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Nonuniform Semi-orthogonal Wavelet Frames on Non-Archimedean Local Fields of Positive Characteristic

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Abstract. In this paper, we introduce the notion of nonuniform semi-orthogonal wavelet frame associated with nonuniform frame multiresolution analysis on non-Archimedean fields and provide their characterization by means of some basic equations in the frequency domain.

Keywords: Nonuniform semi-orthogonal wavelet frame, Fourier transform, Non-Archimedean field.

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1. Introduction

The concept of frame in a Hilbert space was first introduced by Duffin and Schaeffer [14] in connection with some deep problems in non-harmonic Fourier series. Frames are basis-like systems that span a vector space but allow for linear dependency, which can be used to reduce noise, find sparse representations, or obtain other desirable features unavailable with orthonormal bases. An important example about frame is wavelet frame, which is obtained by translating and dilating a finite family of functions. To mention only a few references on wavelet frames, the reader is referred to [9, 11, 10, 12] and many references therein. Multiresolution analysis is an important mathematical tool

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since it provides a natural framework for understanding and constructing discrete wavelet systems. The concept of MRA has been extended in various ways in recent years. These concepts are generalized to $L^2(\mathbb{R}^d)$, to lattices different from $\mathbb{Z}^d$, allowing the subspaces of MRA to be generated by Riesz basis instead of orthonormal basis, admitting a finite number of scaling functions, replacing the dilation factor 2 by an integer $M \geq 2$ or by an expansive matrix $A \in GL_d(\mathbb{R})$ as long as $A \subset A\mathbb{Z}^d$. All these concepts are developed on regular lattices, that is the translation set is always a group. Recently, Gabardo and Nashed [15] considered a generalization of Mallat's [18] celebrated theory of MRA based on spectral pairs, in which the translation set acting on the scaling function associated with the MRA to generate the subspace V_0 is no longer a group, but is the union of $\mathbb{Z}$ and a translate of $\mathbb{Z}$.

On the other hand, there is a substantial body of work that has been concerned with the construction of wavelets on non-Archimedean fiels. For example, R. L. Benedetto and J. J. Benedetto [8] developed a wavelet theory for local fields and related groups. They did not develop the multiresolution analysis (MRA) approach, their method is based on the theory of wavelet sets and only allows the construction of wavelet functions whose Fourier transforms are characteristic functions of some sets. Jiang et al.[16] pointed out a method for constructing orthogonal wavelets on local field K with a constant generating sequence and derived necessary and sufficient conditions for a solution of the refinement equation to generate a multiresolution analysis of $L^2(K)$. Recently, Shah and Abdullah [19] have generalized the concept of multiresolution analysis on Euclidean spaces $\mathbb{R}^n$ to nonuniform multiresolution analysis on local fields of positive characteristic, in which the translation set acting on the scaling function associated with the multiresolution analysis to generate the subspace V_0 is no longer a group, but is the union of $\mathcal{Z}$ and a translate of $\mathcal{Z}$, where $\mathcal{Z} = \{u(n) : n \in \mathbb{N}_0\}$ is a complete list of (distinct) coset representation of the unit disc $\mathfrak{D}$ in the locally compact Abelian group K^+. More precisely, this set is of the form $\Lambda = \{0, r/N\} + \mathcal{Z}$, where $N \geq 1$ is an integer and r is an odd integer such that r and N are relatively prime. They call this a *nonuniform multiresolution analysis* on local fields of positive characteristic.The notion of nonuniform wavelet frames on non-Archimedean fields was introduced by Ahmad and Sheikh [7] and established a complete characterization of tight nonuniform wavelet frames on these fields. More results in this direction can also be found in [1, 2, 3, 4, 5, 6, 17, 20, 21] and the references therein.

Motivated and inspired by the above work,we introduce the notion of nonuniform semi-orthogonal wavelet frames on non-Archimedean fields. By

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using the machinery of Fourier transform, we provide the complete characterization of nonuniform semi-orthogonal wavelets as as generalization of the orthonormal wavelets .

This article is tailored as follows. In Section 2, we discuss some preliminary facts on non-Archimedean fields including some auxillary results that will be used in establishing main results. Section 3 is devoted to the construction of nonuniform semi-orthogonal wavelet frames on non-Archimedean fields.

2. Preliminaries on non-archimedean fields

A non-Archimedean field K is a locally compact, non-discrete and totally disconnected field. If it is of characteristic zero, then it is a field of p-adic numbers $\mathbb{Q}_p$ or its finite extension. If K is of positive characteristic, then K is a field of formal Laurent series over a finite field $GF(p^c)$. If $c = 1$, it is a p-series field, while for $c \neq 1$, it is an algebraic extension of degree c of a p-series field. Let K be a fixed non-Archimedean field with the ring of integers $\mathfrak{D} = \{x \in K : |x| \leq 1\}$. Since K^+ is a locally compact Abelian group, we choose a Haar measure dx for K^+. The field K is locally compact, non-trivial, totally disconnected and complete topological field endowed with non–Archimedean norm $|\cdot| : K \to \mathbb{R}^+$ satisfying

(a) $|x| = 0$ if and only if $x = 0$;

(b) $|x\,y| = |x||y|$ for all $x, y \in K$;

(c) $|x + y| \leq \max\{|x|, |y|\}$ for all $x, y \in K$.

Property (c) is called the ultrametric inequality. Let $\mathfrak{B} = \{x \in K : |x| < 1\}$ be the prime ideal of the ring of integers $\mathfrak{D}$ in K. Then, the residue space $\mathfrak{D}/\mathfrak{B}$ is isomorphic to a finite field $GF(q)$, where $q = p^c$ for some prime p and $c \in \mathbb{N}$. Since K is totally disconnected and $\mathfrak{B}$ is both prime and principal ideal, there exist a prime element $\mathfrak{p}$ of K such that $\mathfrak{B} = \langle \mathfrak{p} \rangle = \mathfrak{p}\mathfrak{D}$. Let $\mathfrak{D}^* = \mathfrak{D} \setminus \mathfrak{B} = \{x \in K : |x| = 1\}$. Clearly, $\mathfrak{D}^*$ is a group of units in K^* and if $x \neq 0$, then can write $x = \mathfrak{p}^n y, y \in \mathfrak{D}^*$. Moreover, if $U = \{a_m : m = 0, 1, \ldots, q-1\}$ denotes the fixed full set of coset representatives of $\mathfrak{B}$ in $\mathfrak{D}$, then every element $x \in K$ can be expressed uniquely as $x = \sum_{\ell=k}^{\infty} c_\ell\, \mathfrak{p}^\ell$ with $c_\ell \in U$. Recall that $\mathfrak{B}$ is compact and open, so each fractional ideal $\mathfrak{B}^k = \mathfrak{p}^k\mathfrak{D} = \left\{x \in K : |x| < q^{-k}\right\}$ is also compact and open and is a subgroup of K^+. We use the notation in Taibleson's book [13]. In the rest of this paper, we use the symbols $\mathbb{N}, \mathbb{N}_0$ and $\mathbb{Z}$ to denote the sets of natural, non-negative integers and integers, respectively.

