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A Note on Acyclic Coloring of Strong Product of Graphs

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Abstract. A vertex coloring of a graph G is called acyclic if no two adjacent vertices have the same color and no cycle in G is bichromatic. The acyclic chromatic number $a(G)$ of a graph G is the least number of colors in an acyclic coloring of G. In this paper, we obtain bound for the acyclic chromatic number of the strong product of a tree and a graph. An exact value for the acyclic chromatic number of the strong product of two trees is derived. Further observations are made on the upper bound for the strong product of three paths.

Keywords: Strong product of graphs, Acyclic coloring, Acyclic chromatic number.

2000 Mathematics subject classification: 05C15, 05C38, 05C76.

1. Introduction

A proper coloring of the vertices of a graph G is an assignment of colors to the vertices so that no two adjacent vertices have the same color. A proper coloring is said to be acyclic if the coloring does not induce any bichromatic cycles. The acyclic chromatic number of G, denoted by $a(G)$, is the minimum number of colors required for its acyclic coloring. The concept of acyclic coloring, acyclic chromatic number, and star coloring was introduced by Grunbaum [5] in 1973

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and mainly studied by Albertson [1], Borodin [3], and amongst others. Acyclic colorings are hereditary in the sense that the restriction of an acyclic coloring to a subgraph is an acyclic coloring. It was also proved by Kostochka [9], that for every $k \geq 3$, the problem of deciding whether a graph is acyclically k-colorable is NP-complete for an arbitrary graph. There exist numerous types of operations on graphs, like graph union, graph intersection, graph join, graph sum, graph product, etc., which are generally named as binary operations on graphs. While there are some other types of operations, called unary operations on graph. Some examples for unary operations on a graph are the complement of a graph, power of a graph, line graph of a graph, middle graph of a graph, total graph of a graph, splitting graph of a graph, central graph of a graph, etc. Other operations of this kind can be found in Harary and Wilcox [6]. The product of graphs and their coloring are an interesting area of work for many researchers, due to its vast applications in different fields of science. A product $G * H$ of two graphs means a graph with vertex set $V(G) \times V(H)$, and the edge set is determined by a function on the edges of the factors. Even though many such products are defined, the most important ones are the strong product, the Cartesian product, the tensor product, and the lexicographic product. These products are respectively denoted by $G \boxtimes H$, $G \Box H$, $G \times H$ and $G[H]$. Sandi Klavzar [8], Greenwell and Lovasz [4] have made studies on some interesting applications of product colorings in their papers.

The strong product of graph was first introduced by the Austrian Mathematician Gert Sabidussi [11] in 1960. The strong product $G_1 \boxtimes G_2$ of two graphs G_1 and G_2 is a graph having vertex set $V(G_1) \times V(G_2)$ and edge set $E(G_1 \boxtimes G_2)$ given by the pairs (u, v), where $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are adjacent in $G = G_1 \boxtimes G_2$ whenever $[u_1 = v_1$ and u_2 adj $v_2]$ or $[u_2 = v_2$ and u_1 adj $v_1]$ or $[u_1$ adj v_1 and u_2 adj $v_2]$. Note that, the strong product of graphs is commutative for unlabeled graphs and also associative. Hence, the graph product $G_1 \boxtimes G_2 \boxtimes \cdots \boxtimes G_n$ is explicitly defined for any n. One of the known application of the strong product of graphs is in the information theory, where the zero-error capacity of a noisy channel is defined in terms of independence numbers of strong products of the graph related to the channel ([10], [12]). The chromatic number of the strong product of cycles and its several consequences has been studied by Zerovnik and Janez [14]. Acyclic colorings of Cartesian products of trees have been studied by Robert E. Jamison et al. [7]. But no work related to the acyclic coloring of strong products of trees has been formulated yet. Determining the exact values of acyclic chromatic number for the strong product of different families of a graph is a hard problem. Even for the simple and highly structured graph classes, the value is still not determined exactly.

