Iranian Journal of Mathematical Sciences and Informatics
Vol. 20, No. 1 (2025), pp 205-229
DOI: 10.61186/ijmsi.20.1.205

New Concepts of Generalized Convexity in Multiobjective Subset Programming

Tadeusz Antczak[a*], Izhar Ahmad[b]

[a]Faculty of Mathematics and Computer Science, University of Łódź, Banacha 22, 90-238 Łódź, Poland
[b]Department of Mathematics and Statistics, King Fahd University of Petroleum and Minerals, Dhahran, 31261, Saudi Arabia

E-mail: tadeusz.antczak@wmii.uni.lodz.pl
E-mail: drizhar@kfupm.edu.sa

Abstract. In this paper, a new class of nonconvex differentiable multiobjective programming problems involving n-set functions with both inequality and equality constraints is considered. Then, under V-r-convexity and/or generalized V-r-convexity hypotheses, several sufficient optimality conditions, saddle point criteria and various mixed duality theorems are proved for such not necessarily convex vector optimization problems involving n-set functions.

Keywords: Vector optimization problem with n-set functions, Optimality conditions, Saddle point criteria, Mixed duality, V-r-convex n-set function.

2000 Mathematics subject classification: 90C29, 90C30, 90C46, 90C48, 90C26.

1. Introduction

Many optimization problems containing set functions arise in situations dealing with optimal constrained selection of measurable subsets. Some extremum problems of this type have been encountered in statistics [11], fluid

*Corresponding Author

Received 07 July 2020; Accepted 28 June 2021

flow [5], electrical insulator design [8], regional design (districting, facility location, warehouse layout, urban planning) [9], earthquake engineering [29], shape optimization [14].

General theory for optimizing n-set functions was first developed by Morris [19] who, for fractions of a single set, obtained results that are similar to the standard mathematical programming problem. Corley [10] developed an optimization theory for mathematical programming problems with n-set functions, established optimality conditions, and obtained Lagrangian duality results. Zalmai [30] considered several practical applications for a class of nonlinear mathematical programming problems involving a single objective and differentiable n-set functions, and established several sufficient optimality conditions and duality results under generalized ρ-convexity conditions. Stancu-Minasian and Preda [28] prepared a survey on optimality conditions and duality results for optimization problems with n-set functions.

Many publications have appeared in the last three decades dealing with optimality conditions and duality results for convex and various nonconvex multiobjective programming problems involving n-set functions (see, for example, [1, 3, 4, 6, 7, 12, 13, 15, 16, 17, 20, 21, 22, 24, 25, 27, 31], and others). Mishra et al. [17] used the concept of vector-valued generalized type-I functions in proving optimality conditions and Mond-Weir type duality results for a multiobjective programming problem involving n-set functions. In [1], Ahmad and Sharma established sufficient optimality conditions for a multiobjective subset programming problem under generalized (F, α, ρ, d)-type-I functions. Jayswal and Minasian [12] introduced various classes of generalized univex n-set functions and, moreover, they proved sufficient optimality conditions and several Mond-Weir duality theorems for the considered multiobjective subset programming problem involving such nonconvex n-set functions. Preda et al. [25] studied optimality conditions and generalized Mond-Weir duality for multiobjective programming involving n-set functions which satisfy appropriate generalized univexity V-type-I conditions.

In the present paper, we consider a new class of differentiable multiobjective programming problems involving n-set functions with both inequality and equality constraints. First, motivated by Avriel [2], we define a new class of n-set functions, called V-r-convex n-set functions, and its generalizations, that is, classes of V-r-pseudo-convex n-set functions and V-r-quasi-convex n-set functions. Then, we prove several sufficient optimality conditions for the considered differentiable multicriteria optimization problem under both V-r-convexity and/or generalized V-r-convexity hypotheses. Further, we define the vector-valued Lagrangian-type function for the considered vector optimization problem with n-set functions and then we introduce the definition of its vector saddle point. Moreover, under appropriate V-r-convexity and/or generalized V-r-convexity hypotheses, we prove saddle point criteria for the considered nonconvex multiobjective programming problem with n-set functions. Furthermore, we define its vector mixed dual subset problem and we prove various mixed dual theorems also under appropriate (generalized) V-r-convexity hypotheses.

2. Preliminaries and problem formulation

In this section, we give a few basic definitions and auxiliary results which will be used frequently throughout the sequel.

The following convention for equalities and inequalities will be used throughout the paper. For any $x = (x_1, x_2, ..., x_n)^T$, $y = (y_1, y_2, ..., y_n)^T$ in R^n, we define:

(i) $x = y$ if and only if $x_i = y_i$ for all $i = 1, 2, ..., n$;
(ii) $x < y$ if and only if $x_i < y_i$ for all $i = 1, 2, ..., n$;
(iii) $x \leqq y$ if and only if $x_i \leqq y_i$ for all $i = 1, 2, ..., n$;
(iv) $x \leq y$ if and only if $x \leqq y$ and $x \neq y$.

Throughout the paper, we assume that $(X, \mathbb{A}, \mu)$ is a finite atomless measure space with $L_1(X, \mathbb{A}, \mu)$ separable. We also assume that S is a subset of $\mathbb{A}^n = \mathbb{A} \times \mathbb{A} \times ... \times \mathbb{A}$, the n-fold product of the σ-algebra $\mathbb{A}$ of subsets of a given set X. Let d be the pseudometric on $\mathbb{A}^n$ (see [19]) defined by

$$d(S,T) = d((S_1, ..., S_n), (T_1, ..., T_n)) = \left[\sum_{k=1}^{n} \mu^2 (S_k \ominus T_k)\right]^{1/2},$$

where $\ominus$ denotes the symmetric difference for S_i and T_i, $i = 1, ..., n$. Thus (A^n, d) is a pseudometric space which will serve as the domain for the most of the functions used in the present paper. For $h \in L_1(X, \mathbb{A}, \mu)$ and $V \in \mathbb{A}^n$ with the indicator (characteristic) function $\chi_V \in L_\infty(X, \mathbb{A}, \mu)$ of V, the integral $\int_V h d\mu$ is denoted by $\langle h, \chi_V \rangle$.

We now give the following definitions along the lines of Zalmai [30].

Definition 2.1. A set function $F : \mathbb{A} \to R$ is differentiable at $S^* \in \mathbb{A}$ if there exists $DF(S^*) \in L_1(X, \mathbb{A}, \mu)$, called the derivative of F at S^*, and $V_F : \mathbb{A} \times \mathbb{A} \to R$, such that

$$F(S) = F(S^*) + \langle DF(S^*), \chi_S - \chi_{S^*} \rangle + V_F(S^*, S), \tag{2.1}$$

where $V_F(S^*, S)$ is $o(d(S^*, S))$, i.e. $\lim\limits_{d(S^*,S)\to 0} \frac{V_F(S^*,S)}{d(S^*,S)} = 0$, and d is a pseudometric on $\mathbb{A}$.

We now give the definition of the partial derivatives of n-set functions.

Definition 2.2. A function $F : \mathbb{A}^n \to R$ is said to have a partial derivative at $S^* = (S_1^*, ..., S_n^*) \in \mathbb{A}$ with respect to its kth argument, $1 \leqq k \leqq n$, if the set function $H(S_k) = F\left(S_1^*, ..., S_{k-1}^* S_k, S_{k+1}^*, ..., S_n^*\right)$ has derivative $DH(S_k^*)$ at S_k^*. In that case, we define kth partial derivative of F at S^* to be $D_k F(S^*) =$

$DH\left(S_k^*\right)$, $1 \leqq k \leqq n$. If $D_kF(S^*)$, $1 \leqq k \leqq n$, all exist, then we put $DF(S^*) = (D_1F(S^*), ..., D_nF(S^*))$.

Using the partial derivatives of the n-set function, we can define the derivative of the vector-valued n-set function.

Definition 2.3. Let $F : \mathbb{A}^n \to R^p$ and $S^* \in \mathbb{A}^n$. Then F is said to be differentiable at S^* if the partial derivatives $D_kF_i\left(S^*\right)$, $k = 1, ..., n$, of F_i exist for each $i = 1, ..., p$ and satisfy

$$F\left(S\right) = F\left(S^*\right) + \left(\sum_{k=1}^{n}\left\langle D_kF_1\left(S^*\right), \chi_{S_k} - \chi_{S_k^*}\right\rangle, ..., \sum_{k=1}^{n}\left\langle D_kF_p\left(S^*\right), \chi_{S_k} - \chi_{S_k^*}\right\rangle\right) + V_F\left(S^*, S\right) \text{ for all } S \in \mathbb{A}^n,$$

where $V_F\left(S^*, S\right)$ is $o\left(d\left(S^*, S\right)\right)$ for all $S \in \mathbb{A}^n$, i.e. $\lim\limits_{d(S^*,S)\to 0} \frac{V_F(S^*,S)}{d(S^*,S)} = 0$.

If F is differentiable at each $S^* \in \mathbb{A}^n$, we say that F is differentiable on $\mathbb{A}^n$.

In this section, we introduce new classes of generalized convex differentiable vector-valued n-set functions. Firstly, we give the definition of a differentiable V-r-convex (strictly V-r-convex) n-set function.

Definition 2.4. Let $F = (F_1, ..., F_p) : \mathbb{A}^n \to R^p$ be a differentiable set function and $S^* \in A^n$ be given. If there exist $\alpha_i : \mathbb{A}^n \times \mathbb{A}^n \to R_+ \setminus \{0\}$, $i = 1, ..., p$, and a real number r such that, for any $i = 1, ..., p$, the inequalities

$$\begin{aligned} &\tfrac{1}{r}e^{rF_i(S)} \geqq \tfrac{1}{r}e^{rF_i(S^*)}\left[1 + r\alpha_i\left(S, S^*\right)\left\langle \textstyle\sum_{k=1}^{n} D_kF_i\left(S^*\right), \chi_{S_k} - \chi_{S_k^*}\right\rangle\right] (>) r \neq 0, \\ &F_i(S) - F_i(S^*) \geqq \alpha_i\left(S, S^*\right)\left\langle \textstyle\sum_{k=1}^{n} D_kF_i\left(S^*\right), \chi_{S_k} - \chi_{S_k^*}\right\rangle \qquad (>) r = 0 \end{aligned} \tag{2.2}$$

hold for each $S \in \mathbb{A}^n$, $(S \neq S^*)$, then F is said to be a V-r-convex (strictly V-r-convex) n-set function at S^* on $\mathbb{A}^n$. If inequalities (2.2) are satisfied at any point S^*, then F is said to be a V-r-convex (strictly V-r-convex) function on $\mathbb{A}^n$. Each function F_i, $i = 1, ..., p$, satisfying (2.2) is said to be an α_i-r-convex n-set function at S^* on $\mathbb{A}^n$.