Let χ be a fixed character on K^+ that is trivial on $\mathfrak{D}$ but non-trivial on $\mathfrak{B}^{-1}$. Therefore, χ is constant on cosets of $\mathfrak{D}$ so if $y \in \mathfrak{B}^k$, then $\chi_y(x) = \chi(y, x), x \in K$. Suppose that χ_u is any character on K^+, then the restriction

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$\chi_u|\mathfrak{D}$ is a character on $\mathfrak{D}$. Moreover, as characters on $\mathfrak{D}, \chi_u = \chi_v$ if and only if $u - v \in \mathfrak{D}$. Hence, if $\{u(n) : n \in \mathbb{N}_0\}$ is a complete list of distinct coset representative of $\mathfrak{D}$ in K^+, then, as it was proved in [13], the set $\{\chi_{u(n)} : n \in \mathbb{N}_0\}$ of distinct characters on $\mathfrak{D}$ is a complete orthonormal system on $\mathfrak{D}$.

We now impose a natural order on the sequence $\{u(n)\}_{n=0}^{\infty}$. We have $\mathfrak{D}/\mathfrak{B} \cong GF(q)$ where $GF(q)$ is a c-dimensional vector space over the field $GF(p)$. We choose a set $\{1 = \zeta_0, \zeta_1, \zeta_2, \dots, \zeta_{c-1}\} \subset \mathfrak{D}^*$ such that $\operatorname{span}\{\zeta_j\}_{j=0}^{c-1} \cong GF(q)$. For $n \in \mathbb{N}_0$ satisfying

$$0 \le n < q, \ n = a_0 + a_1 p + \cdots + a_{c-1} p^{c-1}, \ 0 \le a_k < p, \text{ and } k = 0, 1, \dots, c-1,$$

we define

$$u(n) = (a_0 + a_1\zeta_1 + \cdots + a_{c-1}\zeta_{c-1})\,\mathfrak{p}^{-1}.$$

Also, for $n = b_0 + b_1 q + b_2 q^2 + \cdots + b_s q^s, \ n \in \mathbb{N}_0, \ 0 \le b_k < q, k = 0, 1, 2, \dots, s$, we set

$$u(n) = u(b_0) + u(b_1)\mathfrak{p}^{-1} + \cdots + u(b_s)\mathfrak{p}^{-s}.$$

This defines $u(n)$ for all $n \in \mathbb{N}_0$. In general, it is not true that $u(m+n) = u(m) + u(n)$. But, if $r, k \in \mathbb{N}_0$ and $0 \le s < q^k$, then $u(rq^k + s) = u(r)\mathfrak{p}^{-k} + u(s)$. Further, it is also easy to verify that $u(n) = 0$ if and only if $n = 0$ and $\{u(\ell) + u(k) : k \in \mathbb{N}_0\} = \{u(k) : k \in \mathbb{N}_0\}$ for a fixed $\ell \in \mathbb{N}_0$. Hereafter we use the notation $\chi_n = \chi_{u(n)}$, $n \ge 0$.

Let the local field K be of characteristic $p > 0$ and $\zeta_0, \zeta_1, \zeta_2, \dots, \zeta_{c-1}$ be as above. We define a character χ on K as follows:

$$\chi(\zeta_\mu \mathfrak{p}^{-j}) = \begin{cases} \exp(2\pi i/p), & \mu = 0 \text{ and } j = 1, \\ 1, & \mu = 1, \dots, c-1 \text{ or } j \ne 1. \end{cases}$$

The Fourier transform of $f \in L^1(K)$ is denoted by $\hat{f}(\xi)$ and defined by

$$\mathcal{F}\{f(x)\} = \hat{f}(\xi) = \int_K f(x)\overline{\chi_\xi(x)}\,dx.$$

It is noted that

$$\hat{f}(\xi) = \int_K f(x)\,\overline{\chi_\xi(x)}dx = \int_K f(x)\chi(-\xi x)\,dx.$$

The properties of Fourier transforms on non-Archimedean field K are much similar to those of on the classical field $\mathbb{R}$. In fact, the Fourier transform on non-Archimedean fields of positive characteristic have the following properties:

- The map $f \to \hat{f}$ is a bounded linear transformation of $L^1(K)$ into $L^\infty(K)$, and $\|\hat{f}\|_\infty \le \|f\|_1$.

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- If $f \in L^1(K)$, then $\hat{f}$ is uniformly continuous.
- If $f \in L^1(K) \cap L^2(K)$, then $\|\hat{f}\|_2 = \|f\|_2$.

The Fourier transform of a function $f \in L^2(K)$ is defined by

$$\hat{f}(\xi) = \lim_{k\to\infty} \hat{f}_k(\xi) = \lim_{k\to\infty} \int_{|x|\le q^k} f(x)\overline{\chi_\xi(x)}\,dx,$$

where $f_k = f\,\Phi_{-k}$ and Φ_k is the characteristic function of $\mathfrak{B}^k$. Furthermore, if $f \in L^2(\mathfrak{D})$, then we define the Fourier coefficients of f as

$$\hat{f}\big(u(n)\big) = \int_{\mathfrak{D}} f(x)\overline{\chi_{u(n)}(x)}\,dx.$$

The series $\sum_{n\in\mathbb{N}_0} \hat{f}\big(u(n)\big)\chi_{u(n)}(x)$ is called the Fourier series of f. From the standard L^2-theory for compact Abelian groups, we conclude that the Fourier series of f converges to f in $L^2(\mathfrak{D})$ and Parseval's identity holds:

$$\|f\|_2^2 = \int_{\mathfrak{D}} \big|f(x)\big|^2 dx = \sum_{n\in\mathbb{N}_0} \Big|\hat{f}\big(u(n)\big)\Big|^2.$$

For an integer $N \ge 1$ and an odd integer r with $1 \le r \le qN-1$ such that r and N are relatively prime, we define

$$\Lambda = \left\{0, \frac{u(r)}{N}\right\} + \mathcal{Z},$$

where $\mathcal{Z} = \{u(n) : n \in \mathbb{N}_0\}$. It is easy to verify that Λ is not a group on non-Archimedean field K, but is the union of $\mathcal{Z}$ and a translate of $\mathcal{Z}$. Following is the definition of nonuniform frame multiresolution analysis (NUFMRA) on non-Archimedean fields of positive characteristic.

