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Darbo Type Best Proximity Point Results via $\mathcal{R}$-function using Measure of Noncompactness with an Application

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Abstract. The concept of measure of noncompactness (MNC) allows us to select an important class of mappings which are more general than compact operators. The proposed work exploits the axiomatic definition of MNC and introduces the notion of relatively nonexpansive cyclic (noncyclic) $\mathcal{SR}$-condensing operators along with the aid of $\mathcal{SR}$-functions. The first phase of the paper concentrates on establishing the best proximity point (pair) theorems for such operators. The main results in this manuscript extend and generalize several state of art literature on Darbo type fixed point results. In the second phase, proposed results are applied to show the actuality of optimum solutions for system of second order differential equations with two initial conditions.

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1. Introduction

Fixed point theory has grown up into a rich literature and got utmost importance in some recent decades as the theory is utilized in proving various results in branches of mathematical sciences. As the area of applicability got wider, the new concepts and ideas emerged with the time and made the theory more popular and attractive for the workers in area of research. Banach contraction principle (BCP) and Schauder fixed point theorem (SFPT) for compact operators are the two pioneering results in fixed point theory which attract the highest citations and applications, and remain a constant source of inspiration for large number of generalizations in the theory. Before stating these two results, we recall that, a point u in a set S is a fixed point of a function $T : S \to S$ if $Tu = u$ is satisfied.

Theorem 1.1 (Banach)**.** *Let (S, m) be a complete metric space. Then a mapping $T : S \to S$ admits a unique fixed point provided T is a contraction map, that is, for each $a, b \in S$ there exists $0 \leq \lambda < 1$ such that $m(Ta, Tb) \leq \lambda\, m(a, b)$.*

Theorem 1.2 (Schauder)**.** *A compact (self) operator on a bounded, closed and convex (nonempty) subset of a Banach space admits a fixed point.*

These two results forced to constitute a major part of the literature through a lot of extensions and generalizations. We recall some of the notable ideas due to which major breakthrough occurred in this doctrine of research.

1.1. **Concepts used to generalize BCP.** In order to generalize BCP, various contractive conditions using auxiliary functions have been introduced by several authors. We recall some of them. One of the extension of BCP is due to Meir-Keeler [16], which attracted lot of attention.

Definition 1.3. [16] A self mapping T on a metric space (S, m) is called a Meir-Keeler contraction if for each $\epsilon > 0$ there exists $\delta > 0$ such that for each $a, b \in S$, we have

$$\epsilon \leq m(a, b) < \epsilon + \delta \text{ implies } m(Ta, Tb) < \epsilon.$$

Though this definition is not dependent on any auxiliary function, but Lim in [15] proved its equivalence with following concept called L function.

Definition 1.4. [15] A mapping $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\varphi(p) > 0$ for each $p > 0$, $\varphi(0) = 0$ and satisfying the condition

$$\text{for each } \epsilon > 0, \text{ there exists } \delta > 0 \text{ such that } \varphi(p) \leq \epsilon, \text{ for all } p \in [\epsilon, \epsilon + \delta],$$

is called an L-function.

In 2015, a novel notion called as 'simulation function' is brought into the doctrine of fixed points by Khojasteh et al. [14]. However, R.-L.-de-Hierro and Samet [20] modified this notion slightly and enlarged the class of simulation

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functions. Later, Argoubi et al. [2] found that the first condition is redundant in definition of simulation function in [14, Definition 2.1] and redefined the notion by removing first condition, which we present here.

Definition 1.5. [2] A mapping $\zeta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ satisfying

(ζ_1) $\zeta(t_1, t_2) < t_2 - t_1$ for all $t_1, t_2 > 0$,

(ζ_2) if there are two sequences $\{s_j\}$ and $\{t_j\}$ in $\mathbb{R}^+\backslash\{0\}$ such that $\lim_{j\to\infty} s_j = \lim_{j\to\infty} t_j > 0$ and $t_j < s_j$ then $\limsup_{j\to\infty} \zeta(t_j, s_j) < 0$,

is called a simulation function. For examples of simulation function, refer to [2, 7, 14].

Recently, in order to extend the concept of simulation function and Meir-Keeler contractions, R.-L.-de-Hierro and Sahzad [21] coined the new concept and called it as $\mathcal{R}$-function. Let $\mathcal{A}^+ = \mathcal{A} \cap (0, \infty)$, where $\mathcal{A}$ is a nonempty subset of set of real numbers.

Definition 1.6. [21] A mapping $\rho : \mathcal{A} \times \mathcal{A} \to \mathbb{R}$ is called an $\mathcal{R}$-function if ρ satisfies the following conditions:

(R_1) if a sequence $\{s_j\} \subset \mathcal{A}^+$ satisfies $\rho(s_{j+1}, s_j) > 0$, for all $j \in \mathbb{N}$, then $\{s_j\} \to 0$,

(R_2) if there are two sequences $\{s_j\}$, $\{t_j\}$ in $\mathcal{A}^+$ such that $\lim_{j\to\infty} s_j = \lim_{j\to\infty} t_j = l \geq 0$ with $l < s_j$ and $\rho(s_j, t_j) > 0$, for all $j \in \mathbb{N}$ then $l = 0$.

Let $\mathcal{R}_\mathcal{A}$ denotes the class of all $\mathcal{R}$-functions with domain $\mathcal{A} \times \mathcal{A}$. For examples of $\mathcal{R}$-functions refer to [21].

Following property is also considered on $\mathcal{R}$-functions:

(R_3) if there are two sequences $\{s_j\}$ and $\{t_j\}$ in $\mathcal{A}^+$ such that $\{t_j\} \to 0$ and $\rho(s_j, t_j) > 0$, for all $j \in \mathbb{N}$, then $\{s_j\} \to 0$.

Definition 1.7. [21] A self mapping T on a metric space (S, m) is called an $\mathcal{R}$-contraction if for each $a, b \in S$ there is a $\rho \in \mathcal{R}_\mathcal{A}$ such that range of m is contained in $\mathcal{A}$ and

$$\rho(m(Ty, Tx), m(y, x)) > 0, \ x \neq y.$$

Remark 1.8. (1) Collection of $\mathcal{R}$-functions with (R_3) contains class of all simulation functions [21].
(2) Meir-Keeler contraction can't be covered with Z-contractions which use simulation function [13].
(3) The class of Meir-Keeler contraction is contained in class of R-contractions [21].

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1.2. **Generalizations of Schauder FPT.** SFPT (Theorem 1.2) is very often used in proving existence of solutions to problems related to partial and ordinary differential equations, but still there are lot of generalizations and extensions of this result appeared in literature as compactness is quite strong condition. Darbo [8] and Sadovskii [22] obtain one of the important improvement of SFPT using the concept of measure of noncompactness (MNC). Before going into details about these generalizations we will recall the important notion of measure of noncompactness. We present axiomatic definition of MNC here. Let $(\mathcal{S}, m)$ be a metric space. We use following notations throughout this article.