Definition 2.5. Let $F : A^n \to R^p$ be a differentiable set function and $S^* \in A^n$. If there exist $\alpha_i : \mathbb{A}^n \times \mathbb{A}^n \to R_+ \setminus \{0\}$, $i = 1, ..., p$, and a real number r such that, for any $i = 1, ..., p$, the inequalities

$$\begin{aligned} &\tfrac{1}{r}e^{rF_i(S)} \leqq \tfrac{1}{r}e^{rF_i(S^*)}\left[1 + r\alpha_i\left(S, S^*\right)\left\langle \textstyle\sum_{k=1}^{n} D_kF_i\left(S^*\right), \chi_{S_k} - \chi_{S_k^*}\right\rangle\right] (<) r \neq 0, \\ &F_i(S) - F_i(S^*) \leqq \alpha_i\left(S, S^*\right)\left\langle \textstyle\sum_{k=1}^{n} D_kF_i\left(S^*\right), \chi_{S_k} - \chi_{S_k^*}\right\rangle \qquad (<) r = 0 \end{aligned} \tag{2.3}$$

hold for each $S \in \mathbb{A}^n$, $(S \neq S^*)$, then F is said to be a V-r-concave (strictly V-r-concave) n-set function at S^* on $\mathbb{A}^n$. If inequalities (2.3) are satisfied at any point S^*, then F is said to be a V-r-concave (strictly V-r-concave) function on

$\mathbb{A}^n$. Each function F_i, $i = 1, ..., p$, satisfying (2.3) is said to be an α_i-r-concave n-set function at S^* on $\mathbb{A}^n$.

Now, we introduce the definitions of generalized V-r-convex functions.

Definition 2.6. Let $F : \mathbb{A}^n \to R^p$ be a differentiable set function and $S^* \in \mathbb{A}^n$. If there exist $\alpha_i : \mathbb{A}^n \times \mathbb{A}^n \to R_+ \setminus \{0\}$ and a real number r such that the relation

$$\begin{aligned}
&\tfrac{1}{r}\textstyle\sum_{i=1}^{p} \alpha_i\left(S, S^*\right) e^{rF_i(S)} < (\leqq) \tfrac{1}{r}\sum_{i=1}^{p} \alpha_i\left(S, S^*\right) e^{rF_i(S^*)} \Longrightarrow \\
&\qquad \textstyle\sum_{i=1}^{p} \left\langle \sum_{k=1}^{n} D_k F_i\left(S^*\right), \chi_{S_k} - \chi_{S_k^*} \right\rangle < 0 \qquad \text{for } r \neq 0, \\
&\textstyle\sum_{i=1}^{p} \alpha_i\left(S, S^*\right) F_i(S) < (\leqq) \sum_{i=1}^{p} \alpha_i\left(S, S^*\right) F_i(S^*) \Longrightarrow \\
&\qquad \textstyle\sum_{i=1}^{p} \left\langle \sum_{k=1}^{n} D_k F_i\left(S^*\right), \chi_{S_k} - \chi_{S_k^*} \right\rangle < 0 \qquad \text{for } r = 0
\end{aligned} \tag{2.4}$$

holds for each $S \in \mathbb{A}^n$, $(S \neq S^*)$, then F is said to be a V-r-pseudo-convex (strictly r-pseudo-convex) n-set function at S^* on $\mathbb{A}^n$. If the relation (2.4) is satisfied at any point S^*, then F is said to be a V-r-pseudo-convex (strictly V-r-pseudo-convex) function on $\mathbb{A}^n$. Each function F_i, $i = 1, ..., p$, satisfying (2.4) is said to be an α_i-r-pseudo-convex n-set function at S^* on $\mathbb{A}^n$.

Definition 2.7. Let $F : \mathbb{A}^n \to R^p$ be a differentiable set function and $S^* \in \mathbb{A}^n$. If there exist $\alpha_i : \mathbb{A}^n \times \mathbb{A}^n \to R_+ \setminus \{0\}$ and a real number r such that the relation

$$\begin{aligned}
&\tfrac{1}{r}\textstyle\sum_{i=1}^{p} \alpha_i\left(S, S^*\right) e^{rF_i(S)} \leqq \tfrac{1}{r}\sum_{i=1}^{p} \alpha_i\left(S, S^*\right) e^{rF_i(S^*)} \Longrightarrow \\
&\qquad \textstyle\sum_{i=1}^{p} \left\langle \sum_{k=1}^{n} D_k F_i\left(S^*\right), \chi_{S_k} - \chi_{S_k^*} \right\rangle \leqq 0 \qquad \text{for } r \neq 0, \\
&\textstyle\sum_{i=1}^{p} \alpha_i\left(S, S^*\right) F_i(S) \geqq \sum_{i=1}^{p} \alpha_i\left(S, S^*\right) F_i(S^*) \Longrightarrow \\
&\qquad \textstyle\sum_{i=1}^{p} \left\langle \sum_{k=1}^{n} D_k F_i\left(S^*\right), \chi_{S_k} - \chi_{S_k^*} \right\rangle \leqq 0 \qquad \text{for } r = 0
\end{aligned} \tag{2.5}$$

holds for each $S \in \mathbb{A}^n$, then F is said to be a V-r-quasi-convex n-set function at S^* on $\mathbb{A}^n$.
If the relation (2.5) is satisfied at any point S^*, then F is said to be a V-r-quasi-convex function on $\mathbb{A}^n$.
Each function F_i, $i = 1, ..., p$, satisfying (2.5) is said to be an α_i-r-quasi-convex n-set function at S^* on $\mathbb{A}^n$.

Remark 2.8. All the results in the paper will be proved only in the case when $r \neq 0$ (the case $r = 0$ can be dealt with likewise since the only changes arise from the form of inequality defining the introduced classes of generalized convex functions).

In the paper, we consider the following constrained multiobjective programming problem with n-set functions:

$$\begin{aligned} &V\text{-minimize } F(S) = (F_1(S), ..., F_p(S)) \\ &\text{subject to} \quad Q_j(S) \leqq 0, \ \ j \in J = \{1, ...m\}, \\ &H_t(S) = 0, \ \ t \in T = \{1, ...w\}, \\ &S = (S_1, ..., S_n) \in \mathbb{A}^n, \end{aligned} \tag{MP}$$

where $\mathbb{A}^n$ is the n-fold product of the σ-algebra $\mathbb{A}$, F_i, $i \in I = \{1, ..., p\}$, Q_j, $j \in J$, and H_t, $t \in T$, are differentiable real-valued n-set functions defined on $\mathbb{A}^n$. The term "V-minimize" being used in the problem (MP) is for finding its weakly efficient, efficient and properly efficient solutions.

For the purpose of simplifying our presentation, we will next introduce some notation which will be used frequently throughout this paper. Let Ω (assumed to be nonempty) defined as follows

$$\Omega := \{S \in \mathbb{A}^n : Q_j(S) \leq 0, \, j \in J, \, H_t(S) = 0, \, t \in T\}$$

be the set of all feasible solutions of (MP). Further, we denote by $J(S^*)$ the set of inequality constraint indexes active at $S^* \in \Omega$, that is,

$$J(S^*) = \{j \in J : Q_j(S^*) = 0\}.$$

Definition 2.9. A feasible solution S^* is said to be a weakly efficient solution of (MP) if there is no other $S \in \Omega$ such that

$$F(S) < F(S^*).$$

Definition 2.10. A feasible solution S^* is said to be an efficient solution of (MP) if there is no other $S \in \Omega$ such that

$$F(S) \leq F(S^*).$$

Definition 2.11. An efficient solution $S^* \in \Omega$ is said to be a properly efficient solution of (MP) if there exists $M > 0$ such that, for each i and $S \in \Omega$ satisfying $F_i(S) < F(S^*)$, we have that the inequality

$$\frac{F_i(S^*) - F_i(S)}{F_q(S) - F_q(S^*)} \leqq M$$

holds for at least one $q \in I$ for which $F_q(S^*) < F_q(S)$.

Now, for the considered vector optimization subset problem (MP), we prove sufficient optimality conditions for its weakly efficient solution, efficient solution and properly efficient solution under the introduced generalized convexity notions.

Theorem 2.12. *Let S^* be a feasible solution of the considered multiobjective programming problem (MP) and, moreover, there exist $\xi^* \in R^p$, $\zeta^* \in R^m$ and $\vartheta^* \in R^w$ such that the following optimality conditions*

$$\sum_{i=1}^{p} \xi_i^* \left\langle \sum_{k=1}^{n} D_k F_i\left(S^*\right), \chi_{S_k} - \chi_{S_k^*} \right\rangle + \sum_{j=1}^{m} \zeta_j^* \sum_{k=1}^{n} \left\langle D_k Q_j\left(S^*\right), \chi_{S_k} - \chi_{S_k^*} \right\rangle +$$

$$\sum_{t=1}^{w} \vartheta_t^* \sum_{k=1}^{n} \left\langle D_k H_t\left(S^*\right), \chi_{S_k} - \chi_{S_k^*} \right\rangle \geqq 0, \quad \forall S \in \mathbb{A}^n, \tag{2.6}$$

$$\zeta_j^* Q_j\left(S^*\right) = 0, \quad j \in J, \tag{2.7}$$

$$\xi^* \geq 0, \; \sum_{i=1}^{p} \xi_i^* = 1, \; \zeta^* \geqq 0 \tag{2.8}$$

be satisfied at S^*. Further, we assume that one of the following sets of hypotheses is fulfilled:

A) *each objective function F_i, $i \in I$, is α_i-r-convex at S^* on Ω, each inequality constraint Q_j, $j \in J\left(S^*\right)$,is β_j-r-convex at S^* on Ω, each equality constraint H_t, $t \in T^+\left(S^*\right) = \{t \in T : \vartheta_t^* > 0\}$ is γ_t-r-convex at S^* on Ω, each equality constraint H_t, $t \in T^-\left(S^*\right) = \{t \in T : \vartheta_t^* < 0\}$ is γ_t-r-concave at S^* on Ω,*

B) *$\xi_i^* F_i$, $i \in I$, is an α_i-r-pseudo-convex function at S^* on Ω, $\zeta_j^* Q_j$, $j \in J$, is β_j-r-quasi-convex at S^* on Ω, $\vartheta_t^* H_t$, $t \in T$, is γ_t-r-quasi-convex at S^* on Ω.*

Then S^* is a weakly efficient solution of (MP).