Definition 2.1. For an integer $N \ge 1$ and an odd integer r with $1 \le r \le qN-1$ such that r and N are relatively prime, an associated NUMRA on non-Archimedean field K of positive characteristic is a sequence of closed subspaces $\{V_j : j \in \mathbb{Z}\}$ of $L^2(K)$ such that the following properties hold:

(a) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$;

(b) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(K)$;

(c) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$;

(d) $f(\cdot) \in V_j$ if and only if $f(\mathfrak{p}^{-1}N\cdot) \in V_{j+1}$ for all $j \in \mathbb{Z}$;

(e) There exists a function φ in V_0 such that $\{\varphi(\cdot - \lambda) : \lambda \in \Lambda\}$, forms a frame for V_0.

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It is worth noticing that, when $N = 1$, one recovers from the definition above the definition of an FMRA on non-Archimedean fields of positive characteristic $p > 0$. When, $N > 1$, the dilation is induced by $\mathfrak{p}^{-1}N$ and $|\mathfrak{p}^{-1}| = q$ ensures that $qN\Lambda \subset \mathcal{Z} \subset \Lambda$.

As in the standard scheme, one expects the existence of $qN - 1$ number of functions so that their translation by elements of Λ and dilations by the integral powers of $\mathfrak{p}^{-1}N$ form an orthonormal basis for $L^2(K)$.

For a given $\Psi = \{\psi_1, \psi_2, \dots, \psi_{qN-1}\} \subset L^2(K)$, define the nonuniform wavelet system

$$W(\psi, j, \lambda) = \Big\{\psi_{\ell,j,\lambda} =: (qN)^{j/2}\psi_\ell\big((\mathfrak{p}^{-1}N)^j x - \lambda\big);\ j \in \mathbb{Z},\ \lambda \in \Lambda\Big\} \tag{2.1}$$

We call the wavelet system $W(\psi, j, \lambda)$ a nonuniform wavelet frame for $L^2(K)$, if there exist constants A and $B, 0 < A \leq B < \infty$ such that for all $f \in L^2(K)$

$$A\|f\|_2^2 \leq \sum_{\ell=1}^{qN-1} \sum_{j\in\mathbb{Z}} \sum_{\lambda\in\Lambda} \big|\langle f, \psi_{\ell,j,\lambda}\rangle\big|^2 \leq B\|f\|_2^2. \tag{2.2}$$

The largest constant A and the smallest constant B satisfying () are respectively known as *lower* and *upper frame bound*. A nonuniform wavelet frame is a *tight nonuniform wavelet frame* if A and B are equal and then the set $\Psi = \{\psi_1, \psi_2, \dots, \psi_{qN-1}\}$ is called *nonuniform frame wavelets* for the corresponding tight nonuniform wavelet frame.If $A = B = 1$ then the nonuniform wavelet frame is called *Parseval or normalized tight nonuniform wavelet frame*, i.e.,

$$\sum_{\ell=1}^{qN-1} \sum_{j\in\mathbb{Z}} \sum_{\lambda\in\Lambda} \big|\langle f, \psi_{\ell,j,\lambda}\rangle\big|^2 = \|f\|_2^2.$$

The collection $W(\psi, j, \lambda)$ is said to be *a nonuniform semi-orthogonal wavelet frame* if

$$\langle \psi_{\ell,j,\lambda}, \psi_{\ell,j',\sigma}\rangle = 0 \text{ whenever } j \neq j', \lambda, \sigma \in \Lambda, 1 \leq \ell \leq qN - 1.$$

The characterization of nonuiform wavelet frames on non- Archimedean local fields has been studied by Ahmad and Sheikh [7]. In fact, for a given dilation $\mathfrak{p}^{-1}N$, they characterized all orthonormal wavelets $\{\psi_1, \psi_2, \dots, \psi_{qN-1}\}$ in $L^2(K)$ by means of two basic equations in the frequency domain as:

Theorem 2.1. *Let $\Psi = \{\psi_1, \psi_2, \dots, \psi_{qN-1}\} \subset L^2(K)$. The affine system W_Ψ given by (2.1) is a normalized tight frame for $L^2(K)$ if and only if*

$$\sum_{\ell=1}^{qN-1} \sum_{j\in\mathbb{Z}} \Big|\hat{\psi}_\ell\left((\mathfrak{p}^{-1}N)^j\xi\right)\Big|^2 = 1 \qquad a.e.\ \xi \in K \tag{2.3}$$

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$$\sum_{\ell=1}^{qN-1}\sum_{j=0}^{\infty}\hat{\psi}_\ell\left((\mathfrak{p}^{-1}N)^j\xi\right)\overline{\hat{\psi}_\ell\left((\mathfrak{p}^{-1}N)^j\left(\xi+\lambda\right)\right)}=0 \qquad a.e.\ \xi\in K, \lambda\in\Lambda\setminus qN\Lambda. \tag{2.4}$$

In particular, Ψ is a set of basic wavelets of $L^2(K)$ if and only if $\|\psi_\ell\|_2=1$ for $1\le\ell\le qN-1$ and equations (2.3) and (2.4) hold.

