forall\ t \in [j,k]$.

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On the other hand, by assuming $\lambda = 0$ and $\omega(t) = \frac{1}{b-a}$ in (2.22), we obtain

$$\frac{1}{2}\left[-\rho(j) + 2\rho\left(\frac{j+k}{2}\right) + \rho(k)\right] - \frac{1}{k-j}\int_j^k \rho(t)dt$$
$$\leq \frac{1}{k-j}\left[\int_j^k (t-j)\,\Gamma(t)dt + \frac{k-j}{2}\left(\int_j^{\frac{j+k}{2}} \Gamma(t)dt - \int_{\frac{j+k}{2}}^k \Gamma(t)dt\right)\right] \quad (2.25)$$

provided that $\rho'(t) \leq \Gamma(t)\ \forall\ t \in [j,k]$. Combining the above two inequalities obtain the required result. □

Remark 2.36. If $\gamma(\theta) \leq \rho'(\theta) \leq \Gamma(\theta)$ for any $\theta \in [j,k]$ and $\gamma, \Gamma \in C[j,k]$, and if we choose $\lambda = 0$, then error bound of non-standard quadrature would be

$$m_{30} \leq \frac{1}{2}\left[\rho(j) + 2\rho\left(\frac{j+k}{2}\right) - \rho(k)\right] - \frac{1}{k-j}\int_j^k \rho(t)dt \leq M_{30} \quad (2.26)$$

where

$$\begin{aligned} m_{30} &= \frac{1}{k-j}\left[\int_j^{\frac{j+k}{2}} (t-j)\,\Gamma(t)dt + \int_{\frac{j+k}{2}}^k (t-k)\Gamma(t)dt\right. \\ &- \left.\frac{(k-j)}{2}\int_j^k \Gamma(t)dt\right] \end{aligned}$$

and

$$\begin{aligned} M_{30} &= \frac{1}{k-j}\left[\int_j^{\frac{j+k}{2}} (t-j)\,\gamma(t)dt\right. \\ &+ \left.\int_{\frac{j+k}{2}}^k (t-k)\gamma(t)dt - \frac{(k-j)}{2}\int_j^k \gamma(t)dt\right], \end{aligned}$$

which is Corollary 4 of [18], Corollary 5 of [19] and Corollary 2.11 of [8].

Proof. Proof of (2.26) is similar to that of Remark 2.35, if one replaces $\lambda = 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.18) and (2.22), respectively, and combines them together. □

Remark 2.37. If $\gamma(\theta) \leq \rho'(\theta) \leq \Gamma(\theta)$ for any $\theta \in [j,k]$ and $\gamma, \Gamma \in C[j,k]$ then by replacing $\theta = k$, $\lambda = 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.18) and (2.22), respectively, then error of non-standard quadrature is given as

$$m_{31} \leq \rho(j) - \frac{1}{(k-j)}\int_j^k \rho(t)dt \leq M_{31},$$

where

$$m_{31} = \frac{1}{(k-j)}\int_j^k (t-k)\Gamma(t)dt$$

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and

$$M_{31} = \frac{1}{(k-j)} \int_j^k (t-k)\gamma(t)dt,$$

which is Corollary 5 of [18], Corollary 2 of [19] and Corollary 2.11.

Remark 2.38. If $\gamma(\theta) \leq \rho'(\theta) \leq \Gamma(\theta)$ for any $\theta \in [j,k]$ and $\gamma, \Gamma \in C[j,k]$ then by replacing $\theta = j$, $\lambda = 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.18) and (2.22), respectively, one gets error bounds of non-standard quadrature as follows:

$$m_{32} \leq \rho(k) - \frac{1}{(k-j)} \int_j^k \rho(t)dt \leq M_{32},$$

where

$$m_{32} = \frac{1}{(k-j)} \int_j^k (t-j)\gamma(t)dt \tag{2.27}$$

and

$$M_{32} = \frac{1}{(k-j)} \int_j^k (t-j)\Gamma(t)dt.$$

The inequality presented above is same as the Corollary 6 of [18], Corollary 3 of [19] and Corollary 2.11 of [8].

3. Applications to Numerical Quadrature Rules

Let $I_n : j = z_0 < z_1 < \cdots < z_n = k$ be a partition of interval $[j,k]$ and let $h_i = z_{i+1} - z_i, i \in \{0,1,2,\ldots,n-1\}$. Then

$$\int_j^k \omega(t)\rho(t)dt = Q_n(\rho,\omega,\lambda) + R_n(\rho,\omega,\lambda). \tag{3.1}$$