$\mathbb{R}$: set of real numbers,
$\mathbb{N}$: set of natural numbers,
$\mathcal{B}(\omega, \gamma)$: closed ball of radius γ with center ω,
$\overline{\mathcal{D}}$: closure of the set $\mathcal{D}$,
$\overline{con}(\mathcal{D})$: convex and closed hull of $\mathcal{D}$,
$diam(\mathcal{D})$: diameter of the set $\mathcal{D}$,
$\mathbb{B}(\mathcal{S})$: collection of bounded subsets in metric space $\mathcal{S}$.

Definition 1.9. [3, 5] An MNC is a mapping $\aleph : \mathbb{B}(\mathcal{S}) \to \mathbb{R}^+$ satisfying the following axioms:

(1) $\aleph(P) = 0$ if and only if P is relatively compact,
(2) $\aleph(P) = \aleph(\overline{P})$, $P \in \mathbb{B}(\mathcal{X})$,
(3) $\aleph(P \cup Q) = \max\{\aleph(P), \aleph(Q)\}$, where $P, Q \in \mathbb{B}(\mathcal{X})$.

An MNC $\aleph$ on $\mathbb{B}(\mathcal{S})$ satisfies following properties.

(a) $P \subset Q$ implies $\aleph(P) \leq \aleph(Q)$.
(b) $\aleph(P) = 0$ if P is a finite set.
(c) $\aleph(P \cap Q) = \min\{\aleph(P), \aleph(Q)\}$, for all $P, Q \in \mathbb{B}(\mathcal{X})$.
(d) If $\lim_{n\to\infty} \aleph(P_n) = 0$ for a nonincreasing sequence $\{P_n\}$ of bounded and closed (nonempty) subsets of $\mathcal{X}$, then $P_\infty = \cap_{n\geq 1} P_n$ is compact (nonempty).

On a Banach space $\mathcal{S}$, $\aleph$ has following properties.

(i) $\aleph(\overline{con}(Q)) = \aleph(Q)$, for all $Q \in \mathbb{B}(\mathcal{X})$.
(ii) $\aleph(\lambda Q) = |\lambda|\aleph(Q)$ for any number λ and $Q \in \mathbb{B}(\mathcal{X})$.
(iii) $\aleph(P + Q) \leq \aleph(P) + \aleph(Q)$.

Example 1.10. [4] The non-negative numbers

$$\alpha(\mathcal{C}) = \inf\{r > 0 : P \subset \cup_{i=1}^{N} S_i,\ diam(S_i) \leq r,\ i = 1, 2, \cdots N\}$$

and

$$\beta(\mathcal{C}) = \inf\{r > 0 : \mathcal{C} \subset \cup_{i=1}^{N} B(x_i, r),\ x_i \in \mathcal{X},\ i = 1, ..., N\},$$

assigned with a bounded subset $\mathcal{C}$ of a metric space $\mathcal{S}$ are called Kuratowski MNC (K-MNC) and Hausdorff MNC (H-MNC) respectively.

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The MNC acquired great importance due to its applicability in operator theory. One of the significant fact is that MNC eased the work of selecting a very important class of mappings which are more general than compact operators. The classical results generalizing the Theorem 1.2 using non-compactness measure are due to Darbo [8] and Sadovskii [22]. We present their combine statement in Theorem 1.11. In what follows the term $\mathcal{NBCC}$ set means Nonempty, Bounded, Convex and Closed set and *nls* means Normed Linear Space.

Theorem 1.11. [8, 22] *A continuous self mapping T on a $\mathcal{NBCC}$ subset $\mathcal{C}$ of a Banach space $\mathcal{S}$, for every $\mathcal{M} \subset \mathcal{C}$ satisfying one of the following*

(D) $\exists\ 0 \leq \lambda < 1$ *such that* $\aleph(T(\mathcal{M})) \leq \lambda\ \aleph(\mathcal{M})$,
(S) $\aleph(\mathcal{M}) > 0$, $\aleph(T(\mathcal{M})) < \aleph(\mathcal{M})$,

admits a fixed point.

A mapping satisfying condition (D) is called λ-set contraction (due to Darbo [8]) whereas satisfying (S) is called as $\aleph$-condensing (due to Sadovskii [22]).

Aghajani et al. [1] coined the notion of Meir-Keeler (M-K) condensing operator and obtained the fixed point results for these operators which generalize fixed point theorem of Darbo.

Definition 1.12. A self mapping T on a nonempty subset $\mathcal{C}$ of a Banach space $\mathcal{S}$ is called M-K condensing if for each $\epsilon > 0$ there exists $\delta > 0$ such that for any bounded subset $\mathcal{M}$ of Ω, we have

$$\epsilon \leq \aleph(\mathcal{M}) < \epsilon + \delta \Longrightarrow \aleph(T(\mathcal{M}) < \epsilon.$$

Since Meir-Keeler condensing mapping can be characterized using L function so another analogous version of Definition 1.12 can be obtained with the help of L-function. We skip to define it.

Chen and Tang [7] defined the notion of $Z_\aleph$ contraction using simulation function and obtained the fixed point theorem which generalizes various Darbo type fixed point results. The statement can be given as:

Theorem 1.13. A continuous mapping (self) T which is a $Z_\aleph$ contraction, that is, T satisfies

$$\zeta(\aleph(T(\mathcal{M})), \aleph(\mathcal{M})) \geq 0,$$

where ζ is a simulation function and $\mathcal{M} \subseteq \mathcal{C}$ is nonempty, defined on a $\mathcal{NBCC}$ subset $\mathcal{C}$ of a Banach space $\mathcal{S}$, admits a fixed point.

Moreover, Patle and Patel [18] proved the Krasnoselskii type fixed point result for sum of a compact operator with a Z-contraction.

In recent advancement, Zarinfar et al. [23] defined the notion of $\mathcal{SR}_\aleph$ contractions using $\mathcal{SR}$-function and obtained the fixed point theorem which generalizes Darbo type fixed point results.

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Definition 1.14. $\mathcal{SR}$-function is a mapping $\rho : \mathcal{A} \times \mathcal{A} \to \mathbb{R}$ satisfying property (R_1) of Definition 1.6. The class of all $\mathcal{SR}$-functions with domain $\mathcal{A} \times \mathcal{A}$ is denoted by $\mathcal{SR}_{\mathcal{A}}$.