Proof. Let S^* be a feasible solution solution in the considered multiobjective programming problem (MP) and the optimality conditions (2.6)-(2.8) be satisfied at S^*. Using hypothesis A), by Definitions 2.4 and 2.5, the inequalities

$$\frac{1}{r} e^{rF_i(S)} \geqq \frac{1}{r} e^{rF_i(S^*)} \left[1 + r\alpha_i\left(S, S^*\right) \sum_{k=1}^{n} \left\langle D_k F_i\left(S^*\right), \chi_{S_k} - \chi_{S_k^*} \right\rangle \right], \quad i \in I, \tag{2.9}$$

$$\frac{1}{r} e^{rQ_j(S)} \geqq \frac{1}{r} e^{rQ_j(S^*)} \left[1 + r\beta_j\left(S, S^*\right) \sum_{k=1}^{n} \left\langle D_k Q_j\left(S^*\right), \chi_{S_k} - \chi_{S_k^*} \right\rangle \right], \; j \in J\left(S^*\right), \tag{2.10}$$

$$\frac{1}{r} e^{rH_t(S)} \geqq \frac{1}{r} e^{rH_t(S^*)} \left[1 + r\gamma_t\left(S, S^*\right) \sum_{k=1}^{n} \left\langle D_k H_t\left(S^*\right), \chi_{S_k} - \chi_{S_k^*} \right\rangle \right], \; t \in T^+\left(S^*\right), \tag{2.11}$$

$$\frac{1}{r} e^{rH_t(S)} \leqq \frac{1}{r} e^{rH_t(S^*)} \left[1 + r\gamma_t\left(S, S^*\right) \sum_{k=1}^{n} \left\langle D_k H_t\left(S^*\right), \chi_{S} - \chi_{S^*} \right\rangle \right], \; t \in T^-\left(S^*\right) \tag{2.12}$$

hold for all $S \in \Omega$. We proceed by contradiction. Suppose, contrary to the result, that there exists other $\widetilde{S} \in \Omega$ such that

$$F\left(\widetilde{S}\right) < F\left(S^*\right).$$

Then, (2.9) gives

$$\alpha_i\left(\widetilde{S}, S^*\right)\sum_{k=1}^{n}\left\langle D_kF_i\left(S^*\right), \chi_{\widetilde{S}_k} - \chi_{S_k^*}\right\rangle < 0, \ \ i \in I.$$

Since $\alpha_i\left(\widetilde{S}, S^*\right) > 0$, $i = 1, ..., p$, the foregoing inequalities yield

$$\sum_{k=1}^{n}\left\langle D_kF_i\left(S^*\right), \chi_{\widetilde{S}_k} - \chi_{S_k^*}\right\rangle < 0, \ \ i \in I. \tag{2.13}$$

Hence, by (2.8), (2.13) implies

$$\sum_{i=1}^{p}\xi_i^*\sum_{k=1}^{n}\left\langle D_kF_i\left(S^*\right), \chi_{\widetilde{S}_k} - \chi_{S_k^*}\right\rangle < 0. \tag{2.14}$$

Multiplying the inequalities (2.10)-(2.12) by the associated Lagrange multiplier, respectively, we get

$$\frac{1}{r}\zeta_j^* e^{rQ_j(\widetilde{S})} \geqq \frac{1}{r}\zeta_j^* e^{rQ_j(S^*)}\left[1 + r\beta_j\left(\widetilde{S}, S^*\right)\sum_{k=1}^{n}\left\langle D_kQ_j\left(S^*\right), \chi_{\widetilde{S}_k} - \chi_{S_k^*}\right\rangle\right], \quad j \in J\left(S^*\right), \tag{2.15}$$

$$\frac{1}{r}\vartheta_t^* e^{rH_t(\widetilde{S})} \geqq \frac{1}{r}\vartheta_t^* e^{rH_t(S^*)}\left[1 + r\gamma_t\left(\widetilde{S}, S^*\right)\sum_{k=1}^{n}\left\langle D_kH_t\left(S^*\right), \chi_{\widetilde{S}_k} - \chi_{S_k^*}\right\rangle\right], \quad t \in T^+\left(S^*\right) \cup T^-\left(S^*\right). \tag{2.16}$$

Thus, (2.15) yields

$$\frac{\zeta_j^*}{r}\left(e^{\frac{r}{\zeta_j^*}\left(\zeta_j^*Q_j(\widetilde{S}) - \zeta_j^*Q_j(S^*)\right)} - 1\right) \geqq \zeta_j^*\beta_j\left(\widetilde{S}, S^*\right)\sum_{k=1}^{n}\left\langle D_kQ_j\left(S^*\right), \chi_{\widetilde{S}_k} - \chi_{S_k^*}\right\rangle, \quad j \in J\left(S^*\right). \tag{2.17}$$

Using the optimality conditions (2.7) and (2.8) together with the feasibility of $\widetilde{S}$ in (MP), we obtain

$$\zeta_j^*\beta_j\left(\widetilde{S}, S^*\right)\sum_{k=1}^{n}\left\langle D_kQ_j\left(S^*\right), \chi_{\widetilde{S}_k} - \chi_{S_k^*}\right\rangle \leqq 0, \ j \in J\left(S^*\right). \tag{2.18}$$

Since $\beta_j\left(\widetilde{S}, S^*\right) > 0$, $j \in J\left(S^*\right)$, (2.18) gives

$$\zeta_j^*\sum_{k=1}^{n}\left\langle D_kQ_j\left(S^*\right), \chi_{\widetilde{S}_k} - \chi_{S_k^*}\right\rangle \leqq 0, \ j \in J\left(S^*\right).$$

Then, adding both sides of the inequalities above and taking into account $\zeta_j^* = 0$, $j \notin J(S^*)$, we get

$$\sum_{j=1}^{m} \zeta_j^* \sum_{k=1}^{n} \left\langle D_k Q_j (S^*), \chi_{\widetilde{S}_k} - \chi_{S_k^*} \right\rangle \leqq 0. \tag{2.19}$$

Thus, by the feasibility of $\widetilde{S}$ and S^* in (MP), (2.16) implies

$$\vartheta_t^* \gamma_t \left(\widetilde{S}, S^*\right) \sum_{k=1}^{n} \left\langle D_k H_t (S^*), \chi_{\widetilde{S}_k} - \chi_{S_k^*} \right\rangle \leq 0,\ t \in T^+(S^*) \cup T^-(S^*). \tag{2.20}$$

Since $\gamma_t \left(\widetilde{S}, S^*\right) > 0$, $t \in T^+(S^*) \cup T^-(S^*)$, (2.20) yields

$$\vartheta_t^* \sum_{k=1}^{n} \left\langle D_k H_t (S^*), \chi_{\widetilde{S}_k} - \chi_{S_k^*} \right\rangle \leqq 0,\ t \in T^+(S^*) \cup T^-(S^*).$$

Then, adding both sides of the inequalities above and taking into account $\vartheta_t^* = 0$, $t \notin T^+(S^*) \cup T^-(S^*)$, we get

$$\sum_{t=1}^{w} \vartheta_t^* \sum_{k=1}^{n} \left\langle D_k H_t (S^*), \chi_{\widetilde{S}_k} - \chi_{S_k^*} \right\rangle \leqq 0. \tag{2.21}$$

Combining (2.14), (2.19) and (2.21), we obtain that the inequality

$$\sum_{i=1}^{p} \xi_i^* \sum_{k=1}^{n} \left\langle D_k F_i (S^*), \chi_{\widetilde{S}_k} - \chi_{S_k^*} \right\rangle + \sum_{j=1}^{m} \zeta_j^* \sum_{k=1}^{n} \left\langle D_k Q_j (S^*), \chi_{\widetilde{S}_k} - \chi_{S_k^*} \right\rangle +$$

$$\sum_{t=1}^{w} \vartheta_t^* \sum_{k=1}^{n} \left\langle D_k H_t (S^*), \chi_{\widetilde{S}_k} - \chi_{S_k^*} \right\rangle < 0 \tag{2.22}$$

holds, contradicting (2.6). This means that S^* is a weakly efficient solution of (MP) and completes the proof of this theorem under hypothesis A).

Now, we prove this theorem under hypothesis B).

We proceed by contradiction. Suppose, contrary to the result, that there exists other $\widetilde{S} \in \Omega$ such that

$$F\left(\widetilde{S}\right) < F(S^*). \tag{2.23}$$

Since $\xi^* = \left(\xi_1^*, ..., \xi_p^*\right) \geq 0$, (2.23) yields

$$\xi_i^* F_i \left(\widetilde{S}\right) \leqq \xi_i^* F_i(S^*),\ i \in I, \tag{2.24}$$

$$\xi_i^* F_i \left(\widetilde{S}\right) < \xi_i^* F_i(S^*) \text{ for at least one } i \in I. \tag{2.25}$$

By assumption, $\xi_i^* F_i$, $i \in I$, is an α_i-r-pseudo-convex function at S^* on Ω. Then, by Definition 2.6, there exist functions $\alpha_i : \Omega \times \Omega \to R$, $i = 1, ..., p$, such that $\alpha_i\left(\widetilde{S}, S^*\right) > 0$. Thus, (2.24) and (2.25) imply

$$\frac{1}{r}\sum_{i=1}^{p}\alpha_i\left(\widetilde{S}, S^*\right)e^{r\xi_i^* F_i(\widetilde{S})} < \frac{1}{r}\sum_{i=1}^{p}\alpha_i\left(\widetilde{S}, S^*\right)e^{r\xi_i^* F_i(S^*)}. \tag{2.26}$$

Hence, by Definition 2.6, (2.26) gives

$$\sum_{i=1}^{p}\xi_i^*\left\langle\sum_{k=1}^{n}D_kF_i\left(S^*\right), \chi_{\widetilde{S}_k} - \chi_{S_k^*}\right\rangle < 0. \tag{2.27}$$

By assumption, $\zeta_j^* Q_j$, $j \in J\left(S^*\right)$, is β_j-r-quasi-convex function at S^* on Ω. Hence, by Definition 2.7, there exist functions $\beta_j : \Omega\times\Omega \to R$, $j \in J\left(S^*\right)$, such that $\beta_j\left(\widetilde{S}, S^*\right) > 0$. Thus, taking into account also $\zeta_j^* \notin J\left(S^*\right)$, by $\widetilde{S} \in \Omega$, (2.7) and (2.8) yield

$$\frac{1}{r}\sum_{j=1}^{m}\beta_j\left(\widetilde{S}, S^*\right)e^{r\zeta_j^* Q_j(\widetilde{S})} \leqq \frac{1}{r}\sum_{j=1}^{m}\beta_j\left(\widetilde{S}, S^*\right)e^{r\zeta_j^* Q_j(S^*)}. \tag{2.28}$$

Hence, by Definition 2.7, (2.28) implies

$$\sum_{j=1}^{m}\zeta_j^*\sum_{k=1}^{n}\left\langle D_kQ_j\left(S^*\right), \chi_{\widetilde{S}_k} - \chi_{S_k^*}\right\rangle \leqq 0. \tag{2.29}$$

By assumption,$\vartheta_t^* H_t$, $t \in T$, is γ_t-r-quasi-convex function at S^* on Ω. Hence, by Definition 2.7, there exist functions $\gamma_t : \Omega\times\Omega \to R$, $i = 1, ..., p$, such that $\gamma_t\left(\widetilde{S}, S^*\right) > 0$. Thus, by $\widetilde{S} \in \Omega$, it follows that

$$\frac{1}{r}\sum_{t=1}^{w}\gamma_t\left(\widetilde{S}, S^*\right)e^{r\vartheta_t^* H_t(\widetilde{S})} \leqq \frac{1}{r}\sum_{t=1}^{w}\gamma_t\left(\widetilde{S}, S^*\right)e^{r\vartheta_t^* H_t(S^*)}. \tag{2.30}$$