Let $\Psi=\{\psi_1,\psi_2,\dots,\psi_{qN-1}\}$ be a set of basic frame wavelets of $L^2(K)$. We define the spaces $W_j, j\in\mathbb{Z}$, by $W_j=\overline{\text{span}}\big\{\psi_{\ell,j,\lambda}:1\le\ell\le qN-1,\lambda\in\Lambda\big\}$. We also define $V_j=\bigoplus_{m<j}W_m, j\in\mathbb{Z}$. Then it follows that $\{V_j:j\in\mathbb{Z}\}$ satisfies the properties (a)-(e) in the definition of an FMRA. Hence, $\{V_j:j\in\mathbb{Z}\}$ will form an FMRA of $L^2(K)$ if we can find a function $\varphi\in L^2(K)$ such that the system $\{\tau_\lambda\varphi(x):\lambda\in\Lambda\}$ is a frame for V_0. It is clear that the system of translates $\{\tau_\lambda\varphi(x):\lambda\in\Lambda\}$ constitutes a frame sequence in $L^2(K)$ with frame bounds $0<A\le B<\infty$ if and only if

$$A\le\sum_{\lambda\in\Lambda}\left|\hat{\varphi}\big(\xi-\lambda\big)\right|^2\le B, \qquad \text{a.e. } \xi\in S, \tag{2.5}$$

where $S=\big\{\xi\in\mathfrak{D}:\hat{\varphi}\big(\xi-\lambda\big)\neq0\big\}$.
In general, the collection $\{\tau_\lambda\psi_\ell(x):1\le\ell\le qN-1,\lambda\in\Lambda\}$ forms a wavelet frame for W_0 if and only if there exist positive numbers A and B such that

$$A\le\sum_{\lambda\in\Lambda}\left|\hat{\psi}_\ell\big(\xi-\lambda\big)\right|^2\le B, \quad \text{a.e. } \xi\in S_\ell, \tag{2.6}$$

where $B=\max B_\ell$, $A=\min A_\ell$ and $\Gamma_\ell=\Big\{\xi\in\mathfrak{D}:\hat{\psi}_\ell\big(\xi-\lambda\big)\neq0\Big\}$.

3. Characterization of nonuniform semi-orthogonal wavelet frames

In this section we characterize all the nonuniform semi-orthogonal wavelet frames on non-Archimedean fields.

For each $1\le\ell\le qN-1$, we define

$$\hat{\psi}^*_\ell(\xi)=\begin{cases}\dfrac{\hat{\psi}_\ell(\xi)}{\left\|\hat{\psi}_\ell\big(\xi-\lambda\big)\right\|_{l^2(\mathbb{N}_0)}}, & \text{if } \xi\in\Gamma_\ell,\\ 0, & \text{otherwise,}\end{cases}$$

where $\Gamma_\ell=\Big\{\xi\in\mathfrak{D}:\hat{\psi}_\ell\big(\xi-\lambda)\big)\neq0\Big\}$. Then, the system $W(\Psi^*,j,\lambda)$ obtained by the combined action of dilation and translation a finite number of functions $\Psi^*=\big\{\psi^*_1,\psi^*_2,\dots,\psi^*_{qN-1}\big\}\subset L^2(K)$ is given by

$$W(\Psi^*,j,\lambda)=\big\{\psi^*_{\ell,j,\lambda}:1\le\ell\le qN-1,j\in\mathbb{Z},\lambda\in\Lambda\big\}. \tag{3.1}$$

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The following theorem provides the characterization of nonuniform semi-orthogonal wavelet frames on non-Archimedean fields.

Theorem 3.1. *Let $W(\Psi, j, \lambda)$ and $W(\Psi^*, j, \lambda)$ be as defined in equations (2.1) and (3.1), respectively. Then the following are equivalent:*

(a) $W(\Psi, j, \lambda)$ is a nonuniform semi-orthogonal wavelet frame with frame bounds A and B.

(b) For each $1 \leq \ell \leq qN-1$, there exist positive constants $A = \min A_\ell$ and $B = \max B_\ell$ such that equation (2.6) hold and following two conditions are satisfied:

$$\sum_{\ell=1}^{qN-1} \sum_{j \in \mathbb{Z}} \left| \hat{\psi}^*_\ell\big((\mathfrak{p}^{-1}N)^j \xi\big) \right|^2 = 1 \qquad a.e.\ \xi \in K, \tag{3.2}$$

$$\sum_{\ell=1}^{qN-1} \sum_{j=0}^{\infty} \hat{\psi}^*_\ell\big((\mathfrak{p}^{-1}N)^j \xi\big) \overline{\hat{\psi}^*_\ell\big((\mathfrak{p}^{-1}N)^j (\xi+\lambda)\big)} = 0 \quad a.e.\ \xi \in K,\ \lambda \in \Lambda \setminus qN\Lambda. \tag{3.3}$$

(c) There exist positive numbers A and B such that functions $\psi_1, \psi_2, \ldots, \psi_{qN-1}$ satisfy conditions (2.6) , (3.3) and

$$\sum_{\ell=1}^{qN-1} \sum_{\lambda \in \Lambda} \hat{\psi}_\ell(\xi+\lambda) \overline{\hat{\psi}_\ell\big((\mathfrak{p}^{-1}N)^j (\xi+\lambda)\big)} = 0, \qquad a.e.\ \xi \in K,\ j \in \mathbb{Z}, \tag{3.4}$$

$$A \leq \sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \left| \hat{\psi}_\ell((\mathfrak{p}^{-1}N)^j \xi) \right|^2 \leq B, \qquad a.e.\ \xi \in K. \tag{3.5}$$

Proof. For each $j \in \mathbb{Z}$, we define the following spaces

$$W_j = \overline{\text{span}}\big\{\psi_{\ell,j,\lambda} : 1 \leq \ell \leq qN-1, \lambda \in \Lambda\big\}$$

and

$$W_j^* = \overline{\text{span}}\big\{\psi^*_{\ell,j,\lambda} : 1 \leq \ell \leq qN-1, \lambda \in \Lambda\big\}.$$

Suppose that the nonuniform affine system $W(\Psi, j, \lambda)$ given by (2.1) is a nonuniform semi-orthogonal wavelet frame for $L^2(K)$ with bounds A and B, i.e.,

$$A\big\|f\big\|_2^2 \leq \sum_{\ell=1}^{qN-1} \sum_{j \in \mathbb{Z}} \sum_{\lambda \in \Lambda} |\langle f, \psi_{\ell,j,\lambda}\rangle|^2 \leq B\big\|f\big\|_2^2, \quad \text{for all } f \in L^2(K).$$