Theorem 1.15. *A continuous mapping (self) T which is a $SR_{\aleph}$ contraction, that is, T satisfies*

$$\rho(\aleph(T(\mathcal{M})), \aleph(\mathcal{M})) \geq 0,$$

where ρ is SR-function and $\mathcal{M} \subseteq \mathcal{C}$ is nonempty, defined on a $\mathcal{NBCC}$ subset $\mathcal{C}$ of a Banach space $\mathcal{S}$, admits a fixed point.

Very recently Gabeleh and Markin [11] initiated to study the case of best proximity points in the absence of fixed points for cyclic relatively condensing operators. Before going into detail of this result, we recall the concept of best proximity points.

Let us take two subsets (nonempty) P and Q of an *nls* $\mathcal{S}$. Assume that a pair (P, Q) satisfies a property, if P and Q individually satisfy that property, e.g, we say a pair (P, Q) is compact if and only if P and Q are compact. We define distance between two sets P and Q as,

$$dist(P, Q) = \inf\{\|a - b\| : a \in P,\ b \in Q\}.$$

For the pair (P, Q), let us define

$$P_0 = \{a \in P :\ \exists\, b' \in Q \mid \|a - b'\| = dist(P, Q)\},$$

$$Q_0 = \{b \in Q :\ \exists\, a' \in P \mid \|a' - b\| = dist(P, Q)\}.$$

In Banach space $\mathcal{S}$, (P_0, Q_0) is convex and weakly compact (nonempty) pair if (P, Q) is convex and weakly compact (nonempty). If $P = P_0$ and $Q = Q_0$ then the pair (P, Q) of nonempty subsets in an *nls* $\mathcal{S}$ is called proximinal.

A mapping $T : P \cup Q \to P \cup Q$ is called cyclic if T maps P into Q and Q into P whereas if $T(P) \subseteq P$ and $T(Q) \subseteq Q$ then it is called noncyclic. T is called relatively nonexpansive if $\|Ta - Tb\| \leq \|a - b\|$ holds, whenever $a \in P$ and $b \in Q$. T is called nonexpansive mapping (self) if $P = Q$. We consider a best proximity point for a cyclic mapping T, which is defined as, a point $w^* \in P \cup Q$ satisfying

$$\|w^* - Tw^*\| = dist(P, Q).$$

In case of a noncyclic mapping T we consider existence of a pair $(b, a) \in (P, Q)$ for which $a = Ta$, $b = Tb$ and $\|a - b\| = dist(P, Q)$. Such pairs are called best proximity pairs.

Eldred et al. in [9] coined the idea of cyclic (noncyclic) relatively nonexpansive mappings and obtained the best proximity point (pair) results in Banach spaces. In doing so, they have used the concept which is called as proximal normal structure (in short, PNS). In 2017, Gabeleh [10] proved that every convex and compact (nonempty) pair in a Banach space has PNS. Considering

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this fact Gabeleh obtains following result. Recall that $T : P \cup Q \to P \cup Q$ is compact means $(\overline{T(P)}, \overline{T(Q)})$ is compact.

Theorem 1.16. [10] *A relatively nonexpansive cyclic mapping $T : P \cup Q \to P \cup Q$ admits a best proximity point provided T is compact and P_0 is nonempty, where (P, Q) is a $\mathcal{NBCC}$ pair in a Banach space $\mathcal{S}$.*

Before stating the result for noncyclic mappings, let us recall a mathematical concept of strictly convex Banach space. A Banach space $\mathcal{S}$ is strictly convex if for $p, q, r \in \mathcal{S}$ and $\Lambda > 0$,

$$\big[\|p - r\| \leq \Lambda,\ \|q - r\| \leq \Lambda,\ p \neq q\big] \Rightarrow \Big\| \frac{p+q}{2} - r \Big\| < \Lambda$$

holds. The L^p space $(1 < p < \infty)$ and Hilbert space are examples of strictly convex Banach spaces.

Theorem 1.17. [10] *Let Banach space $\mathcal{S}$ be strictly convex. A relatively nonexpansive noncyclic mapping $T : P \cup Q \to P \cup Q$ admits a best proximity pair provided it is compact and P_0 is nonempty, where (P, Q) is a $\mathcal{NBCC}$ pair in $\mathcal{S}$.*

We are now in a position to state the best proximity point result for relatively nonexpansive cyclic condensing operator presented in [11].

Theorem 1.18. *A relatively nonexpansive cyclic mapping $T : P \cup Q \to P \cup Q$ which is condensing, that is, if there exists $r \in (0, 1)$ for any $\mathcal{NBCC}$ proximinal and T-invariant pair $(\mathcal{M}_1, \mathcal{M}_2) = dist(P, Q)$, such that*

$$\aleph(T(\mathcal{M}_1) \cup T(\mathcal{M}_2)) \leq r\aleph(\mathcal{M}_1 \cup \mathcal{M}_2), \tag{1.1}$$

admits a best proximity point provided P_0 is nonempty.

In the same article they have proved the existence of best proximity pairs for a relatively nonexpansive noncyclic condensing operator in a uniformly convex Banach space. In sequel, Gabeleh and Vetro [12] have published another article in which they obtained best proximity point (pair) results for relatively nonexpansive cyclic (noncyclic) Meir-Keeler condensing operators. Also in [17, 19], the best proximity point (pair) results have been obtained for some different classes of cyclic (concyclic) pair of mappings using measure of noncompactness.

In this article, first in Section 2 we present the notion of relatively nonexpansive cyclic (noncyclic) $\mathcal{SR}$-condensing operators via $\mathcal{SR}$-function and prove the best proximity point (pair) theorems using the concept of measure of noncompactness. The obtained results generalize and extend results of Aghajani et al. [1], Chen and Tang [7], Darbo [8], Gabeleh and Markin [11], Gabeleh and Vetro [12], Zarinfar et al. [23], etc. In Section 3 the main results are applied to actualize the optimum solutions of a system of second order differential equations with two initial conditions.

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2. Main results

Let us enunciate with the following concept of cyclic (noncyclic) $\mathcal{SR}$-condensing operator. Consider, $\mathcal{S}$ a Banach space, $\aleph$ be an MNC on $\mathcal{S}$ and P, Q be nonempty and convex subsets of $\mathcal{S}$, throughout this section.

Definition 2.1. An operator $T : P \cup Q \to P \cup Q$ is called a cyclic (noncyclic) $\mathcal{SR}$-condensing if for each $\mathcal{NBCC}$, proximinal and T invariant pair $(\mathcal{M}_1, \mathcal{M}_2) \subseteq (P, Q)$ with $dist(\mathcal{M}_1, \mathcal{M}_2) = dist(P, Q)$ there is a function $\rho \in \mathcal{SR}_A$ such that

$$\rho(\aleph(T(\mathcal{M}_1) \cup T(\mathcal{M}_2)), \aleph(\mathcal{M}_1 \cup \mathcal{M}_2)) \geq 0.$$

Following theorem for relatively nonexpansive cyclic $\mathcal{SR}$-condensing operator is our first main result. Some part of the proof is adopted from [12], for the sake of completeness we are giving complete proof. In sequel we consider (P, Q), a $\mathcal{NBCC}$ pair in $\mathcal{S}$.