Hence, by Definition 2.7, (2.28) implies

$$\sum_{t=1}^{w}\vartheta_t^*\sum_{k=1}^{n}\left\langle D_kH_t\left(S^*\right), \chi_{\widetilde{S}_k} - \chi_{S_k^*}\right\rangle \leqq 0. \tag{2.31}$$

Combining (2.27), (2.29) and (2.31), we get that the inequality (2.22) holds, contradicting (2.6). This means that S^* is a weakly efficient solution of (MP) and completes the proof of this theorem under hypothesis B). □

Theorem 2.13. *Let S^* be a feasible solution of the considered multiobjective programming problem (MP). Further, assume that there exist $\xi^* \in R^p$, $\zeta^* \in R^m$ and $\vartheta^* \in R^w$ such that the conditions (2.6)-(2.8) be satisfied at S^*. If one of the set of the following hypotheses*

a) *each objective function F_i, $i \in I$, is strictly α_i-r-convex at S^* on Ω, each inequality constraint Q_j, $j \in J\left(S^*\right)$,is β_j-r-convex at S^* on Ω, each equality constraint H_t, $t \in T^+\left(S^*\right) = \{t \in T : \vartheta_t^* > 0\}$ is γ_t-r-convex at S^* on Ω, each equality constraint H_t, $t \in T^-\left(S^*\right) = \{t \in T : \vartheta_t^* < 0\}$ is γ_t-r-concave at S^* on Ω,*

b) *$\xi_i^* F_i$, $i \in I$, is a strictly α_i-r-pseudo-convex function at S^* on Ω, $\zeta_j^* Q_j$, $j \in J$, is β_j-r-quasi-convex at S^* on Ω, $\vartheta_t^* H_t$, $t \in T$, is γ_t-r-quasi-convex at S^* on Ω*

is fulfilled, then S^* is an efficient solution of (MP).

Proof. The proof of this theorem follows on the lines of Theorem 2.12. □

Theorem 2.14. *Let all hypotheses of Theorem 2.13 be fulfilled. If $\xi^* > 0$, then S^* is a properly efficient solution of (MP).*

Proof. Since all the hypotheses of Theorem 2.13 are fulfilled, therefore, S^* is an efficient solution of (MP).

Now, we prove that S^* is a properly efficient solution of (MP). Suppose, contrary to the result, that S^* is not a properly efficient solution of (MP). If we assume that $p \geqq 2$, then we choose

$$M = (p-1) \max_{i,q \in I,\ i \neq q} \frac{\xi_q^*}{\xi_i^*}. \tag{2.32}$$

Then, there exist other $\widetilde{S} \in \Omega$ and $i \in I$, such that $F_i(S^*) > F_i\left(\widetilde{S}\right)$ and, moreover,

$$\frac{F_i(S^*) - F_i\left(\widetilde{S}\right)}{F_q\left(\widetilde{S}\right) - F_q(S^*)} > M \tag{2.33}$$

for each $q \neq i$ such that $F_q\left(\widetilde{S}\right) > F_q(S^*)$. Thus, for each $q \neq i$, (2.32) and (2.33) yield

$$F_i(S^*) - F_i\left(\widetilde{S}\right) > (p-1) \max_{i,q \in I, i \neq q} \frac{\xi_q^*}{\xi_i^*} \left(F_q\left(\widetilde{S}\right) - F_q(S^*)\right). \tag{2.34}$$

Since $F_q\left(\widetilde{S}\right) - F_q(S^*) > 0$, (2.34) implies

$$(p-1) \max_{i,q \in P, i \neq q} \frac{\xi_q^*}{\xi_i^*} \left(F_q\left(\widetilde{S}\right) - F_q(S^*)\right) \geqq \tag{2.35}$$
$$(p-1) \frac{\xi_q^*}{\xi_i^*} \left(F_q\left(\widetilde{S}\right) - F_q(S^*)\right), \forall i \in I \backslash \{q\}.$$

By (2.34) and (2.35), it follows that

$$\xi_i^* \left(F_i(S^*) - F_i\left(\widetilde{S}\right)\right) > (p-1)\, \xi_q^* \left(F_q\left(\widetilde{S}\right) - F_q(S^*)\right).$$

Summing over $q \neq i$ both sides of the inequalities above, we get

$$(p-1)\, \xi_i^* \left(F_i(S^*) - F_i\left(\widetilde{S}\right)\right) > (p-1) \sum_{q \neq i} \xi_q^* \left(F_q\left(\widetilde{S}\right) - F_q(S^*)\right). \tag{2.36}$$

Hence, by (2.36), we conclude that the inequality

$$\xi_i^* F_i(S^*) + \sum_{q \neq i} \xi_q^* F_q(S^*) > \xi_i^* F_i\left(\widetilde{S}\right) dt + \sum_{q \neq i} \xi_q^* F_q\left(\widetilde{S}\right)$$

holds. This is a contradiction to the assumption that S^* is an efficient solution of (MP). Hence, S^* is a properly efficient solution of (MP) and the proof of this theorem is completed. □

3. Saddle-point criteria

In this section, for the considered multicriteria optimization problem (MP) with n-set functions, we prove vector saddle point criteria. First, we define the vector-valued Lagrangian-type function L for the considered vector optimization problem (MP) with n-set functions and then we introduce the definition of its saddle point.

Definition 3.1. For the considered multiobjective programming problem (MP), the Lagrange-type function $L(\cdot,\zeta^*,\vartheta^*) : \mathbb{A}^n \to R^p$ is defined, for fixed ζ^*, ϑ^*, as follows:

$$L(S,\zeta^*,\vartheta^*) = F(S) + \left[\sum_{j=1}^{m} \zeta_j^* Q_j(S) + \sum_{t=1}^{w} \vartheta_t^* H_t(S)\right] e,$$

where $e = [1, ..., 1]^T \in R^p$.

Definition 3.2. A point $(S^*,\zeta^*,\vartheta^*) \in \Omega \times R_+^m \times R^w$ is said to be a saddle point of the vector-valued Lagrange function L defined for the multiobjective programming problem (MP) if,

i): $L(S^*,\zeta,\vartheta) \leqq L(S^*,\zeta^*,\vartheta^*) \quad \forall \zeta \in R_+^m,\ \vartheta \in R^w$,
ii): $L(S^*,\zeta^*,\vartheta^*) \nleq L(S,\zeta^*,\vartheta^*) \quad \forall S \in \Omega$.

Theorem 3.3. *Let $(S^*,\zeta^*,\vartheta^*) \in \Omega \times R_+^m \times R^w$ be a saddle point of the vector-valued Lagrangian-type function defined for the considered multicriteria optimization problem (MP) with n-set functions. Then S^* is a weakly efficient solution of (MP).*

Proof. By assumption, $(S^*,\zeta^*,\vartheta^*) \in \Omega \times R_+^m \times R^w$ is a saddle point of the partial Lagrange function defined for the considered multiobjective programming problem (MP). By Definition 3.2 i), we have that, for each $\zeta \in R_+^m$ and $\vartheta \in R^w$, the inequality

$$L(S^*,\zeta,\vartheta) \leqq L(S^*,\zeta^*,\vartheta^*)$$

holds. Then, by a definition of the Lagrange function L, it follows that

$$F(S^*) + \left[\sum_{j=1}^{m} \zeta_j Q_j(S^*) + \sum_{t=1}^{w} \vartheta_t H_t(S^*)\right] e \leqq$$

$$F(S^*) + \left[\sum_{j=1}^{m} \zeta_j^* Q_j(S^*) + \sum_{t=1}^{w} \vartheta_t^* H_t(S^*)\right] e.$$

In the above inequality, let $\xi = 0$ and $\zeta = 0$. Then,

$$\sum_{i=1}^{m} \zeta_i^* Q_i \left(S^*\right) + \sum_{t=1}^{w} \vartheta_t^* H_t \left(S^*\right) \geqq 0. \tag{3.1}$$

Using the feasibility of S^* in the problem (MP) again together with $\zeta^* \in R_+^m$, we get

$$\sum_{j=1}^{m} \zeta_j^* Q_j \left(S^*\right) + \sum_{t=1}^{w} \vartheta_t^* H_t \left(S^*\right) \leqq 0. \tag{3.2}$$

Combining (3.1) and (3.2), we obtain

$$\sum_{j=1}^{m} \zeta_j^* Q_j \left(S^*\right) + \sum_{t=1}^{w} \vartheta_t^* H_t \left(S^*\right) = 0. \tag{3.3}$$

By means of contradiction, suppose, contrary to the result, that S^* is not a weakly efficient solution of (MP). This means that there exists $\widetilde{S} \in \Omega$ such that

$$F\left(\widetilde{S}\right) < F\left(S^*\right).$$

Thus, the above inequality yields

$$F_i\left(\widetilde{S}\right) < F_i(S^*),\ i = 1, ..., p. \tag{3.4}$$

Using the feasibility of $\widetilde{S}$ in (MP) together with $\zeta^* \geqq 0$, we get

$$\sum_{j=1}^{m} \zeta_j^* Q_j \left(\widetilde{S}\right) + \sum_{t=1}^{w} \vartheta_t^* H_t \left(\widetilde{S}\right) \leqq 0. \tag{3.5}$$

Combining (3.3), (3.4) and (3.5), we obtain

$$F_i\left(\widetilde{S}\right) + \sum_{j=1}^{m} \xi_j^* Q_j \left(\widetilde{S}\right) + \sum_{t=1}^{w} \zeta_t^* H_t \left(\widetilde{S}\right) < \tag{3.6}$$

$$F_i(S^*) + \sum_{j=1}^{m} \xi_j^* Q_j \left(S^*\right) + \sum_{t=1}^{w} \zeta_t^* H_t \left(S^*\right),\ i = 1, ..., p.$$

By the definition of the vector-valued Lagrangian-type function, (3.6) implies that the inequality

$$L\left(\widetilde{S}, \zeta^*, \vartheta^*\right) < L\left(S^*, \zeta^*, \vartheta^*\right)$$

holds, contradicting the inequality ii) in Definition 3.2. This means that S^* is a weakly efficient solution of the considered multiobjective programming problem (MP). □

Remark 3.4. Note that we have established the necessary optimality condition in Theorem 3.3 for a saddle point of the vector-valued Lagrangian-type function L without any V-r-convexity assumptions imposed on the functions constituting (MP).

Theorem 3.5. *Let S^* be a feasible point in (MP) and there exist Lagrange multipliers $\xi^* \in R^p$, $\zeta^* \in R^m$ and $\vartheta^* \in R^w$ such that the optimality conditions (2.6)-(2.8) are satisfied at S^*. Further, we assume that at least one of the hypotheses are fulfilled:*

A) *the function $F(\cdot) + \left[\sum_{j=1}^{m} \zeta_j^* Q_j(\cdot) + \sum_{t=1}^{w} \vartheta_t^* H_t(\cdot)\right] e$ is strictly V-r-convex at S^* on Ω with $\alpha(S, S^*) = (\alpha_1(S, S^*), ..., \alpha_p(S, S^*)) > 0$ for any $S \in \Omega$.*

B) *each function $\xi_i^* \left[F_i(\cdot) + \sum_{j=1}^{m} \zeta_j Q_j(\cdot) + \sum_{t=1}^{w} \vartheta_t H_t(\cdot)\right]$, $i \in \{1, ..., p : \xi_i^* > 0\}$, is strictly α_i-r-pseudo-convex at S^* on Ω with $\alpha_i(S, S^*) > 0$.*

Then $(S^*, \zeta^*, \vartheta^*) \in \Omega \times R_+^m \times R^w$ is a saddle point of the vector-valued Lagrangian-type function defined for the considered multiobjective programming problem (MP).