By using scaling property of W_j spaces, we write

$$\sum_{\ell=1}^{qN-1} \sum_{j \in \mathbb{Z}} \sum_{\lambda \in \Lambda} |\langle f, \psi_{\ell,j,\lambda}\rangle|^2 = \sum_{\ell=1}^{qN-1} \sum_{\lambda \in \Lambda} |\langle f, \psi_{\ell,0,\lambda}\rangle|^2, \quad \text{for all } f \in W_0.$$

Using the frame condition of the nonuniform affine system $W(\Psi, j, \lambda)$, we have

$$A\|f\|_2^2 \leq \sum_{\ell=1}^{qN-1} \sum_{\lambda\in\Lambda} \left|\langle f, \psi_{0,\lambda}^{\ell}\rangle\right|^2 \leq B\|f\|_2^2, \quad \text{for all } f \in W_0,$$

Therefore, it follows that the system of translates $\{\tau_\lambda \psi_\ell : 1 \leq \ell \leq qN-1, \lambda \in \Lambda\}$ is a frame for the space W_0 which is equivalent to equation (2.6).
Since the spaces W_j^*'s are orthogonal to each other, i.e., $W_{j_1}^* \perp W_{j_2}^*, j_1 \neq j_2$, we have

$$\sum_{\lambda\in\Lambda} \left|\hat{\psi}_\ell^*(\xi - \lambda)\right|^2 = 1, \qquad \text{a.e. } \xi \in \mathfrak{D} \setminus \Gamma_\ell. \tag{3.6}$$

Therefore, the system $W(\Psi^*, j, \lambda)$ is a tight nonuniform frame with frame bound 1 for $L^2(K)$. Hence, conditions (3.2) and (3.3) are satisfied by Theorem 2.1.
To prove (b)$\Rightarrow$ (c), we use equation (2.6) for each $1 \leq \ell \leq qN-1$ to get

$$\frac{1}{B_\ell}\left|\hat{\psi}_\ell(\xi)\right|^2 \leq \left|\hat{\psi}_\ell^*(\xi)\right|^2 \leq \frac{1}{A_\ell}\left|\hat{\psi}_\ell(\xi)\right|^2. \tag{3.7}$$

Or equivalently,

$$\sum_{\ell=1}^{qN-1}\sum_{j\in\mathbb{Z}} \frac{1}{B_\ell}\left|\hat{\psi}_\ell\left((\mathfrak{p}^{-1}N)\xi\right)\right|^2 \leq \sum_{\ell=1}^{qN-1}\sum_{j\in\mathbb{Z}} \left|\hat{\psi}_\ell^*((\mathfrak{p}^{-1}N)^j\xi)\right|^2 \leq \sum_{\ell=1}^{qN-1}\sum_{j\in\mathbb{Z}} \frac{1}{A_\ell}\left|\hat{\psi}_\ell((\mathfrak{p}^{-1}N)^j\xi)\right|^2.$$

On taking $\max B_\ell = B$, $\min A_\ell = A$ and applying equation (3.2), we get

$$A \leq \sum_{\ell=1}^{qN-1}\sum_{j\in\mathbb{Z}} \left|\hat{\psi}_\ell((\mathfrak{p}^{-1}N)^j\xi)\right|^2 \leq B, \quad \text{a.e.}\, \xi \in K$$

which shows equality (3.5). Moreover, it follows from identities (3.2) and (3.3) that the system $\left\{\psi_{\ell,0,k}^* : \lambda \in \Lambda\right\}$ forms a tight frame for W_0^* with frame bound 1. By Theorem 2.1, $\left\{\psi_{\ell,j,\lambda}^* : j \in \mathbb{Z}, \lambda \in \Lambda\right\}$ is a normalised tight frame for $L^2(K)$. Since each ψ_ℓ^* lies in W_0^*, it follows from the tightness of both systems $\left\{\psi_{\ell,0,\lambda}^* : \lambda \in \Lambda\right\}$ and $\left\{\psi_{\ell,j,\lambda}^* : j \in \mathbb{Z}, \lambda \in \Lambda\right\}$ that

$$\|\psi_\ell^*\|_2^2 = \sum_{j\in\mathbb{Z}}\sum_{\lambda\in\Lambda} \left|\langle \psi_\ell^*, \psi_{\ell,j,\lambda}^*\rangle\right|^2 = \sum_{\lambda\in\Lambda} \left|\langle \psi_\ell^*, \psi_{\ell,0,\lambda}^*\rangle\right|^2.$$

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Therefore, $\left\langle \psi_\ell^*, \psi_{\ell,j,\lambda}^* \right\rangle = 0$ for $j \neq 0$. Also, for each $1 \leq \ell \leq qN-1$ and $j \in \mathbb{N}$, we have

$$\begin{aligned}
0 &= \left\langle \psi_\ell^*, \psi_{\ell,j,k}^* \right\rangle \\
&= \left\langle \hat{\psi}_\ell^*, \hat{\psi}_{\ell,j,\lambda}^* \right\rangle \\
&= (qN)^{j/2} \int\limits_K \hat{\psi}_\ell^*(\xi)\, \overline{\hat{\psi}_\ell^*\left(\frac{\xi}{(\mathfrak{p}^{-1}N)^j}\right)}\, \overline{\chi_\lambda\left(\frac{\xi}{(\mathfrak{p}^{-1}N)^j}\right)} d\xi \\
&= (qN)^{-j/2} \int\limits_K \hat{\psi}_\ell^*\left((\mathfrak{p}^{-1}N)^j\xi\right) \overline{\hat{\psi}_\ell^*(\xi)}\, \overline{\chi_\lambda(\xi)} d\xi \\
&= (qN)^{-j/2} \sum_{r\in\mathbb{N}_0} \int\limits_{rN\mathfrak{D}} \hat{\psi}_\ell^*\left((\mathfrak{p}^{-1}N)^j\xi\right) \overline{\hat{\psi}_\ell^*(\xi)}\, \overline{\chi_k(\xi)} d\xi \\
&= (qN)^{-j/2} \int\limits_{\mathfrak{D}} \left\{ \sum_{r\in\mathbb{N}_0} \hat{\psi}_\ell^*\left((\mathfrak{p}^{-1}N)^j(\xi + Nu(r)\right) \overline{\hat{\psi}_\ell^*\big(\xi + Nu(r)\big)} \right\} \overline{\chi_\lambda(\xi)} d\xi.
\end{aligned}$$