Theorem 2.2. *A relatively nonexpansive cyclic $\mathcal{SR}$-condensing operator T : $P \cup Q \to P \cup Q$ admits a best proximity point provided P_0 is nonempty.*

Proof. Clearly (P_0, Q_0) is nonempty because P_0 is nonempty. Taking into account the conditions on T, one can also show that (P_0, Q_0) is convex, closed, T-invariant and proximinal pair. For $a \in P_0$, there is a $b \in Q_0$ such that $\|a - b\| = dist(P, Q)$. Since T is relatively non-expansive

$$\|Ta - Tb\| \leq \|a - b\| = dist(P, Q),$$

which gives $Ta \in Q_0$, that is, $T(P_0) \subseteq Q_0$. Similarly, $T(Q_0) \subseteq P_0$ and so T is cyclic on $P_0 \cup Q_0$.

Let us define a pair (G_n, H_n) as $G_n = \overline{con}(T(G_{n-1}))$ and $H_n = \overline{con}(T(H_{n-1}))$, $n \geq 1$ with $G_0 = P_0$ and $H_0 = Q_0$. We claim that $G_{n+1} \subseteq H_n$ and $H_n \subseteq G_{n-1}$ for all $n \in \mathbb{N}$. We have $H_1 = \overline{con}(T(H_0)) = \overline{con}(TQ_0)) = \overline{con}(P_0) \subseteq P_0 = G_0$. Therefore, $T(H_1) \subseteq T(G_0)$. So $H_2 = \overline{con}(T(H_1)) \subseteq \overline{con}(T(G_0)) = G_1$. Continuing this pattern, we get $H_n \subseteq G_{n-1}$ by using induction. Similarly, we can see that $G_{n+1} \subseteq H_n$ for all $n \in \mathbb{N}$. Thus $G_{n+2} \subseteq H_{n+1} \subseteq G_n \subseteq H_{n-1}$ for all $n \in \mathbb{N}$. Hence, we get a decreasing sequence $\{(G_{2n}, H_{2n})\}$ of closed and convex (nonempty) pairs in $P_0 \times Q_0$. Moreover, $T(H_{2n}) \subseteq T(G_{2n-1}) \subseteq \overline{con}(T(G_{2n-1})) = G_{2n}$ and $T(G_{2n}) \subseteq T(H_{2n-1}) \subseteq \overline{con}(T(H_{2n-1})) = H_{2n}$. Therefore for all $n \in \mathbb{N}$, the pair (G_{2n}, H_{2n}) is T-invariant. Now if $(u, v) \in P_0 \times Q_0$ is a proximinal pair then

$$dist(G_{2n}, H_{2n}) \leq \|T^{2n}u - T^{2n}v\| \leq \|u - v\| = dist(P, Q).$$

Next, we show that the pair (G_n, H_n) is proximinal using mathematical induction. Obviously for $n = 0$, the pair (G_0, H_0) is proximinal. Suppose that (G_k, H_k) is proximinal. We show that (G_{k+1}, H_{k+1}) is also proximinal. Let x be an arbitrary member in $G_{k+1} = \overline{con}(T(G_k))$. Then it is represented as $x = \sum_{l=1}^{m} \lambda_l T(x_l)$ with $x_l \in G_k$, $m \in \mathbb{N}$, $\lambda_l \geq 0$ and $\sum_{l=1}^{m} \lambda_l = 1$. Due to the

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proximinality of the pair (G_k, H_k), there exists $y_l \in H_k$ for $1 \leq l \leq m$ such that $\|x_l - y_l\| = dist(G_k, H_k) = dist(P, Q)$. Take $y = \sum_{l=1}^{m} \lambda_l T(y_l)$. Then $y \in \overline{con}(T(H_k)) = H_{k+1}$ and

$$\|x - y\| = \| \sum_{l=1}^{m} \lambda_l T(x_l) - \sum_{l=1}^{m} \lambda_l T(y_l) \| \leq \sum_{l=1}^{m} \lambda_l \|x_l - y_l\| = dist(P, Q).$$

This means that the pair (G_{k+1}, H_{k+1}) is proximinal and mathematical induction does the rest to prove (G_n, H_n) is proximinal for all $n \in \mathbb{N}$. Now, it is understood that there arise two cases: namely either $\max\{\aleph(G_{2j}), \aleph(H_{2j})\} = 0$ for some $j \in \mathbb{N}$ or $\max\{\aleph(G_{2n}), \aleph(H_{2n})\} > 0$ for all $n \in \mathbb{N}$.
First, let $\max\{\aleph(G_{2j}), \aleph(H_{2j})\} = 0$ for some $j \in \mathbb{N}$, then $T : G_{2j} \cup H_{2j} \to G_{2j} \cup H_{2j}$ is compact, so the outcome of Theorem 1.16 yields our result.
Second, let $\max\{\aleph(G_n), \aleph(H_n)\} > 0$ for all $n \in \mathbb{N}$. As $G_{2n+1} \subseteq T(G_{2n})$ and $H_{2n+1} \subseteq T(H_{2n})$, we have

$$\begin{aligned}
&\rho\big(\aleph(G_{2n+1} \cup H_{2n+1}), \aleph(G_{2n} \cup H_{2n})\big) \\
=&\rho\big(\max\{\aleph(G_{2n+1}), \aleph(H_{2n+1})\}, \aleph(G_{2n} \cup H_{2n})\big) \\
=&\rho\big(\max\{\aleph(\overline{con}(T(G_{2n}))), \aleph(\overline{con}(T(H_{2n})))\}, \aleph(G_{2n} \cup H_{2n})\big) \\
=&\rho\big(\max\{\aleph(T(G_{2n})), \aleph(T(H_{2n}))\}, \aleph(G_{2n} \cup H_{2n})\big) \\
=&\rho\big(\aleph(T(G_{2n}) \cup T(H_{2n})), \aleph(G_{2n} \cup H_{2n})\big) \geq 0.
\end{aligned}$$

By definition of $\mathcal{SR}$-function we have

$$\lim_{n\to\infty} \aleph(G_{2n} \cup H_{2n}) = 0. \tag{2.1}$$

Also it is easy to show that $\{(\aleph(G_{2n} \cup H_{2n}))\}$ is a decreasing sequence of positive real numbers. Thus (2.1) yields, $\max\{\lim_{n\to\infty} \aleph(G_{2n}), \lim_{n\to\infty} \aleph(H_{2n})\} = 0$. Now let $G_\infty = \cap_{n=0}^{\infty} G_{2n}$ and $H_\infty = \cap_{n=0}^{\infty} H_{2n}$. By property (d) of MNC, the pair (G_∞, H_∞) is nonempty, convex, compact and T-invariant with $dist(G_\infty, H_\infty) = dist(P, Q)$. All this is sufficient to ensure that T admits a best proximity point. $\square$

We now present the second main result of the section which is analogous to the above theorem for relatively nonexpansive noncyclic $\mathcal{SR}$-condensing mapping.