Proof. Let S^* be a feasible solution in (MP) and let the optimality conditions (2.6)-(2.8) be fulfilled at S^* with Lagrange multipliers $\xi^* \in R^p$, $\zeta^* \in R^m$ and $\vartheta^* \in R^w$.

Firstly, we prove the inequality i) in Definition 3.2.
Using the feasibility of S^* in (MP) together with the optimality conditions (2.7) and (2.8), we get that the inequality

$$\sum_{i=1}^{m} \zeta_j Q_j(S^*) \leqq \sum_{j=1}^{m} \zeta_j^* Q_j(S^*) \tag{3.7}$$

holds for each $\zeta \in R^m$, $\zeta \geqq 0$. Using the feasibility of S^* in the problem (MP) again, we obtain that the relation

$$\sum_{t=1}^{w} \vartheta_t H_t(S^*) = \sum_{t=1}^{w} \vartheta_t^* H_t(S^*) \tag{3.8}$$

holds for each $\vartheta \in R^w$. Combining (3.7) and (3.8), we get that the inequalities

$$F_i(S^*) + \sum_{j=1}^{m} \zeta_j Q_j(S^*) + \sum_{t=1}^{w} \vartheta_t H_t(S^*) \leqq$$

$$F_i(S^*) + \sum_{j=1}^{m} \zeta_j^* Q_j(S^*) + \sum_{t=1}^{w} \vartheta_t^* H_t(S^*),\ i = 1, ..., p$$

hold for any $\zeta \in R^m$, $\zeta \geqq 0$, and any $\vartheta \in R^w$. Hence, by the definition of the Lagrange function, the inequality above imply that the inequality

$$L(S^*, \zeta, \vartheta) \leqq L(S^*, \zeta^*, \vartheta^*) \tag{3.9}$$

holds for any $\zeta \in R^m$, $\zeta \geqq 0$, and any $\vartheta \in R^w$.

We proceed by contradiction in order to prove the second relation in Definition 3.2. Suppose, contrary to the result, that the inequality ii) in Definition

3.2 is not satisfied. This means that there exists other $\widetilde{S} \in \Omega$ such that

$$L\left(\widetilde{S}, \zeta^*, \vartheta^*\right) \leq L\left(S^*, \zeta^*, \vartheta^*\right). \tag{3.10}$$

By the definition of L, (3.10) implies that the inequalities

$$F_i\left(\widetilde{S}\right) + \sum_{j=1}^{m} \zeta_j^* Q_j\left(\widetilde{S}\right) + \sum_{t=1}^{w} \vartheta_t^* H_t\left(\widetilde{S}\right) \leqq \tag{3.11}$$

$$F_i(S^*) + \sum_{j=1}^{m} \zeta_j^* Q_j\left(S^*\right) + \sum_{t=1}^{w} \vartheta_t^* H_t\left(S^*\right),\ i = 1, ..., p,$$

$$F_i\left(\widetilde{S}\right) + \sum_{j=1}^{m} \zeta_j^* Q_j\left(\widetilde{S}\right) + \sum_{t=1}^{w} \vartheta_t^* H_t\left(\widetilde{S}\right) < \tag{3.12}$$

$$F_i(S^*) + \sum_{j=1}^{m} \zeta_j^* Q_j\left(S^*\right) + \sum_{t=1}^{w} \vartheta_t^* H_t\left(S^*\right) \text{ for at least one } i \in \{1, ..., p\}$$

hold.

Now, we prove the inequality ii) in Definition 3.2 under hypothesis A).

By assumption, the function $F(\cdot) + \left[\sum_{j=1}^{m} \zeta_j^* Q_j(\cdot) + \sum_{t=1}^{w} \vartheta_t^* H_t(\cdot)\right] e$ is strictly V-r-convex at S^* on Ω with $\alpha(S, S^*) = (\alpha_1(S, S^*), ..., \alpha_p(S, S^*)) > 0$ for any $S \in \Omega$. Thus, by Definition 2.4, (3.11) and (3.12) yield, respectively,

$$\alpha_i\left(\widetilde{S}, S^*\right) \sum_{k=1}^{n} \left\langle D_k F_i\left(S^*\right) + \sum_{j=1}^{m} \zeta_j^* D_k Q_j\left(S^*\right) + \right. \tag{3.13}$$

$$\left. + \sum_{t=1}^{w} \vartheta_t^* D_k H_t\left(S^*\right), \chi_{\widetilde{S}_k} - \chi_{S_k^*} \right\rangle \leqq 0,\ i = 1, ..., p,$$

$$\alpha_i\left(\widetilde{S}, S^*\right) \sum_{k=1}^{n} \left\langle D_k F_i\left(S^*\right) + \sum_{j=1}^{m} \zeta_j^* D_k Q_j\left(S^*\right) + \right. \tag{3.14}$$

$$\left. + \sum_{t=1}^{w} \vartheta_t^* D_k H_t\left(S^*\right), \chi_{\widetilde{S}_k} - \chi_{S_k^*} \right\rangle < 0 \text{ for at least one } i \in \{1, ..., p\}.$$

By $\alpha_i\left(\widetilde{S}, S^*\right) > 0$, $i = 1, ..., p$, (3.13) and (3.14) give, respectively,

$$\sum_{k=1}^{n} \left\langle D_k F_i\left(S^*\right) + \sum_{j=1}^{m} \zeta_j^* D_k Q_j\left(S^*\right) + \right. \tag{3.15}$$

$$\left. + \sum_{t=1}^{w} \vartheta_t^* D_k H_t\left(S^*\right), \chi_{\widetilde{S}_k} - \chi_{S_k^*} \right\rangle \leqq 0,\ i = 1, ..., p,$$

$$\sum_{k=1}^{n} \left\langle D_k F_i\left(S^*\right) + \sum_{j=1}^{m} \zeta_j^* D_k Q_j\left(S^*\right) + \right. \tag{3.16}$$

$$+\sum_{t=1}^{w}\vartheta_t^* D_k H_t\left(S^*\right), \chi_{\widetilde{S}_k}-\chi_{S_k^*}\Big\rangle<0 \text{ for at least one } i\in\{1,...,p\}.$$

Since $\xi^*\geq 0$, $\sum_{i=1}^{p}\xi_i^*=1$, (3.15) and (3.16) imply that the inequality

$$\sum_{k=1}^{n}\left\langle D_k\left[\sum_{i=1}^{p}\xi_i^* F_i\left(S^*\right)+\sum_{j=1}^{m}\zeta_j^* Q_j\left(S^*\right)+\sum_{t=1}^{w}\vartheta_t^* H_t\left(S^*\right)\right], \chi_{\widetilde{S}_k}-\chi_{S_k^*}\right\rangle<0$$

holds, contradicting (2.6).

Now, we prove the inequality ii) in Definition 3.2 under hypothesis B)

By assumption, each function $\xi_i^*\left[F_i\left(\cdot\right)+\sum_{j=1}^{m}\zeta_j^* Q_j\left(\cdot\right)+\sum_{t=1}^{w}\vartheta_t^* H_t\left(\cdot\right)\right]$, $i\in\{1,...,p:\xi_i^*>0\}$, is strictly α_i-r-pseudo-convex at S^* on Ω, where $\alpha_i\left(S,S^*\right)>0$ for any $S\in\Omega$. Thus, (3.11) and (3.12) yield

$$\frac{1}{r}\sum_{i=1}^{p}\alpha_i\left(\widetilde{S},S^*\right)e^{r\xi_i^*\left[F_i\left(\widetilde{S}\right)+\sum_{j=1}^{m}\zeta_j^* Q_j\left(\widetilde{S}\right)+\sum_{t=1}^{w}\vartheta_t^* H_t\left(\widetilde{S}\right)\right]}< \tag{3.17}$$

$$\frac{1}{r}\sum_{i=1}^{p}\alpha_i\left(\widetilde{S},S^*\right)e^{r\xi_i^*\left[F_i\left(S^*\right)+\sum_{j=1}^{m}\zeta_j^* Q_j\left(S^*\right)+\sum_{t=1}^{w}\vartheta_t^* H_t\left(S^*\right)\right]}.$$

Hence, by Definition 2.6, (3.17) implies that

$$\sum_{i=1}^{p}\xi_i^*\sum_{k=1}^{n}\left\langle D_k\left[\sum_{i=1}^{p}F_i\left(S^*\right)+\sum_{j=1}^{m}\zeta_j^* Q_j\left(S^*\right)+\sum_{t=1}^{w}\vartheta_t^* H_t\left(S^*\right)\right], \chi_{\widetilde{S}_k}-\chi_{S_k^*}\right\rangle<0. \tag{3.18}$$

Since $\sum_{i=1}^{p}\xi_i^*=1$, (3.18) gives that the inequality

$$\sum_{k=1}^{n}\left\langle D_k\left[\sum_{i=1}^{p}\xi_i^* F_i\left(S^*\right)+\sum_{j=1}^{m}\zeta_j^* Q_j\left(S^*\right)+\sum_{t=1}^{w}\vartheta_t^* H_t\left(S^*\right)\right], \chi_{\widetilde{S}_k}-\chi_{S_k^*}\right\rangle<0$$

holds, contradicting (2.6). Then, the proof of this theorem is completed. □

Remark 3.6. If $\xi_i^*>0$, $i=1,...,p$, then the hypotheses A) and B) can be weakened. Namely, it is sufficient to assume that, in place of hypotheses A) and B), one of the following hypotheses a) and b) is fulfilled:

a) the function $F\left(\cdot\right)+\left[\sum_{j=1}^{m}\zeta_j^* Q_j\left(\cdot\right)+\sum_{t=1}^{w}\vartheta_t^* H_t\left(\cdot\right)\right]e$ is V-r-convex at S^* on Ω with $\alpha\left(S,S^*\right)=\left(\alpha_1\left(S,S^*\right),...,\alpha_p\left(S,S^*\right)\right)>0$ for any $S\in\Omega$,

b) each function $\xi_i^*\left[F_i\left(\cdot\right)+\sum_{j=1}^{m}\zeta_j Q_j\left(\cdot\right)+\sum_{t=1}^{w}\vartheta_t H_t\left(\cdot\right)\right]$, $i=1,...,p$, is α_i-r-pseudo-convex at S^* on Ω with $\alpha_i\left(S,S^*\right)>0$.