This shows that

$$\sum_{r\in\mathbb{N}_0} \hat{\psi}_\ell^*\left((\mathfrak{p}^{-1}N)^j(\xi + Nu(r)\right) \overline{\hat{\psi}_\ell^*\big(\xi + Nu(r)\big)} = 0, \quad \text{for a.e. } \xi \in K, j \in \mathbb{Z}.$$

Therefore, we have

$$\begin{aligned}
&\sum_{\lambda\in\Lambda} \hat{\psi}_\ell\left((\mathfrak{p}^{-1}N)^j(\xi+\lambda\right) \overline{\hat{\psi}_\ell\big(\xi+\lambda\big)} \\
&\qquad = \left\{ \sum_{\lambda\in\Lambda} \left| \hat{\psi}_\ell\left((\mathfrak{p}^{-1}N)^j(\xi-\lambda\right)\right|^2 \right\}^{1/2} \left\{ \sum_{\lambda\in\Lambda} \left| \hat{\psi}_\ell\big(\xi-\lambda\big)\right|^2 \right\}^{1/2} \\
&\qquad\qquad \times \sum_{k\in\mathbb{N}_0} \hat{\psi}_\ell^*\left((\mathfrak{p}^{-1}N)^j(\xi+\lambda\right) \overline{\hat{\psi}_\ell^*\big(\xi+\lambda\big)} = 0.
\end{aligned}$$

Hence, condition (3.4) is satisfied.

To show (c) implies (a), we use condition (2.6) which shows that the nonuniform affine system $W(\Psi, j, \lambda)$ constitutes a frame for W_j and identity (3.4) shows that W_0 is orthogonal to W_j for $j \neq 0$. Therefore, by the change of variables, we have

$$\langle \psi_{\ell,j,\lambda}, \psi_{\ell,n,\sigma} \rangle = \left\langle \psi_{\ell,0,\lambda-(qN)^{1-j}\sigma}, \psi_{\ell,n-j,0} \right\rangle, \quad 1 \leq \ell \leq qN-1, \lambda, \sigma \in \Lambda.$$

It is immediate from the above relation that $W_j \perp W_n$ for $j \neq n$. Thus, we conclude that $W(\Psi, j, \lambda)$ forms a frame for $W = \overline{\text{span}}\{\psi_{\ell,j,\lambda} : 1 \leq \ell \leq qN-1, j \in \mathbb{Z}, \lambda \in \Lambda\big\}$. Next, we claim that $W = L^2(K)$. In order to prove this, it is sufficient to show that the system $W(\Psi^*, j\lambda)$ given by equation (3.1) constitutes a frame for $L^2(K)$. To do so, we set $\Omega_0 = \big\{f \in \Omega : \sup \hat{f} \subset K \setminus \{0\}\big\}$ which is also dense in $L^2(K)$. So it is sufficient to say

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that $\left\{\psi^*_{\ell,j,\lambda} : 1 \le \ell \le qN-1, j \in \mathbb{Z}, \lambda \in \Lambda\right\}$ is a wavelet frame for $L^2(K)$ if the frame condition holds for all $f \in \Omega_0$.

Let f be in Ω_0 and $\{\psi^*_1, \psi^*_2, \dots, \psi^*_L\} \subset L^2(K)$. Applying Parseval's formula, we obtain

$$\begin{aligned}
\sum_{j\in\mathbb{Z}}\sum_{\lambda\in\Lambda}\left|\left\langle f, \psi^*_{\ell,j,\lambda}\right\rangle\right|^2 &= \sum_{j\in\mathbb{Z}}\sum_{\lambda\in\Lambda}\left|\left\langle \hat{f}, \hat{\psi}^*_{\ell,j,\lambda}\right\rangle\right|^2 \\
&= \sum_{j\in\mathbb{Z}}\sum_{\lambda\in\Lambda}(qN)^{j/2}\left|\int_K \hat{f}(\xi)\,\overline{\hat{\psi}^*_\ell\left(\frac{\xi}{(\mathfrak{p}^{-1}N)^j}\right)}\,\overline{\chi_\lambda\left(\frac{\xi}{(\mathfrak{p}^{-1}N)^j}\right)}\right|^2 d\xi \\
&= \sum_{j\in\mathbb{Z}}(qN)^{-j/2}\sum_{\lambda\in\Lambda}\left|\int_K \hat{f}\left((\mathfrak{p}^{-1}N)^j\xi\right)\,\overline{\hat{\psi}^*_\ell(\xi)}\chi_\lambda(\xi)\right|^2 d\xi \\
&= \sum_{j\in\mathbb{Z}}(qN)^{-j/2}\int_{N\mathfrak{D}}\left|\sum_{s\in\mathbb{N}_0}\hat{f}\left((\mathfrak{p}^{-1}N)^j(\xi+Nu(s)\right)\,\overline{\hat{\psi}^*_\ell(\xi+u(s))}\right|^2 d\xi \\
&= \sum_{j\in\mathbb{Z}}(qN)^{-j/2}\int_{N\mathfrak{D}}\left|R^\ell_j(\xi)\right|^2 d\xi, \qquad (3.8)
\end{aligned}$$

where

$$R^\ell_j(\xi) = \sum_{s\in\mathbb{N}_0}\hat{f}\left((\mathfrak{p}^{-1}N)^j(\xi+Nu(s)\right)\,\overline{\hat{\psi}^*_\ell\big(\xi+Nu(s)\big)}, \quad 1 \le \ell \le qN-1.$$