Theorem 2.3. *Let Banach space $\mathcal{S}$ be strictly convex and (P, Q) be a $\mathcal{NBCC}$ pair in $\mathcal{S}$ such that P_0 is nonempty. Then every relatively nonexpansive noncyclic $\mathcal{SR}$-condensing operator $T : P \cup Q \to P \cup Q$ admits a best proximity pair.*

Proof. It is clear that (P_0, Q_0) is $\mathcal{NBCC}$ pair which is proximinal and T-invariant. Let $(p, q) \in P_0 \times Q_0$ be such that $\|p - q\| = dist(P, Q)$. As T is relatively nonexpansive noncyclic mapping

$$\|Tp - Tq\| \leq \|p - q\| = dist(P, Q),$$

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which gives $Tp \in P_0$, that is, $T(P_0) \subseteq P_0$. Similarly, $T(Q_0) \subseteq Q_0$ and so T is noncyclic on $P_0 \cup Q_0$. Let us define a pair (G_n, H_n) as $G_n = \overline{con}(T(G_{n-1}))$ and $H_n = \overline{con}(T(H_{n-1}))$, $n \geq 1$ with $G_0 = P_0$ and $H_0 = Q_0$. We have $H_1 = \overline{con}(T(H_0)) = \overline{con}(T(Q_0)) \subseteq Q_0 = H_0$. Therefore, $T(H_1) \subseteq T(H_0)$. Thus $H_2 = \overline{con}(T(H_1)) \subseteq \overline{con}(T(H_0)) = H_1$. Continuing this pattern, we get $H_n \subseteq H_{n-1}$ by using induction. Similarly we can see that $G_n \subseteq G_{n-1}$ for all $n \in \mathbb{N}$. Hence we get a decreasing sequence $\{(G_n, H_n)\}$ of closed and convex (nonempty) pairs in $P_0 \times Q_0$. Also, $T(H_n) \subseteq T(H_{n-1}) \subseteq \overline{con}(T(H_{n-1})) = H_n$ and $T(G_n) \subseteq T(G_{n-1}) \subseteq \overline{con}(T(G_{n-1})) = G_n$. Therefore for all $n \in \mathbb{N}$, the pair (G_n, H_n) is T-invariant. From the proof of Theorem 2.2, we have (G_n, H_n) is a proximinal pair such that $dist(G_n, H_n) = dist(P, Q)$ for all $n \in \mathbb{N} \cup \{0\}$. Now as the case in Theorem 2.2, there arise two situations: namely either $\max\{\aleph(G_j), \aleph(H_j)\} = 0$ for some $j \in \mathbb{N}$ or $\max\{\aleph(G_n), \aleph(H_n)\} > 0$ for every $n \in \mathbb{N}$. First, suppose $\max\{\aleph(G_j), \aleph(H_j)\} = 0$ for some $j \in \mathbb{N}$, then $T : G_j \cup H_j \to G_j \cup H_j$ is a compact. Then Theorem 1.17 does the rest to prove the theorem as T is relatively nonexpansive noncyclic mapping.

Next, we assume that $\max\{\aleph(G_n), \aleph(H_n)\} > 0$ for all $n \in \mathbb{N}$. Since $G_{n+1} \subseteq T(G_n)$ and $H_{n+1} \subseteq T(H_n)$, we have

$$\begin{aligned}
\rho\big(\aleph(G_{n+1} \cup H_{n+1}), \aleph(G_n \cup H_n)\big) &= \rho\big(\max\{\aleph(G_{n+1}), \aleph(H_{n+1})\}, \aleph(G_n \cup H_n)\big) \\
&= \rho\big(\max\{\aleph(\overline{con}(T(G_n))), \aleph(\overline{con}(T(H_n)))\}, \aleph(G_n \cup H_n)\big) \\
&= \rho\big(\max\{\aleph(T(G_n)), \aleph(T(H_n))\}, \aleph(G_n \cup H_n)\big) \\
&= \rho\big(\aleph(T(G_n) \cup T(H_n)), \aleph(G_n \cup H_n)\big) \geq 0.
\end{aligned}$$

Thus by definition of $\mathcal{SR}$-function, we get

$$\lim_{n\to\infty} \aleph(G_n \cup H_n) = 0.$$

That is, $\max\{\lim_{n\to\infty} \aleph(G_n), \lim_{n\to\infty} \aleph(H_n)\} = 0$. Now let $G_\infty = \cap_{n=0}^{\infty} G_n$ and $H_\infty = \cap_{n=0}^{\infty} H_n$. By property (d) of MNC, (G_∞, H_∞) is convex, compact and T-invariant (nonempty) pair with $dist(G_\infty, H_\infty) = dist(P, Q)$. All this is sufficient to ensures that T admits a best proximity pair. $\square$

Keeping in mind Remark 1.8 and equivalence of Meir-Keeler contraction and L-function, we get following corollaries as consequences of above results which generalize Darbo fixed point theorem.

Corollary 2.4. *A relatively nonexpansive cyclic mapping $T : P \cup Q \to P \cup Q$ which is Meir-Keeler condensing, that is, if for any $\epsilon > 0$ there exists $\delta > 0$ such that for any $\mathcal{NBCC}$ proximinal and T-invariant pair $(\mathcal{M}_1, \mathcal{M}_2)$ with $dist(\mathcal{M}_1, \mathcal{M}_2) = dist(P, Q)$, such that*

$$\epsilon \leq \aleph(\mathcal{M}_1 \cup \mathcal{M}_2) < \epsilon + \delta \Longrightarrow \aleph(T(\mathcal{M}_1) \cup T(\mathcal{M}_2)) < \epsilon, \tag{2.2}$$

admits a best proximity point provided P_0 is nonempty.