4. Mixed duality

In this section, for the considered differentiable multiobjective programming subset problem (MP), we formulate its mixed dual problem. We prove several families of duality results under both V-r-convexity and/or generalized V-r-convexity hypotheses imposed on certain combinations of the functions constituting (MP). This is accomplished by employing a certain partitioning scheme which was originally proposed Mond and Weir [18] for the purpose of constructing generalized dual problems for nonlinear scalar optimization problems. To do this, we need some additional notations.

Let $\{J_0, J_1, ..., J_\tau\}$ and $\{T_0, T_1, ..., T_\gamma\}$ be partitions of the indexes sets J and T, respectively. Thus, $J_q \subseteq J$ for each $q = 0, 1, \ldots, \tau$, $J_{q_1} \cap J_{q_2} = \varnothing$ for each q_1, $q_2 = 0, 1, \ldots, \tau$ with $q_1 \neq q_2$, and $\bigcup_{i=0}^{\tau} J_i = J$. Obviously, similar properties hold for $\{T_0, T_1, ..., T_\tau\}$. Moreover, if q_1, q_2 are the numbers of partitioning sets of J and T, respectively, then $\Omega = max\{q_1, q_2\}$ and $J_q = \varnothing$ or $T_q = \varnothing$ for $q > min\{q_1, q_2\}$.

Now, we state, for the considered multiobjective programming subset problem (MP), the following vector mixed dual problems is defined as follows:

$$V\text{-maximize } \psi\left(Z, \zeta, \vartheta\right) = F(Z) + \left[\sum_{j \in J_0} \zeta_j Q_j\left(Z\right) + \sum_{t \in T_0} \vartheta_t H_t\left(Z\right)\right] e$$

$$\text{s.t. } \left\langle \sum_{i=1}^{p} \xi_i \sum_{k=1}^{n} D_k F_i\left(Z\right) + \sum_{j=1}^{m} \zeta_j \sum_{k=1}^{n} D_k Q_j\left(Z\right) + \sum_{t=1}^{w} \vartheta_t \sum_{k=1}^{n} D_k H_t\left(Z\right), \chi_{S_k} - \chi_{Z_k} \right\rangle \geqq 0,\ \forall S \in \mathbb{A}^n, \tag{4.1}$$

$$\sum_{j \in J_r} \zeta_j Q_j\left(Z\right) + \sum_{t \in T_r} \vartheta_t H_t\left(Z\right) \geqq 0,\ r = 1, ..., \tau,\ \text{(MD)} \tag{4.2}$$

$$Z \in A^n,\ \xi \geq 0,\ \sum_{i=1}^{p} \xi_i = 1,\ \zeta \geqq 0, \tag{4.3}$$

where $e = [1, ..., 1] \in R^p$. We denote by Δ the set of all feasible solutions in the vector mixed dual subset problem (MD), that is,
$\Delta = \{(Z, \xi, \zeta, \vartheta) \in \mathbb{A}^n \times R^p \times R^m \times R^w : (Z, \xi, \zeta, \vartheta)$ satisfying the constraints (4.1)-(4.3)$\}$. Further, denote $\Gamma = \{Z \in \mathbb{A}^n : (Z, \xi, \zeta, \vartheta) \in \Delta\}$.

Remark 4.1. In view of the vector Wolfe type dual, one can see that its objective function contains all constraint functions of the original multiobjective programming problem, while the vector Mond-Weir type dual contains no constraint function of the original multiobjective programming problem in its objective function. In this section, we formulate a vector mixed-type dual (MD) like an incomplete Lagrangian dual in its objective function which will involve the Wolfe type dual and Mond-Weir type dual as the special cases. In fact, we see that the Wolfe and Mond-Weir duals follows as special cases of mixed duality. Namely, in the case when $J_0 = \varnothing$ and $T_0 = \varnothing$, the vector mixed dual problem (MD) reduces to the vector dual problem in the sense of Mond-Weir. Whereas in the case when $J_0 = J$ and $T_0 = T$, we obtain the definition of the vector dual problem in the sense of Wolfe.

Theorem 4.2. *(Weak duality): Let S and $(Z, \xi, \zeta, \vartheta)$ be feasible solutions for the vector optimization problems (MP) and (MD), respectively. Further, we assume that one of the hypotheses are fulfilled:*

A) *the function $F(\cdot) + \left[\sum_{j=1}^{m} \zeta_j Q_j(\cdot) + \sum_{t=1}^{w} \vartheta_t H_t(\cdot)\right] e$ is V-r-convex at Z on $\Omega \cup \Gamma$ with $\alpha(S, Z) = (\alpha_1(S, Z), ..., \alpha_p(S, Z)) > 0$.*

B) *each function $\xi_i \left[F_i(\cdot) + \sum_{j=1}^{m} \zeta_j Q_j(\cdot) + \sum_{t=1}^{w} \vartheta_t H_t(\cdot)\right]$, $i \in \{1, ..., p : \xi_i > 0\}$, is α_i-r-pseudo-convex at Z on $\Omega \cup \Gamma$ with $\alpha_i(S, Z) > 0$.*

Then $F(S) \not< \psi(Z, \zeta, \vartheta)$.

Proof. We proceed by contradiction. Suppose, contrary to the result, that there exist $S \in \Omega$ and $(Z, \xi, \zeta, \vartheta) \in \Delta$ such that

$$F(S) < \psi(Z, \zeta, \vartheta).$$

Thus,

$$F(S) < F(Z) + \left[\sum_{j \in J_0} \zeta_j Q_j(Z) + \sum_{t \in T_0} \vartheta_t H_t(Z)\right] e. \tag{4.4}$$

Since $S \in \Omega$ and $\zeta \geqq 0$, (4.4) implies

$$\begin{gathered} F_i(S) + \textstyle\sum_{j \in J_0} \zeta_j Q_j(S) + \sum_{t \in T_0} \vartheta_t H_t(S) < \\ F_i(Z) + \textstyle\sum_{j \in J_0} \zeta_j Q_j(Z) + \sum_{t \in T_0} \vartheta_t H_t(Z),\ i = 1, ..., p. \end{gathered} \tag{4.5}$$

By $S \in \Omega$ and $(Z, \xi, \zeta, \vartheta) \in \Delta$, it follows that

$$\sum_{j \in J_r} \zeta_j Q_j(S) + \sum_{t \in T_r} \vartheta_t H_t(S) \leqq \left(\sum_{j \in J_r} \zeta_j Q_j(Z) + \sum_{t \in T_r} \vartheta_t H_t(Z)\right),\ r = 1, ..., \tau. \tag{4.6}$$

Hence, (4.5) and (4.6) yield

$$\begin{gathered} F_i(S) + \textstyle\sum_{j=1}^{m} \zeta_j Q_j(S) + \sum_{t=1}^{w} \vartheta_t H_t(S) < \\ F_i(Z) + \textstyle\sum_{j=1}^{m} \zeta_j Q_j(Z) + \sum_{t=1}^{w} \vartheta_t H_t(Z),\ i = 1, ..., p. \end{gathered} \tag{4.7}$$

Proof under hypothesis A).

By assumption, $F(\cdot) + \left[\sum_{j=1}^{m} \zeta_j Q_j(\cdot) + \sum_{t=1}^{w} \vartheta_t H_t(\cdot)\right] e$ is V-r-convex at Z on $\Omega \cup \Gamma$. Hence, by Definition 2.4, each its component $F_i(\cdot) + \sum_{j=1}^{m} \zeta_j Q_j(\cdot) +$

$\sum_{t=1}^{w} \vartheta_t H_t(\cdot)$ is α_i-r-convex at Z on $\Omega \cup \Gamma$. Then, by Definition 2.4, for any $i = 1, ..., p$,

$$\frac{1}{r} e^{r\left[F_i(S)+\sum_{j=1}^{m} \zeta_j Q_j(S)+\sum_{t=1}^{w} \vartheta_t H_t(S)\right]} \geqq$$

$$\frac{1}{r} e^{r\left[F_i(Z)+\sum_{j=1}^{m} \zeta_j Q_j(Z)+\sum_{t=1}^{w} \vartheta_t H_t(Z)\right]} [1+$$

$$r\alpha_i(S,Z) \sum_{k=1}^{n} \left\langle D_k \left[F_i(Z) + \sum_{j=1}^{m} \zeta_j Q_j(Z) + \sum_{t=1}^{w} \vartheta_t H_t(Z)\right], \chi_{S_k} - \chi_{Z_k} \right\rangle \Bigg].$$

Thus, the above inequality can be re-written as follows

$$\frac{1}{r}\left(e^{r\left(\left[F_i(S)+\sum_{j=1}^{m} \zeta_j Q_j(S)+\sum_{t=1}^{w} \vartheta_t H_t(S)\right]-\left[F_i(Z)+\sum_{j=1}^{m} \zeta_j Q_j(Z)+\sum_{t=1}^{w} \vartheta_t H_t(Z)\right]\right)} - 1\right) \tag{4.8}$$

$$\geqq \alpha_i(S,Z) \sum_{k=1}^{n} \left\langle D_k \left[F_i(Z) + \sum_{j=1}^{m} \zeta_j Q_j(Z) + \sum_{t=1}^{w} \vartheta_t H_t(Z)\right], \chi_{S_k} - \chi_{Z_k} \right\rangle.$$

Combining (4.7) and (4.8), we get

$$\alpha_i(S,Z) \sum_{k=1}^{n} \left\langle D_k \left[F_i(Z) + \sum_{j=1}^{m} \zeta_j Q_j(Z) + \sum_{t=1}^{w} \vartheta_t H_t(Z)\right], \chi_{S_k} - \chi_{Z_k} \right\rangle < 0,\ i \in I. \tag{4.9}$$

Since $\alpha_i(S,Z) > 0$, $i = 1, ..., p$, (4.9) gives

$$\sum_{k=1}^{n} \left\langle D_k \left[F_i(Z) + \sum_{j=1}^{m} \zeta_j Q_j(Z) + \sum_{t=1}^{w} \vartheta_t H_t(Z)\right], \chi_{S_k} - \chi_{Z_k} \right\rangle < 0,\ i \in I. \tag{4.10}$$

Multiplying each inequality (4.10) by the corresponding Lagrange multiplier ξ_i, by the constraint (4.3), we obtain that the inequality

$$\left\langle \sum_{i=1}^{p} \xi_i \sum_{k=1}^{n} D_k F_i(Z) + \sum_{j=1}^{m} \zeta_j \sum_{k=1}^{n} D_k Q_j(Z) + \sum_{t=1}^{w} \vartheta_t \sum_{k=1}^{n} D_k H_t(Z), \chi_{S_k} - \chi_{Z_k} \right\rangle < 0 \tag{4.11}$$

holds, contradicting the constraint (4.2).

Proof under hypothesis B).