Moreover, we have

$$\begin{aligned}
\int_{N\mathfrak{D}}\left|R^\ell_j(\xi)\right|^2 d\xi &= \int_{N\mathfrak{D}}\left|\sum_{s\in\mathbb{N}_0}\hat{f}\left((\mathfrak{p}^{-1}N)^j(\xi+Nu(s)\right)\,\overline{\hat{\psi}^*_\ell\big(\xi+Nu(s)\big)}\right|^2 d\xi \\
&\le \int_{N\mathfrak{D}}\sum_{s\in\mathbb{N}_0}\left|\hat{f}\left((\mathfrak{p}^{-1}N)^j(\xi+Nu(s)\right)\,\overline{\hat{\psi}^*_\ell\big(\xi+Nu(s)\big)}\right|^2 d\xi \\
&= \sum_{s\in\mathbb{N}_0}\int_{N\mathfrak{D}}\left|\hat{f}\left((\mathfrak{p}^{-1}N)^j(\xi+Nu(s)\right)\,\overline{\hat{\psi}^*_\ell\big(\xi+Nu(s)\big)}\right|^2 d\xi
\end{aligned}$$

$$\begin{aligned}
&= \int_K\left|\hat{f}\left((\mathfrak{p}^{-1}N)^j\xi\right)\,\overline{\hat{\psi}^*_\ell(\xi)}\right|^2 d\xi \\
&\le \left(\int_K\left|\hat{f}\left((\mathfrak{p}^{-1}N)^j\xi\right)\right|^2 d\xi\right)^{1/2}\left(\int_K\left|\hat{\psi}^*_\ell(\xi)\right|^2 d\xi\right)^{1/2}.
\end{aligned}$$

Since both f and $\hat{f}$ are compactly supported and each ψ^*_ℓ lies in $L^2(K)$, it follows from the above inequality that $R^\ell_j(\xi) \in L^2(\mathfrak{D})$. Also, for any $j \in \mathbb{Z}$, we have

$$\left|F^\ell_j(\xi)\right|^2 \le \sum_{s\in\mathbb{N}_0}\left|\hat{f}\left((\mathfrak{p}^{-1}N)^j(\xi+Nu(s)\right)\right|^2\sum_{s\in\mathbb{N}_0}\left|\hat{\psi}^*_\ell\left(\xi+Nu(s)\right)\right|^2, \quad 1 \le \ell \le qN-1,$$

and

$$\int\limits_{K} \hat{f}\left((\mathfrak{p}^{-1}N)^{j}\xi\right)\overline{\hat{\psi}_{\ell}^{*}(\xi)}\,\chi_{\lambda}(\xi)\,d\xi = \int\limits_{N\mathfrak{D}} R_{j}^{\ell}(\xi)\,\chi_{\lambda}(\xi)\,d\xi.$$

By applying Plancheral's formula, we get

$$\sum_{\lambda\in\Lambda}\left|\int\limits_{N\mathfrak{D}} R_{j}^{\ell}(\xi)\chi_{\lambda}(\xi)\,d\xi\right|^{2} = \int\limits_{N\mathfrak{D}} \left|R_{j}^{\ell}(\xi)\right|^{2}d\xi.$$

Hence equation (3.8) becomes

$$\begin{aligned}
\sum_{j\in\mathbb{Z}}\sum_{\lambda\in\Lambda}\left|\left\langle f,\psi_{\ell,j,\lambda}^{*}\right\rangle\right|^{2} &= \sum_{j\in\mathbb{Z}}(qN)^{-j/2}\int\limits_{N\mathfrak{D}}\left|\sum_{s\in\mathbb{N}_0}\hat{f}\left((\mathfrak{p}^{-1}N)^{j}(\xi+Nu(s)\right)\overline{\hat{\psi}_{\ell}^{*}(\xi+Nu(s))}\right|^{2}d\xi\\
&= \sum_{j\in\mathbb{Z}}(qN)^{-j/2}\int\limits_{N\mathfrak{D}}\sum_{s\in\mathbb{N}_0}\overline{\hat{f}\left((\mathfrak{p}^{-1}N)^{j}(\xi+Nu(s)\right)}\hat{\psi}_{\ell}^{*}(\xi+Nu(s))\\
&\qquad\times\sum_{s\in\mathbb{N}_0}\hat{f}\left((\mathfrak{p}^{-1}N)^{j}(\xi+Nu(s)\right)\overline{\hat{\psi}_{\ell}^{*}(\xi+Nu(s))}d\xi\\
&= \sum_{j\in\mathbb{Z}}(qN)^{-j/2}\int\limits_{K}\overline{\hat{f}\left((\mathfrak{p}^{-1}N)^{j}\xi\right)}\hat{\psi}_{\ell}^{*}(\xi)\sum_{s\in\mathbb{N}_0}\hat{f}\left((\mathfrak{p}^{-1}N)^{j}(\xi+Nu(s)\right)\\
&\qquad\times\overline{\hat{\psi}_{\ell}^{*}(\xi+Nu(s))}d\xi\\
&= \int\limits_{K}\left|\hat{f}(\xi)\right|^{2}\sum_{j\in\mathbb{Z}}\left|\hat{\psi}_{\ell}^{*}\left((\mathfrak{p}^{-1}N)^{j}\xi\right)\right|^{2}d\xi+H(f),
\end{aligned}\tag{3.9}$$

where

$$H(f)=\sum_{j\in\mathbb{Z}}\int\limits_{K}\overline{\hat{f}(\xi)}\,\hat{\psi}_{\ell}^{*}\left(\frac{\xi}{(\mathfrak{p}^{-1}N)^{j}}\right)\sum_{s\in\mathbb{N}_0}\hat{f}\left(\xi+\frac{Nu(s)}{(\mathfrak{p}^{-1}N)^{j}}\right)\overline{\hat{\psi}_{\ell}^{*}\left(\frac{\xi}{(\mathfrak{p}^{-1}N)^{j}}+Nu(s)\right)}d\xi. \tag{3.10}$$

We now estimate $H(f)$ by means of the decomposition of the set $\{u(s): s\in\Lambda\}$. For given $s\in\Lambda$, there is a unique pair (λ,σ) with $\lambda\in\Lambda$ and $\sigma\in\Lambda\setminus qN\Lambda$, such that $s=(qN)^{\lambda}\sigma$. Therefore, we have $\{u(s): s\in\Lambda\}=\left\{\left((\mathfrak{p}^{-1}N)^{k}\right)u(\sigma)\right\}$. Since the series $H(f)$ is absolutely convergent, we can estimate $H(f)$ by rearranging the series, changing the orders of summation and integration by Levi