Consider a general quadrature rule

$$\begin{aligned} Q_n(\rho,\omega,\lambda) &= \sum_{i=0}^{n-1} \Bigg[\rho(z_{i+1}) \int_{\beta_i}^{z_{i+1}} \omega(t)dt - \rho(z_i) \int_{\alpha_i}^{z_i} \omega(t)dt \\ &+ \rho(\theta_i) \int_{\alpha_i}^{\frac{\alpha_i+\beta_i}{2}} \omega(t)dt + \rho(z_i + z_{i+1} - \theta_i) \int_{\frac{\alpha_i+\beta_i}{2}}^{\beta_i} \omega(t)dt \Bigg] \end{aligned} \tag{3.2}$$

where $\lambda \in [0,1]$ and $\theta_i \in [z_i, z_{i+1}]$. Then we get following result:

Theorem 3.1. *Let all the assumptions of Theorem* 2.1 *be valid. Then* (3.1) *holds where* $Q_n(\rho,\omega,\lambda)$ *is given by formula* (3.2) *and remainder* $R_n(I_n,\rho,\omega)$ *satisfies estimates*

$$|R_n(\rho,\omega,\lambda)| \leq \sum_{i=0}^{n-1} \sup\{|R_1|, |R_2|\}, \tag{3.3}$$

where

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$$
\begin{aligned}
R_1 &= \int_{z_i}^{\theta_i} \left(\left(\int_{\alpha_i}^{t} \omega(t)dt - \left| \int_{\alpha_i}^{t} \omega(t)dt \right| \right) \frac{\Gamma(t)}{2} \right. \\
&\quad \left. + \left(\int_{\alpha_i}^{t} \omega(t)dt + \left| \int_{\alpha_i}^{t} \omega(t)dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\
&\quad + \int_{\theta_i}^{z_i+z_{i+1}-\theta_i} \left(\left(\int_{\frac{\alpha_i+\beta_i}{2}}^{t} \omega(t)dt - \left| \int_{\frac{\alpha_i+\beta_i}{2}}^{t} \omega(t)dt \right| \right) \frac{\Gamma(t)}{2} \right. \\
&\quad \left. + \left(\int_{\frac{\alpha_i+\beta_i}{2}}^{t} \omega(t)dt + \left| \int_{\frac{\alpha_i+\beta_i}{2}}^{t} \omega(t)dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\
&\quad + \int_{z_i+z_{i+1}-\theta_i}^{z_{i+1}} \left(\left(\int_{\beta_i}^{t} \omega(t)dt - \left| \int_{\beta_i}^{t} \omega(t)dt \right| \right) \frac{\Gamma(t)}{2} \right. \\
&\quad \left. + \left(\int_{\beta_i}^{t} \omega(t)dt + \left| \int_{\beta_i}^{t} \omega(t)dt \right| \right) \frac{\gamma(t)}{2} \right) dt
\end{aligned}
$$

and

$$
\begin{aligned}
R_2 &= \int_{z_i}^{\theta_i} \left(\left(\int_{\alpha_i}^{t} \omega(t)dt + \left| \int_{\alpha_i}^{t} \omega(t)dt \right| \right) \frac{\Gamma(t)}{2} \right. \\
&\quad \left. + \left(\int_{\alpha_i}^{t} \omega(t)dt - \left| \int_{\alpha_i}^{t} \omega(t)dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\
&\quad + \int_{\theta_i}^{z_i+z_{i+1}-\theta_i} \left(\left(\int_{\frac{\alpha_i+\beta_i}{2}}^{t} \omega(t)dt + \left| \int_{\frac{\alpha_i+\beta_i}{2}}^{t} \omega(t)dt \right| \right) \frac{\Gamma(t)}{2} \right. \\
&\quad \left. + \left(\int_{\frac{\alpha_i+\beta_i}{2}}^{t} \omega(t)dt - \left| \int_{\frac{\alpha_i+\beta_i}{2}}^{t} \omega(t)dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\
&\quad + \int_{z_i+z_{i+1}-\theta_i}^{z_{i+1}} \left(\left(\int_{\beta_i}^{t} \omega(t)dt + \left| \int_{\beta_i}^{t} \omega(t)dt \right| \right) \frac{\Gamma(t)}{2} \right. \\
&\quad \left. + \left(\int_{\beta_i}^{t} \omega(t)dt - \left| \int_{\beta_i}^{t} \omega(t)dt \right| \right) \frac{\gamma(t)}{2} \right) dt,
\end{aligned}
$$

$\forall\ \theta_i \in [z_i, z_{i+1}]$.