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Corollary 2.5. *Let Banach space $\mathcal{S}$ be strictly convex and (P, Q) be a $\mathcal{NBCC}$ pair in $\mathcal{S}$ such that P_0 is nonempty. Then a relatively nonexpansive noncyclic mapping $T : P \cup Q \to P \cup Q$ which is Meir-Keeler condensing, admits a best proximity pair.*

Corollary 2.6. *A relatively nonexpansive cyclic mapping $T : P \cup Q \to P \cup Q$ which is ζ-condensing that is, if for every $\mathcal{NBCC}$, proximinal and T invariant pair $(\mathcal{M}_1, \mathcal{M}_2) \subseteq (P, Q)$ with $dist(\mathcal{M}_1, \mathcal{M}_2) = dist(P, Q)$ there exists a $\zeta \in \mathcal{Z}_{ASV}$ such that*

$$\zeta(\aleph(T(\mathcal{M}_1) \cup T(\mathcal{M}_2)), \aleph(\mathcal{M}_1 \cup \mathcal{M}_2)) \geq 0.$$

admits a best proximity point provided P_0 is nonempty.

Corollary 2.7. *Let Banach space $\mathcal{S}$ be strictly convex and (P, Q) be a $\mathcal{NBCC}$ pair in $\mathcal{S}$ such that P_0 is nonempty. Then a relatively nonexpansive noncyclic ζ-condensing mapping $T : P \cup Q \to P \cup Q$ admits a best proximity pair.*

Corollary 2.8. *A relatively nonexpansive cyclic mapping $T : P \cup Q \to P \cup Q$ which is L-condensing, that is, if there exists an L-function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ and for any $\mathcal{NBCC}$, proximinal and T-invariant pair $(\mathcal{M}_1, \mathcal{M}_2) = dist(P, Q)$, such that*

$$\aleph(T(\mathcal{M}_1) \cup T(\mathcal{M}_2)) \leq \varphi(\aleph(\mathcal{M}_1 \cup \mathcal{M}_2)), \tag{2.3}$$

admits a best proximity point.

Corollary 2.9. *Let Banach space $\mathcal{S}$ be strictly convex and (P, Q) be a $\mathcal{NBCC}$ pair in $\mathcal{S}$ such that P_0 is nonempty. Then a relatively nonexpansive noncyclic mapping $T : P \cup Q \to P \cup Q$ which is φ-condensing, admits a best proximity pair.*

Corollaries 2.10 and 2.11 which are main results of Gabeleh and Markin [11] are direct consequence of Theorems 2.2 and 2.3 respectively if one takes $\rho(q, p) = rp - q$, $0 < r < 1$.

Corollary 2.10. *A relatively nonexpansive cyclic mapping $T : P \cup Q \to P \cup Q$ which is condensing, that is, if there exists $r \in (0, 1)$ for any $\mathcal{NBCC}$ proximinal and T-invariant pair $(\mathcal{M}_1, \mathcal{M}_2) = dist(P, Q)$, such that*

$$\aleph(T(\mathcal{M}_1) \cup T(\mathcal{M}_2)) \leq r\aleph(\mathcal{M}_1 \cup \mathcal{M}_2), \tag{2.4}$$

admits a best proximity point provided P_0 is nonempty.

Corollary 2.11. *Let Banach space $\mathcal{S}$ be strictly convex, (P, Q) be a $\mathcal{NBCC}$ pair in $\mathcal{S}$ and P_0 is nonempty. Then a relatively nonexpansive noncyclic mapping $T : P \cup Q \to P \cup Q$ which is condensing, admits a best proximity pair.*

If we take $\rho(q, p) = \varphi(p) - q$ where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $\limsup_{p \to r^+} \varphi(p) < 1$, for all $p > 0$ in Theorem 2.2 and 2.3, then we get the Corollaries 2.12 and 2.13.

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Corollary 2.12. A relatively nonexpansive cyclic mapping $T : P \cup Q \to P \cup Q$ admits a best proximity point if it satisfy following condition

$$\aleph(T(\mathcal{M}_1) \cup T(\mathcal{M}_2)) \leq \varphi(\aleph(\mathcal{M}_1 \cup \mathcal{M}_2))\ \aleph(\mathcal{M}_1 \cup \mathcal{M}_2), \tag{2.5}$$

for any $\mathcal{NBCC}$, proximinal and T-invariant pair $(\mathcal{M}_1, \mathcal{M}_2) = dist(P, Q)$ where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $\limsup_{p \to r^+} \varphi(p) < 1$, for all $p > 0$.

Corollary 2.13. *Let Banach space $\mathcal{S}$ be strictly convex and (P, Q) be a $\mathcal{NBCC}$ pair in $\mathcal{S}$ such that P_0 is nonempty. Then a relatively nonexpansive noncyclic mapping $T : P \cup Q \to P \cup Q$ satisfying condition (2.5), admits a best proximity pair.*

If we take $\rho(q, p) = \kappa(p)p - q$ where $\kappa : \mathbb{R}^+ \to [0, 1)$ is a mapping such that for every sequence $\{a_j\}$, $a_j > 0$ we have

$$\lim_{j \to \infty} \kappa(a_j) < 1 \Longrightarrow \lim_{j \to \infty} a_j = 0$$

in Theorem 2.2 and 2.3, then we get the Corollaries 2.14 and 2.15.

Corollary 2.14. *A relatively nonexpansive cyclic mapping $T : P \cup Q \to P \cup Q$ admits a best proximity point if it satisfy following condition*

$$\aleph(T(\mathcal{M}_1) \cup T(\mathcal{M}_2)) \leq \kappa(\aleph(\mathcal{M}_1 \cup \mathcal{M}_2))\ \aleph(\mathcal{M}_1 \cup \mathcal{M}_2), \tag{2.6}$$

for any $\mathcal{NBCC}$, proximinal and T-invariant pair $(\mathcal{M}_1, \mathcal{M}_2) = dist(P, Q)$ where $\kappa : \mathbb{R}^+ \to [0, 1)$ is a mapping such that for every sequence $\{a_j\}$, $a_j > 0$ we have

$$\lim_{j \to \infty} \kappa(a_j) < 1 \Longrightarrow \lim_{j \to \infty} a_j = 0.$$

Corollary 2.15. *Let Banach space $\mathcal{S}$ be strictly convex and (P, Q) be a $\mathcal{NBCC}$ pair in $\mathcal{S}$ such that P_0 is nonempty. Then a relatively nonexpansive noncyclic mapping $T : P \cup Q \to P \cup Q$ satisfying condition (2.6), admits a best proximity pair.*

If we take $\rho(q, p) = \psi(p) - \psi(q) - \phi(q)$ where $\psi, \phi : \mathbb{R}^+ \to \mathbb{R}^+$ are two mapping such that ψ is increasing and continuous from right and ϕ is lower semicontinuous with $\phi(\{0\}) = \{0\}$, in Theorem 2.2 and 2.3, then we get the Corollaries 2.16 and 2.17.