By assumption, the function $\xi_i \left[F_i(\cdot) + \sum_{j \in J_0} \zeta_j Q_j(\cdot) + \sum_{t \in T_0} \vartheta_t H_t(\cdot)\right]$, $i \in \{1, ..., p : \xi_i > 0\}$, is α_i-r-pseudo-convex at Z on $\Omega \cup \Gamma$ with $\alpha_i(S,Z) > 0$. Hence, (4.5) gives

$$\begin{aligned} &\tfrac{1}{r} \textstyle\sum_{i=1}^{p} \alpha_i(S,Z)\, e^{r\xi_i\left[F_i(S)+\sum_{j=1}^{m} \zeta_j Q_j(S)+\sum_{t=1}^{w} \vartheta_t H_t(S)\right]} < \\ &\tfrac{1}{r} \textstyle\sum_{i=1}^{p} \alpha_i(S,Z)\, e^{r\xi_i\left[F_i(Z)+\sum_{j=1}^{m} \zeta_j Q_j(Z)+\sum_{t=1}^{w} \vartheta_t H_t(Z)\right]}. \end{aligned} \tag{4.12}$$

Thus, by Definition 2.6, (4.12) implies that the inequality

$$\sum_{i=1}^{p}\xi_i\sum_{k=1}^{n}\left\langle D_k\left[F_i\left(Z\right)+\sum_{j=1}^{m}\zeta_jQ_j\left(Z\right)+\sum_{t=1}^{w}\vartheta_tH_t\left(Z\right)\right],\chi_{S_k}-\chi_{Z_k}\right\rangle<0 \tag{4.13}$$

holds. By $(Z,\xi,\zeta,\vartheta)\in\Delta$, it follows that $\sum_{i=1}^{p}\xi_i=1$. Thus, (4.13) yields the inequality (4.11), which is a contradiction to the feasibility of (Z,ξ,ζ,ϑ) in (MD). Hence, the proof of this theorem is completed. □

If some stronger (generalized) V-r-convexity hypotheses are imposed on the functions, then the following result is true:

Theorem 4.3. *(Weak duality): Let S and (Z,ξ,ζ,ϑ) be any feasible solutions for the problems (MP) and (MD), respectively. Further, we assume that one of the hypotheses are fulfilled:*

A) *the function $F(\cdot)+\left[\sum_{j=1}^{m}\zeta_jQ_j(\cdot)+\sum_{t=1}^{w}\vartheta_tH_t(\cdot)\right]e$ is strictly V-r-convex at Z on $\Omega\cup\Gamma$ with $\alpha(S,Z)=(\alpha_1(S,Z),...,\alpha_p(S,Z))>0$.*

B) *each function $\xi_i\left[F_i(\cdot)+\sum_{j=1}^{m}\zeta_jQ_j(\cdot)+\sum_{t=1}^{w}\vartheta_tH_t(\cdot)\right]$, $i\in\{1,...,p:\xi_i>0\}$, is strictly α_i-r-pseudo-convex at Z on $\Omega\cup\Gamma$ with $\alpha(S,Z)=(\alpha_1(S,Z),...,\alpha_p(S,Z))>0$.*

Then $F(S)\nleq\psi(Z,\zeta,\vartheta)$.

Theorem 4.4. *(Direct duality). Let $S^*\in\Omega$ and the optimality conditions (2.6)-(2.8) be satisfied at S^* with Lagrange multipliers $\xi^*\in R^p$, $\zeta^*\in R^m$ and $\vartheta^*\in R^w$. If all the hypotheses of the weak duality theorem (Theorem 4.2 or Theorem 4.3) are satisfied, then $(S^*,\xi^*,\zeta^*,\vartheta^*)$ is a weak efficient solution (an efficient solution) of a maximum type in (MD). If $(S^*,\xi^*,\zeta^*,\vartheta^*)$ is an efficient solution of (MD) with $\xi^*>0$, then $(S^*,\xi^*,\zeta^*,\vartheta^*)$ is also a properly efficient solution in (MD).*

Proof. The feasibility of $(S^*,\xi^*,\zeta^*,\vartheta^*)$ in (MD) follows from the optimality conditions (2.6)-(2.8). Then, if all the hypotheses of weak duality (Theorem 4.2 or Theorem 4.3) are fulfilled, then $(S^*,\xi^*,\zeta^*,\vartheta^*)$ is a weakly efficient solution (or an efficient solution) of a maximum type in (MD), respectively.

Now, we assume that $\xi^*>0$. Then we shall prove that $(S^*,\xi^*,\zeta^*,\vartheta^*)$ is a properly efficient solution in (MD) by the method of contradiction. Suppose, contrary to the result, that $(S^*,\xi^*,\zeta^*,\vartheta^*)$ is not a properly efficient solution of (MD). Then, for some criterion i and $\left(\widetilde{Z},\widetilde{\xi},\widetilde{\zeta},\widetilde{\vartheta}\right)\in\Delta$, the following inequality

$$F_i\left(\widetilde{Z}\right)-F_i\left(S^*\right)>M\left[F_j\left(S^*\right)-F_j\left(\widetilde{Z}\right)\right] \tag{4.14}$$

holds for every scalar $M>0$ and for all $j\neq i$ such that $F_j\left(\widetilde{Z}\right)<F_j\left(S^*\right)$. Assume that $p\geqq 2$ and let $M=(p-1)\max\limits_{i,j}\frac{\xi_i^*}{\xi_j^*}$. Hence, (4.14) gives that the

inequality

$$F_i\left(\widetilde{Z}\right) - F_i\left(S^*\right) > (p-1)\frac{\xi_j^*}{\xi_i^*}\left[F_j\left(S^*\right) - F_j\left(\widetilde{Z}\right)\right] \tag{4.15}$$

holds for all $j \neq i$. Then, if we rewrite (4.15), then we get

$$\frac{\xi_i^*}{p-1}F_i\left(\widetilde{Z}\right) - F_i\left(S^*\right) > \xi_j^*\left[F_j\left(S^*\right) - F_j\left(\widetilde{Z}\right)\right].$$

Now, summing the above inequalities over $j \neq i$, we obtain

$$\xi_i^* F_i\left(\widetilde{Z}\right) - F_i\left(S^*\right) > \sum_{j\neq i}\xi_j^*\left[F_j\left(S^*\right) - F_j\left(\widetilde{Z}\right)\right].$$

Then, if we re-write the above inequality, we get that the inequality

$$\sum_{i=1}^{p}\xi_i^* F_i\left(\widetilde{Z}\right) > \sum_{i=1}^{p}\xi_j^* F_j\left(S^*\right)$$

holds, which is a contradiction to the assumption that $(S^*, \xi^*, \zeta^*, \vartheta^*)$ is an efficient solution of a maximum type in (MD). $\square$

Theorem 4.5. *(Strict duality): Let S^* and $(Z^*, \xi^*, \zeta^*, \vartheta^*)$ be feasible solutions in (MP) and (MD), respectively, such that*

$$F(S^*) = \psi\left(Z^*, \zeta^*, \vartheta^*\right). \tag{4.16}$$

Further, we assume that one of the hypotheses are fulfilled:

A) *the function $F(\cdot) + \left[\sum_{j=1}^{m}\zeta_j^* Q_j(\cdot) + \sum_{t=1}^{w}\vartheta_t^* H_t(\cdot)\right]e$ is strictly V-r-convex at Z^* on $\Omega\cup\Gamma$ with $\alpha\left(S^*, Z^*\right) = \left(\alpha_1\left(S^*, Z^*\right), ..., \alpha_p\left(S^*, Z^*\right)\right) > 0$.*

B) *each function $\xi_i^*\left[F_i(\cdot) + \sum_{j=1}^{m}\zeta_j^* Q_j(\cdot) + \sum_{t=1}^{w}\vartheta_t^* H_t(\cdot)\right]$, $i \in \{1, ..., p : \xi_i^* > 0\}$, is strictly α_i-r-pseudo-convex at Z^* on $\Omega\cup\Gamma$ with $\alpha_i\left(S^*, Z^*\right) > 0$.*

Then $S^* = Z^*$.

Proof. Let S^* and $(Z^*, \xi^*, \zeta^*, \vartheta^*)$ be feasible solutions in (MP) and (MD), respectively, such that (4.16) is fulfilled. We proceed by contradiction. Suppose, contrary to the result, that $S^* \neq Z^*$. By the definition of ψ, (4.16) gives

$$F(S^*) = F\left(Z^*\right) + \left[\sum_{j\in J_0}\zeta_j^* Q_j\left(Z^*\right) + \sum_{t\in T_0}\vartheta_t^* H_t\left(Z^*\right)\right]e. \tag{4.17}$$

Using $S^* \in \Omega$ and $(Z^*, \xi^*, \zeta^*, \vartheta^*) \in \Delta$ together with (4.2), (4.17) gives

$$F\left(S^*\right) + \left[\sum_{i=1}^{m}\zeta_j^* Q_j\left(S^*\right) + \sum_{t=1}^{w}\vartheta_t^* H_t\left(S^*\right)\right]e \leqq$$
$$F\left(Z^*\right) + \left[\sum_{i=1}^{m}\zeta_j^* Q_j\left(Z^*\right) + \sum_{t=1}^{w}\vartheta_t^* H_t\left(Z^*\right)\right]e.$$

Hence, we have for any $i = 1, ..., p$,

$$F_i(S^*) + \sum_{i=1}^{m} \zeta_j^* Q_j(S^*) + \sum_{t=1}^{w} \vartheta_t^* H_t(S^*) \leqq F_i(Z^*) + \sum_{i=1}^{m} \zeta_j^* Q_j(Z^*) + \sum_{t=1}^{w} \vartheta_t^* H_t(Z^*) . \tag{4.18}$$

Proof under hypothesis A).

By assumption, $F(\cdot) + \left[\sum_{j=1}^{m} \zeta_j^* Q_j(\cdot) + \sum_{t=1}^{w} \vartheta_t^* H_t(\cdot)\right] e$ is strictly V-r-convex at Z^* on $\Omega \cup \Gamma$. Hence, by Definition 2.4, each its component $F_i(\cdot) + \sum_{j=1}^{m} \zeta_j^* Q_j(\cdot) + \sum_{t=1}^{w} \vartheta_t^* H_t(\cdot)$ is strictly α_i-r-convex at Z^* on $\Omega \cup \Gamma$. Then, by Definition 2.4, for any $i = 1, ..., p$,

$$\frac{1}{r} e^{r\left[F_i(S^*) + \sum_{j=1}^{m} \zeta_j^* Q_j(S^*) + \sum_{t=1}^{w} \vartheta_t^* H_t^*(S^*)\right]} > \frac{1}{r} e^{r\left[F_i(Z^*) + \sum_{j=1}^{m} \zeta_j^* Q_j(Z^*) + \sum_{t=1}^{w} \vartheta_t^* H_t(Z^*)\right]}$$

$$\left[1 + \alpha_i(S^*, Z^*) \sum_{k=1}^{n} \left\langle D_k \left[F_i(Z^*) + \sum_{j=1}^{m} \zeta_j^* Q_j(Z^*) + \sum_{t=1}^{w} \vartheta_t^* H_t(Z^*)\right], \chi_{S_k^*} - \chi_{Z_k^*} \right\rangle\right] .$$