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Lemma as follows:

$$\begin{aligned}
H(f) &= \sum_{j\in\mathbb{Z}}\int_K \overline{\hat{f}(\xi)}\,\hat{\psi}_\ell^*\left(\frac{\xi}{(\mathfrak{p}^{-1}N)^j}\right)\left\{\sum_{s\in\mathbb{N}_0}\hat{f}\left(\xi+\left(\frac{u(s)}{(\mathfrak{p}^{-1}N)^j}\right)\right)\overline{\hat{\psi}_\ell^*\left(\left(\left(\frac{\xi}{(\mathfrak{p}^{-1}N)^j}\right)+u(s)\right)\right)}\right\}d\xi\\
&= \sum_{j\in\mathbb{Z}}\int_K \overline{\hat{f}(\xi)}\left\{\sum_{\lambda\in\Lambda}\sum_{\sigma\in\Lambda\setminus qN\Lambda}\hat{\psi}_\ell^*\left(\frac{\xi}{(\mathfrak{p}^{-1}N)^j}\right)\hat{f}\left(\xi+\left(\frac{u\left((qN)^\lambda\sigma\right)}{(\mathfrak{p}^{-1}N)^j}\right)\right)\right.\\
&\qquad\left.\times\overline{\hat{\psi}_\ell^*\left(\left(\left(\frac{\xi}{(\mathfrak{p}^{-1}N)^j}\right)+u\left((qN)^\lambda\sigma\right)\right)\right)}\right\}d\xi\\
&= \sum_{j\in\mathbb{Z}}\int_K \overline{\hat{f}(\xi)}\left\{\sum_{\lambda\in\Lambda}\sum_{\sigma\in\Lambda\setminus qN\Lambda}\hat{\psi}_\ell^*\left(\frac{\xi}{(\mathfrak{p}^{-1}N)^j}\right)\hat{f}\left(\xi+\left(\mathfrak{p}^{-1}N\right)^{\lambda-j}u(\sigma)\right)\right.\\
&\qquad\left.\times\overline{\hat{\psi}_\ell^*\left(\frac{\xi}{(\mathfrak{p}^{-1}N)^j}+(\mathfrak{p}^{-1}N)^\lambda u(\sigma)\right)}\right\}d\xi\\
&= \int_K \overline{\hat{f}(\xi)}\left\{\sum_{\lambda\in\Lambda}\sum_{\sigma\in\Lambda\setminus qN\Lambda}\sum_{j\in\mathbb{Z}}\hat{\psi}_\ell^*\left(\left(\frac{\xi}{(\mathfrak{p}^{-1}N)^{j-\lambda}}\right)\right)\hat{f}\left(\xi+(\mathfrak{p}^{-1}N)^j u(\sigma)\right)\right.\\
&\qquad\left.\times\overline{\hat{\psi}_\ell^*\left(\left(\left(\frac{\xi}{(\mathfrak{p}^{-1}N)^{j-\lambda}}\right)+(\mathfrak{p}^{-1}N)^\lambda u(\sigma)\right)\right)}\right\}d\xi\\
&= \int_K \overline{\hat{f}(\xi)}\left\{\sum_{j\in\mathbb{Z}}\sum_{\sigma\in\Lambda\setminus qN\Lambda}\hat{f}\left(\xi+(\mathfrak{p}^{-1}N^j u(\sigma)\right)\sum_{\lambda\in\Lambda}\hat{\psi}_\ell^*\left(\frac{\xi}{(\mathfrak{p}^{-1}N)^{j-\lambda}}\right)\right.\\
&\qquad\left.\times\overline{\hat{\psi}_\ell^*\left((\mathfrak{p}^{-1}N)^\lambda\left(\frac{\xi}{(\mathfrak{p}^{-1}N)^j}+u(\sigma)\right)\right)}\right\}d\xi\\
&= \sum_{j\in\mathbb{Z}}\sum_{\sigma\in\Lambda\setminus qN\Lambda}\int_K \overline{\hat{f}(\xi)}\hat{f}\left(\xi+(\mathfrak{p}^{-1}N)^j u(\sigma)\right)\sum_{\lambda\in\Lambda}\hat{\psi}_\ell^*\left(\frac{\xi}{(\mathfrak{p}^{-1}N)^{j-\lambda}}\right)\\
&\qquad\times\overline{\hat{\psi}_\ell^*\left((\mathfrak{p}^{-1}N)^\lambda\left(\frac{\xi}{(\mathfrak{p}^{-1}N)^j}+u(\sigma)\right)\right)}d\xi.
\end{aligned}$$

Using the above estimate of $H(f)$ in equation (3.9), we obtain

$$\begin{aligned}
\sum_{\ell=1}^{qN-1}\sum_{j\in\mathbb{Z}}\sum_{\lambda\in\Lambda}\left|\left\langle f,\psi_{\ell,j,\lambda}^*\right\rangle\right|^2 &= \int_K\left|\hat{f}(\xi)\right|^2\sum_{\ell=1}^{qN-1}\sum_{j\in\mathbb{Z}}\left|\hat{\psi}_\ell^*\left(\frac{\xi}{(\mathfrak{p}^{-1}N)^j}\right)\right|^2 d\xi\\
&\quad+\sum_{j\in\mathbb{Z}}\sum_{\sigma\in\Lambda\setminus qN\Lambda}\int_K\overline{\hat{f}(\xi)}\hat{f}\left(\xi+(\mathfrak{p}^{-1}N)^j u(\sigma)\right)\sum_{\ell=1}^{qN-1}\sum_{\lambda\in\Lambda}\\
&\quad\times\hat{\psi}_\ell^*\left((\mathfrak{p}^{-1}N)^{\lambda-j}\xi\right)\Big)\overline{\hat{\psi}_\ell^*\left((\mathfrak{p}^{-1}N)^\lambda\left(\frac{\xi}{(\mathfrak{p}^{-1}N)^j}+u(\sigma)\right)\right)}d\xi. \quad (3.11)
\end{aligned}$$

Implementation of equations (3.3) and (3.7) in (3.11) yields

$$A/B\|f\|_2^2\leq\sum_{\ell=1}^{qN-1}\sum_{j\in\mathbb{Z}}\sum_{\lambda\in\Lambda}\left|\left\langle f,\psi_{\ell,j,\lambda}^*\right\rangle\right|^2\leq B/A\|f\|_2^2.$$

It follows that the system $\left\{\psi_{\ell,j,\lambda}^*:1\leq\ell\leq qN-1,j\in\mathbb{Z},\lambda\in\Lambda\right\}$ is a frame for $L^2(K)$ and hence, we obtain the desired result. □

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