Proof. Applying inequality (2.1) on the intervals $[z_i, z_{i+1}]$ for $r \in \{1, 2, \ldots, n-1\}$, and using (3.3), we get

$$
\begin{aligned}
R_i(\rho, \omega, \lambda) &= \int_{z_i}^{z_{i+1}} \omega(t)\rho(t)dt - \left[\rho(z_{i+1}) \int_{\beta_i}^{z_{i+1}} \omega(t)dt - \rho(z_i) \int_{\alpha_i}^{z_i} \omega(t)dt \right. \\
&\quad \left. + \rho(\theta_i) \int_{\alpha_i}^{\frac{\alpha_i+\beta_i}{2}} \omega(t)dt + \rho(z_i + z_{i+1} - \theta_i) \int_{\frac{\alpha_i+\beta_i}{2}}^{\beta_i} \omega(t)dt \right].
\end{aligned}
$$

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Summing it over i from 0 to $n-1$ we get

$$R_n(\rho,\omega,\lambda) = \int_j^k \rho(t)\omega(t)dt - \sum_{i=0}^{n-1}\left[\rho(z_{i+1})\int_{\beta_i}^{z_{i+1}}\omega(t)dt - \right.$$
$$\left.\rho(z_i)\int_{\alpha_i}^{z_i}\omega(t)dt + \rho(\theta_i)\int_{\alpha_i}^{\frac{\alpha_i+\beta_i}{2}}\omega(t)dt + \rho(z_i+z_{i+1}-\theta_i)\int_{\frac{\alpha_i+\beta_i}{2}}^{\beta_i}\omega(t)dt\right].$$

It follows from (2.1) that

$$\begin{aligned}
|R_n(\rho,\omega,\lambda)| &= \left|\int_j^k \rho(t)\omega(t)dt\right.\\
&- \sum_{i=0}^{n-1}\left[\rho(z_{i+1})\int_{\beta_i}^{z_{i+1}}\omega(t)dt - \rho(z_i)\int_{\alpha_i}^{z_i}\omega(t)dt\right.\\
&+ \left.\left.\rho(\theta_i)\int_{\alpha_i}^{\frac{\alpha_i+\beta_i}{2}}\omega(t)dt + \rho(z_i+z_{i+1}-\theta_i)\int_{\frac{\alpha_i+\beta_i}{2}}^{\beta_i}\omega(t)dt\right]\right|\\
&\le \sum_{i=0}^{n-1}\sup\left\{\left|\left[\int_{z_i}^{\theta_i}\left(\left(\int_{\alpha_i}^t\omega(t)dt - \left|\int_{\alpha_i}^t\omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right.\right.\right.\\
&+ \left.\left.\left(\int_{\alpha_i}^t\omega(t)dt + \left|\int_{\alpha_i}^t\omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt\\
&+ \int_{\theta_i}^{z_i+z_{i+1}-\theta_i}\left(\left(\int_{\frac{\alpha_i+\beta_i}{2}}^t\omega(t)dt - \left|\int_{\frac{\alpha_i+\beta_i}{2}}^t\omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right.\\
&+ \left.\left(\int_{\frac{\alpha_i+\beta_i}{2}}^t\omega(t)dt + \left|\int_{\frac{\alpha_i+\beta_i}{2}}^t\omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt\\
&+ \int_{z_i+z_{i+1}-\theta_i}^{z_{i+1}}\left(\left(\int_{\beta_i}^t\omega(t)dt - \left|\int_{\beta_i}^t\omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right.\\
&+ \left.\left.\left(\int_{\beta_i}^t\omega(t)dt + \left|\int_{\beta_i}^t\omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt\right]\Bigg|,\\
&\left|\int_{z_i}^{\theta_i}\left(\left(\int_{\alpha_i}^t\omega(t)dt + \left|\int_{\alpha_i}^t\omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right.\right.\\
&+ \left.\left.\left(\int_{\alpha_i}^t\omega(t)dt - \left|\int_{\alpha_i}^t\omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt\right.\\
&+ \int_{\theta_i}^{z_i+z_{i+1}-\theta_i}\left(\left(\int_{\frac{\alpha_i+\beta_i}{2}}^t\omega(t)dt + \left|\int_{\frac{\alpha_i+\beta_i}{2}}^t\omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right.\\
&+ \left.\left(\int_{\frac{\alpha_i+\beta_i}{2}}^t\omega(t)dt - \left|\int_{\frac{\alpha_i+\beta_i}{2}}^t\omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt\\
&\quad + \int_{z_i+z_{i+1}-\theta_i}^{z_{i+1}}\left(\left(\int_{\beta_i}^t\omega(t)dt + \left|\int_{\beta_i}^t\omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right.\\
&\quad + \left.\left.\left.\left(\int_{\beta_i}^t\omega(t)dt - \left|\int_{\beta_i}^t\omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt\right]\right|\Bigg\}.
\end{aligned}$$

□

Remark 3.2. Similarly, we can state applications of other results and their cases as given in Section 2.

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4. Conclusion

In this paper, weighted Ostrowski type inequality is discussed for function differentiable functions with variable bounds. Applications to solve error bounds of midpoint, trapezoidal, Simpson's and Simpson's quadrature and some non-standard quadrature rules are presented. We also have many proven results as our special cases. In particularly our results would generalization of [8, 18, 19, 27].

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