Thus, the above inequality can be re-written as follows

$$\frac{1}{r}\left(e^{r\left(\left[F_i(S^*) + \sum_{j=1}^{m} \zeta_j^* Q_j(S^*) + \sum_{t=1}^{w} \vartheta_t^* H_t^*(S^*)\right] - \left[F_i(Z^*) + \sum_{j=1}^{m} \zeta_j^* Q_j(Z^*) + \sum_{t=1}^{w} \vartheta_t^* H_t(Z^*)\right]\right)} - 1\right) > \tag{4.19}$$

$$\alpha_i(S^*, Z^*) \sum_{k=1}^{n} \left\langle D_k \left[F_i(Z^*) + \sum_{j=1}^{m} \zeta_j^* Q_j(Z^*) + \sum_{t=1}^{w} \vartheta_t^* H_t(Z^*)\right], \chi_{S_k^*} - \chi_{Z_k^*} \right\rangle .$$

Combining (4.18) and (4.19), we obtain

$$\alpha_i(S^*, Z^*) \sum_{k=1}^{n} \left\langle D_k \left[F_i(Z^*) + \sum_{j=1}^{m} \zeta_j^* Q_j(Z^*) + \sum_{t=1}^{w} \vartheta_t^* H_t(Z^*)\right], \chi_{S_k^*} - \chi_{Z_k^*} \right\rangle < 0, \quad i = 1, ..., p. \tag{4.20}$$

Since $\alpha_i(S^*, Z^*) > 0$, $i = 1, ..., p$, (4.20) yields

$$\sum_{k=1}^{n} \left\langle D_k \left[\sum_{i=1}^{p} F_i(Z^*) + \sum_{j=1}^{m} \zeta_j^* Q_j(Z^*) + \sum_{t=1}^{w} \vartheta_t^* H_t(Z^*)\right], \chi_{S_k^*} - \chi_{Z_k^*} \right\rangle < 0, \quad i = 1, ..., p. \tag{4.21}$$

Multiplying each inequality (4.21) by the corresponding Lagrange multiplier ξ_i^*, we get

$$\sum_{i=1}^{p} \xi_i^* \sum_{k=1}^{n} \left\langle D_k \left[F_i(Z^*) + \sum_{j=1}^{m} \zeta_j^* Q_j(Z^*) + \sum_{t=1}^{w} \vartheta_t^* H_t(Z^*)\right], \chi_{S_k^*} - \chi_{Z_k^*} \right\rangle < 0 \tag{4.22}$$

Using the constraint (4.3), we obtain that the inequality

$$\sum_{k=1}^{n}\left\langle D_k\left[\sum_{i=1}^{p}\xi_i^* F_i\left(Z^*\right)+\sum_{j=1}^{m}\zeta_j^* Q_j\left(Z^*\right)+\sum_{t=1}^{w}\vartheta_t^* H_t\left(Z^*\right)\right],\chi_{S_k^*}-\chi_{Z_k^*}\right\rangle<0$$

holds, contradicting the constraint (4.2). This means that $S^*=Z^*$.

Proof under hypothesis B).

By assumption, each function $\xi_i^*\left[F_i\left(\cdot\right)+\sum_{j=1}^m\zeta_j^* Q_j\left(\cdot\right)+\sum_{t=1}^w\vartheta_t^* H_t\left(\cdot\right)\right]$, $i\in\{1,...,p:\xi_i^*>0\}$, is strictly α_i-r-pseudo-convex at Z^* on $\Omega\cup\Gamma$ with $\alpha_i\left(S^*,Z^*\right)>0$. Hence, (4.18) yields

$$\frac{1}{r}\sum_{i=1}^{p}\alpha_i\left(S^*,Z^*\right)e^{r\xi_i^*\left[F_i(S^*)+\sum_{j=1}^m\zeta_j^*Q_j(S^*)+\sum_{t=1}^w\vartheta_t^*H_t(S^*)\right]}<$$
$$\frac{1}{r}\sum_{i=1}^{p}\alpha_i\left(S^*,Z^*\right)e^{r\xi_i^*\left[F_i(Z^*)+\sum_{j=1}^m\zeta_j^*Q_j(Z^*)+\sum_{t=1}^w\vartheta_t^*H_t(Z^*)\right]},\ i=1,...,p. \tag{4.23}$$

Thus, by Definition 2.6, (4.21) implies (4.22). The last part of the proof is the same as in the proof under hypothesis A). Hence, the proof of this theorem is completed. □

Acknowledgments

Authors are grateful to anonymous Referees for their constructive comments and valuable suggestions on the first version of this work.

References

1. I. Ahmad, S. Sharma, Sufficiency in Multiobjective Subset Srogramming Involving Generalized Type-I Functions, *Journal of Global Optimization*, **39**, (2007), 473-481.
2. M. Avriel, r-Convex Functions, *Mathematical Programming*, **2**, (1972), 309-323.
3. A. Bătătorescu, Optimality Conditions Involving V-Type-I Univexity and Set-Functions, *Mathematical Reports*, **7**, (2005) 1-11.
4. C. R. Bector, M. Singh, Duality for Multiobjective B-vex Programming Involving n-Set Functions, Journal of Mathematical Analysis and Applications, **202**, (1996), 701-726.
5. D. Begis, R. Glowinski, Applications de la Méthode des éléments Finis à L'approximation d'un Problème de Domaine Optimal, Mèthodes de Rèsolution de Problèmes Approchès, *Applied Mathematics and Optimization*, **2**, (1975), 130-169.
6. M. Beldiman, A. Paraschiv, O. Cojocaru, On Multiobjective Programming Problems Containing n-Set Functions, *Analele Universităţii Bucuresti, Matematică Anul*, **LVII**, (2008), 189-206.
7. D. Bhatia, A. Mehra, Lagrange Duality in Multiobjective Fractional Programming Problems with n-Set Functions, *Journal of Mathematical Analysis and Applications*, **236**, (1999), 300-311.
8. J. Cea, A. Gioan, J. Michel, Quelque Résultats sur i'Identification de Domaines, *Calcolo*, **10**, (1973), 133-145.
9. H. W. Corley, S. D. Roberts, A Partitioning Problem with Applications in Regional Design, *Operations Research*, **20**, (1982), 1010-1019.
10. H. W. Corley, Optimization Theory for n-set Functions, *Journal of Mathematical Analysis and Applications*, **127**, (1987), 193-205.

11. G. Dantzing, A. Wald, On the Fundamental Lemma of Neyman and Pearson, *The Annals of Mathematical Statistics*, **22**, (1951), 87-93.
12. A. Jayswal, I. M. Stancu-Minasian, Multiobjective Subset Programming Problems Involving Generalized D-Type I Univex Functions, *Proceedings of the Romanian Academy, Series A*, **11**, (2010) 19-24.
13. C. L. Jo, D. S. Kim, G. M. Lee, Duality for Multiobjective Programming Involving n-Set Functions, *Optimization*, **29**, (1994), 45-54.
14. R. Larsson, *Methodology for Topology and Shape Optimization: Application to a Rear-Lower Control Arm*, Chalmers University of Technology Göteborg, Sweden, 2016.
15. L. J. Lin, Optimality of Differentiable Vector-Valued n-set Functions, *Journal of Mathematical Analysis and Applications*, **149**, (1990), 255-270.
16. L. J. Lin, On the Optimality Conditions of Vector-Valued n-set Functions, *Journal of Mathematical Analysis and Applications*, **161**, (1991), 367-387.
17. S. K. Mishra, S. Y. Wang, K. K. Lai, J. Shi, New Generalized Invexity for Duality in Multiobjective Programming Problems Involving n-set, in: A. Eberhard, N. Hadjisavvas, D. T. Luc (eds.), *Generalized Convexity, Generalized Monotonicity and Applications*, Nonconvex Optimization and Applications, **77**, Springer, New York, 2005, pp. 321-339.
18. B. Mond, T. Weir, Generalized Concavity and Duality, in: S. Schaible, W.T. Ziemba (eds.), *Generalized Concavity in Optimization and Economics*, Academic Press, New York, 1981, pp. 263-279.
19. R. J. T. Morris, Optimal Constrained Selection of a Measurable Subset, *Journal of Mathematical Analysis and Applications*, **70**, (1979), 546-562.
20. V. Preda, Some Optimality Conditions for Multiobjective Programming Problems with Set Functions, *Revue Roumaine de Mathmatiques Pures et Appliques*, **39**, (1994), 233-247.
21. V. Preda, On Duality of Multiobjective Fractional Measurable Subsets Selection Problems, *Journal of Mathematical Analysis and Applications*, **196**, (1995), 514-525.
22. V. Preda, Duality for Multiobjective Fractional Programming Problems Involving n-Set Functions. In: C. A. Cazacu, C. W. E. Lehto, T. M. Rassias (eds.), *Analysis and Topology*, World Scientific Publishing Co., River Edge, NJ, 1998, pp. 569-583.
23. V. Preda, A. Bătătorescu, On Duality for Minmax Generalized B-vex Programming Involving n-Set Functions. *Journal of Convex Analysis*, **9**, (2002), 609-623.
24. V. Preda, I. M. Stancu-Minasian, Optimality and Wolfe Duality for Multiobjective Programming Problems Involving n-Set Functions, In: N. Hadjisavvas, J. E. Martínez-Legaz, J.-P. Penot (eds.), *Generalized Convexity and/or Generalized Monotonicity*, Lecture Notes in Economics and Mathematical Systems, **502**, Springer-Verlag, Berlin, 2001, pp. 349-361.
25. V. Preda, I. M. Stancu-Minasian, M. Beldiman, A. M. Stancu, Generalized V-Univexity Type-I for Multiobjective Programming with n-Set Functions, *Journal of Global Optimization*, **44**, (2009), 131-148.
26. V. Preda, I. M. Stancu-Minasian, E. Koller, On Optimality and Duality for Multiobjective Programming Problems Involving Generalized d-Type-I and Related n-Set Functions, *Journal of Mathematical Analysis and Applications*, **283**, (2003), 114-128.
27. A. M. Stancu, Optimality and Duality for Multiobjective Fractional Programming Problems with n-Set Functions and Generalized V-Type-I Univexity, *Revue Roumaine de Mathmatiques Pures et Appliques*, **57**, (2012), 401-421.
28. I. M. Stancu-Minasian, V. Preda, Optimality Conditions and Duality for Programming Problems Involving Set and n-Set Functions - a Survey, *Journal of Statistics and Management Systems*, **5**, (2002), 175-207.

29. A. A. Taflanidis, Robust Stochastic Design of Viscous Dampers for Base Isolation Applications, in: M. Papadrakakis, M. Fragiadakis, N. D. Lagaros (eds.), *Computational Methods in Earthquake Engineering*, Computational Methods in Applied Sciences, **21**, Springer 2011, pp. 305-329.
30. G. J. Zalmai, Sufficiency Criteria and Duality for Nonlinear Programs Involving n-Set Functions, *Journal of Mathematical Analysis and Applications*, **149**, (1990), 322-338.
31. G. J. Zalmai, Optimality Conditions and Duality for Multiobjective Measurable Subset Selection Problems, *Optimization*, **22**, (1991), 221-238.