Corollary 2.16. *A relatively nonexpansive cyclic mapping $T : P \cup Q \to P \cup Q$ admits a best proximity point if it satisfy following condition*

$$\psi(\aleph(T(\mathcal{M}_1) \cup T(\mathcal{M}_2))) < \psi(\aleph(\mathcal{M}_1 \cup \mathcal{M}_2)) - \phi(\aleph(\mathcal{M}_1 \cup \mathcal{M}_2)), \tag{2.7}$$

for any $\mathcal{NBCC}$, proximinal and T-invariant pair $(\mathcal{M}_1, \mathcal{M}_2) = dist(P, Q)$ where $\psi, \phi : \mathbb{R}^+ \to \mathbb{R}^+$ are two mapping such that ψ is increasing and continuous from right and ϕ is lower semicontinuous with $\phi(\{0\}) = \{0\}$.

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Corollary 2.17. *Let Banach space $\mathcal{S}$ be strictly convex and (P,Q) be a $\mathcal{NBCC}$ pair in $\mathcal{S}$ such that P_0 is nonempty. Then a relatively nonexpansive noncyclic mapping $T : P\cup Q \to P\cup Q$ satisfying condition (2.7), admits a best proximity pair.*

Remark 2.18. As we know, the best proximity point theorems reduce to the case of fixed point if the two sets P and Q under consideration are same. Keeping this fact in view, we can reduce all the above best proximity theorems to the case of fixed points. In addition to the above consequences our theorem generalize the results of Aghajani et al. [1], Chen and Tang [7], Darbo [8] and Zarinfar et al. [23].

3. An application

This section is dedicated to prove a result which shows the existence of optimum solutions of system of second order differential equation with two initial conditions.

Let $\tau,\gamma \in \mathbb{R}^+$, $\mathcal{I} = [0,\tau]$ and $(E,\|.\|)$ be a Banach space. Let $B_1 = B(\alpha_0,\gamma)$, $B_2 = B(\beta_0,\gamma)$ where $\alpha_0,\beta_0 \in E$. We consider the following system of second order differential equation with two initial conditions

$$\begin{aligned} x^{''}(s) &= f(s,x(s)),\ x(0)=\alpha_0,\ x^{'}(0)=\alpha_1,\\ y^{''}(s) &= g(s,y(s)),\ y(0)=\beta_0,\ y^{'}(0)=\beta_1, \end{aligned} \tag{3.1}$$

where, $f : \mathcal{I}\times B_1 \to \mathbb{R}$, $g : \mathcal{I}\times B_2 \to \mathbb{R}$ are continuous functions such that $\|f(s,x)\| \leq A_1$, $\|g(s,y)\| \leq A_2$, $s\in\mathcal{I}$ and $\alpha_1,\beta_1 \in E$. Twice integrating (3.1) and usage of initial conditions yields us

$$\begin{aligned} x(s) &= \alpha_0 + \textstyle\int_0^s (\alpha_1 + (s-r)f(r,x(r))dr,\\ y(s) &= \beta_0 + \textstyle\int_0^s (\beta_1 + (s-r)g(r,x(r))dr. \end{aligned} \tag{3.2}$$

It is clear that the systems (3.1) and (3.2) are equivalent to each other. Let $\mathcal{J} \subseteq \mathcal{I}$, $\mathcal{S} = C(\mathcal{J},E)$ be a Banach space of continuous mappings from $\mathcal{J}$ into E endowed with supremum norm and consider

$$\mathcal{S}_1 = C(\mathcal{J},B_1) = \{x : \mathcal{J} \to B_1 : x\in\mathcal{S},\ x(0)=\alpha_0,\ x'(0)=\alpha_1\},$$

$$\mathcal{S}_2 = C(\mathcal{J},B_2) = \{y : \mathcal{J} \to B_2 : y\in\mathcal{S},\ y(0)=\beta_0,\ y'(0)=\beta_1\}.$$

So, $(\mathcal{S}_1,\mathcal{S}_2)$ is $\mathcal{NBCC}$ pair in $\mathcal{S}$. Now, for every $x\in\mathcal{S}_1$ and every $y\in\mathcal{S}_2$, we have

$$\|x-y\| = \sup_{s\in\mathcal{J}} \|x(s)-y(s)| \geq \|\alpha_0-\beta_0\|.$$

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So, $dist(\mathcal{S}_1, \mathcal{S}_2) = \|\alpha_0 - \beta_0\|$. Let us define operator $T : \mathcal{S}_1 \cup \mathcal{S}_2 \to \mathcal{S}$ as follows:

$$Tx(s) = \begin{cases} \beta_0 + \int_0^s \Big(\beta_1 + (s-r)g(r, x(r))\Big)dr, \ x \in \mathcal{S}_1, \\ \\ \alpha_0 + \int_0^s \Big(\alpha_1 + (s-r)f(r, x(r))\Big)dr, \ x \in \mathcal{S}_2. \end{cases}$$

It is clear that T is cyclic operator. It is known that $w \in \mathcal{S}_1 \cup \mathcal{S}_2$ is an optimum solution of the system (3.2) if $\|w - Tw\| = dist(\mathcal{S}_1 \cup \mathcal{S}_2)$ is satisfied. Equivalently, w is the best proximity point of the operator T. Before proving the actuality of optimum solution of system (3.2) we recall mean value theorem's extension for integral, which is presented according to our notations.

Theorem 3.1. [11] *For $\mathcal{I}, \mathcal{J}, B_1, B_2, f$ and g as given in above discussion with $s \in J$ we have*

$$\alpha_0 + \int_0^s (\alpha_1 + (s-r)f(r, x(r)))dr \in \alpha_0 + s\ \overline{con}\big(\{\alpha_1 + (s-r)f(r, x(r)) : r \in [0, s]\}\big)$$

and

$$\beta_0 + \int_0^s (\beta_1 + (s-r)g(r, x(r)))dr \in \beta_0 + s\ \overline{con}\big(\{\beta_1 + (s-r)g(r, x(r)) : r \in [0, s]\}\big).$$

The following theorem shows the actuality of optimum solutions for the system (3.2).

Theorem 3.2. *Let $\aleph$ be an arbitrary MNC on $\mathcal{S}$, $\tau(\tau A_2 + \|\beta_1\|) \leq \gamma$, $\tau(\tau A_1 + \|\alpha_1\|) \leq \gamma$ and $\tau \leq 1$. The system* (3.1) *has an optimal solution if the following condition holds true:*

(1) *For any bounded pair $(N_1, N_2) \subseteq (B_1, B_2)$, there is a upper semi-continuous function $\kappa : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $\kappa(p) < p$ such that*

$$\aleph(f(\mathcal{J} \times N_1) \cup g(\mathcal{J} \times N_2)) < \frac{\kappa(\aleph(N_1 \cup N_2))}{s}.$$

(2) *For each $x \in \mathcal{S}_1$ and for all $y \in \mathcal{S}_2$,*

$$\|g(r, x(r)) - f(r, y(r))\| \leq \frac{1}{s^2}(\|x(r) - y(r)\| - \|\beta_0 - \alpha_0\| + \|\beta_1 - \alpha_1\| s).$$

Proof. As the system (3.1) and (3.2) are equivalent to each other, in order to show (3.1) has an optimal solution it is sufficient to show (3.2) has optimal solution. From the above discussion it is clear that the operator T is cyclic. Our first task is to show that $T(\mathcal{S}_1)$ is bounded and equicontinuous subset of $\mathcal{S}_2$. For each $x \in \mathcal{S}_1$,

$$\begin{aligned} \|Tx(t)\| &= \|\beta_0 + \int_0^s (\beta_1 + (s-r)g(r, x(r))dr\| \\ &\leq \|\beta_0\| + \int_0^s \|\beta_1 + (s-r)g(r, x(r)\|dr \\ &\leq \|\beta_0\| + \tau(\|\beta_1\| + \tau A_2) \\ &\leq \|\beta_0\| + \gamma. \end{aligned}$$

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Thus $T(\mathcal{S}_1)$ is bounded. Now for $s, s' \in J$ and $x \in \mathcal{S}_1$,

$$\begin{aligned}
\|Tx(s) - Tx(s')\| =& \left\| \int_0^s (\beta_1 + (s-r)g(r, x(r))dr - \int_0^{s'} (\beta_1 + (s-r)g(r, x(r))dr \right\| \\
\leq& \left| \int_s^{s'} \| \beta_1 + (s-r)g(r, x(r) \| dr \right| \\
\leq& (\tau A_2 + \|\beta_1\|) \; \|s - s'\| \\
\leq& M|s - s'|, \text{ where } M = \tau A_2 + \|\beta_1\|,
\end{aligned}$$

this means $T(\mathcal{S}_1)$ is equicontinuous. With the similar argument $T(\mathcal{S}_2)$ is bounded and equicontinuous subset of $\mathcal{S}_1$. Thus application of Arzela-Ascoli theorem concludes $(\mathcal{S}_1, \mathcal{S}_2)$ is relatively compact.

Now our aim is to show T a relatively nonexpansive cyclic $\mathcal{SR}$-condensing operator. For each $(x, y) \in \mathcal{S}_1 \times \mathcal{S}_2$ with the help of assumption (2), we have

$$\begin{aligned}
&\|Tx(s) - Ty(s)\| \\
=& \left\| \beta_0 + \int_0^s (\beta_1 + (s-r)g(r, x(r))dr - [\alpha_0 + \int_0^s (\alpha_1 + (s-r)f(r, x(r))dr] \right\| \\
\leq& \|\beta_0 - \alpha_0\| + \left\| \int_0^s \big[(\beta_1 - \alpha_1) + (s-r)(g(r, x(r)) - f(r, x(r))\big] dr \right\| \\
\leq& \|\beta_0 - \alpha_0\| + \|\beta_1 - \alpha_1\| \; s + \left\| s \int_0^s \big(g(r, x(r)) - f(r, x(r)\big) dr \right\| \\
\leq& \|\beta_0 - \alpha_0\| + \|\beta_1 - \alpha_1\| \; s + (\|x(s) - y(s)\| - \|\beta_0 - \alpha_0\| - \|\beta_1 - \alpha_1\| \; s) \\
=& \|x(s) - y(s)\|.
\end{aligned}$$

This means T is relatively nonexpansive. In order to show T is cyclic $\mathcal{SR}$-condensing, suppose that the pair $(N_1, N_2) \subseteq \big(\mathcal{S}_1, \mathcal{S}_2\big)$ is $\mathcal{NBCC}$, proximinal, T-invariant and $dist(N_1, N_2) = dist(\mathcal{S}_1, \mathcal{S}_2) (= \|\alpha_0 - \beta_0\|)$. Now using Theorem

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3.1 and assumption (1) we have

$$\begin{aligned}
&\aleph(T(N_1)\cup T(N_2)) = \max\{\aleph(T(N_1)), \aleph(T(N_2))\}\\
=&\max\left\{\sup_{s\in\mathcal{J}}\left\{\aleph\big(\{Tx(s): x\in N_1\}\big)\right\},\ \sup_{s\in\mathcal{J}}\left\{\aleph\big(\{Ty(s): y\in N_2\}\big)\right\}\right\}\\
=&\max\left\{\sup_{s\in\mathcal{J}}\left\{\aleph\Big(\Big\{\beta_0+\int_0^s(\beta_1+(s-r)g(r,x(r)))dr : x\in N_1\Big\}\Big)\right\},\right.\\
&\left.\sup_{s\in\mathcal{J}}\left\{\aleph\Big(\Big\{\alpha_0+\int_0^s(\alpha_1+(s-r)f(r,x(r)))dr : x\in N_1\Big\}\Big)\right\}\right\}\\
=&\max\left\{\sup_{s\in\mathcal{J}}\left\{\aleph\big(\big\{\beta_0+s\ \overline{con}\big(\{\beta_1+(s-r)g(r,x(r)) : r\in[0,s]\}\big)\right\},\right.\\
&\left.\sup_{s\in\mathcal{J}}\left\{\aleph\big(\big\{\alpha_0+s\ \overline{con}\big(\{\alpha_1+(s-r)f(r,x(r)) : r\in[0,s]\}\big)\right\}\right\}\\
=&\max\left\{\aleph\big(\big\{\beta_0+s\ \big(\beta_1+\overline{con}\big(\{g(r,x(r)) : r\in[0,s]\}\big)\big\}\big),\right.\\
&\left.\aleph\big(\big\{\alpha_0+s\ \big(\alpha_1+\overline{con}\big(\{f(r,x(r)) : r\in[0,s]\}\big)\big\}\big)\right\}\\
\le&\max\left\{s\aleph\big(\{g(\mathcal{J}\times N_1)\}\big), s\aleph\big(\{f(\mathcal{J}\times N_2)\}\big)\right\}\\
=&s\aleph\big(\{f(\mathcal{J}\times N_1)\cup g(\mathcal{J}\times N_2)\}\big)\le s\frac{\kappa(\aleph(N_1\cup N_2))}{s}.
\end{aligned}$$

Thus we get

$$\kappa(\aleph(N_1\cup N_2)) - \aleph(T(N_1\cup T(N_2))) \geq 0.$$

Taking $\rho(t,s)=\kappa(s)-t$, we have

$$\rho(\aleph(T(N_1\cup T(N_2))), \aleph(N_1\cup N_2)) \geq 0.$$

Thus necessary requirements of Theorem 2.2 are satisfied. So the operator T has best proximity point and hence the system (3.1) has an optimal solution. □

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