In this paper, we obtain bound for the acyclic chromatic number of the strong product of a tree and a graph. The exact value obtained for the acyclic

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chromatic number of strong product of paths, lead to the acyclic chromatic number of strong product of two trees. Observations are made on the upper bound for the strong product of three paths; its generalization is a scope of future work. In the strong product of paths and trees, the most of the subgraphs are of the form kings graphs, that is, a graph whose vertices are squares of a chess board and whose edges represent possible moves of a chess king. The strong product of two trees resembles like a tree, where each edge is a king's graph. While discussing the proofs of theorems, we have to deal with various king's graph at different branches of such trees. Throughout this paper graphs means simple connected graphs. In figures the symbol i represents the color c_i. P_m denotes the path on m vertices and C_n denotes the cycle on n vertices. Diameter of a graph G is defined by $diam(G) = max\{d(x, y) : x, y \in V(G)\}$, where $d(x, y)$ is the distance between x and y. The removal of a vertex v_i from a graph G results in that subgraph $G - v_i$ of G consisting of all vertices of G except v_i and all edges incident with v_i. $G - v_i$ is the maximal subgraph of G not containing v_i.

2. Acyclic chromatic number of strong product of two paths

In this section an exact value for the acyclic chromatic number of $P_3 \boxtimes P_n$, $n \geq 2$ and $P_m \boxtimes P_n$,$m, n \geq 4$ are computed.

Proposition 2.1. [13] *Let $G = G_1 \boxtimes G_2$, and Δ_i is the maximum degree of G_i for $i = 1, 2$. Then,*
(i) the maximum degree, $\Delta(G) = (\Delta_1 + 1)(\Delta_2 + 1) - 1$.
(ii) the number of edges, $\varepsilon(G) = 2\varepsilon(G_1)\varepsilon(G_2) + \varepsilon(G_1)\upsilon(G_2) + \varepsilon(G_2)\upsilon(G_1)$.

Proposition 2.2. [2] *Let $G = P_m \boxtimes P_n$. Then*
(i) *$a(G) = 2$, for $m = 1$ and $n \geq 2$.*
(ii) *$a(G) = 4$, for $m = 2$ and $n \geq 2$.*

Theorem 2.3. *For $n \geq 2$, the acyclic chromatic number $a(P_3 \boxtimes P_n) = 4$.*

Proof. Let $G = P_3 \boxtimes P_n$ and $V(G) = \{v_1, v_2, v_3, \cdots, v_n, u_n, u_{n-1}, \cdots, u_1, w_1, w_2, \cdots, w_n\}$ be the vertex set of G which are marked in the same order as they appear in $V(G)$. Consider the set $C = \{c_1, c_2, c_3, c_4\}$, where c_1, c_2, c_3, c_4 are distinct colors. Assign the color c_i to the vertices of G as follows.
For odd values of i, the color c_1 is assigned to v_i and w_{i+1} and c_3 to u_i. For even values of i, the color c_2 is assigned to v_i and w_{i-1} and c_4 to u_i.
Now we prove that the coloring is acyclic. That is the coloring does not induce a bichromatic cycle. The coloring is in such a way that the subgraphs induced by each pair of colors are listed below.

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Sl. No.	**Subgraph Induced by**	ω	**Odd n** ν	ϵ	**Even n** ν	ϵ	**Value of** $\nu-\epsilon$
1	$< c_1, c_3 >$	1	$3\lceil\frac{n}{2}\rceil - 1$	$3\lceil\frac{n}{2}\rceil - 2$	$3\frac{n}{2}$	$3\frac{n}{2} - 1$	1
2	$< c_1, c_4 >$	1	$3\lceil\frac{n}{2}\rceil - 2$	$3\lceil\frac{n}{2}\rceil - 3$	$3\frac{n}{2}$	$3\frac{n}{2} - 1$	1
3	$< c_2, c_3 >$	1	$3\lceil\frac{n}{2}\rceil - 1$	$3\lceil\frac{n}{2}\rceil - 2$	$3\frac{n}{2}$	$3\frac{n}{2} - 1$	1
4	$< c_2, c_4 >$	1	$3\lceil\frac{n}{2}\rceil - 2$	$3\lceil\frac{n}{2}\rceil - 3$	$3\frac{n}{2}$	$3\frac{n}{2} - 1$	1

Here in each case the result $\epsilon = \nu - \omega$, (ω is the number of components, ν the number of vertices and ϵ is the number of edges) is verified, which is the necessary and sufficient condition for a forest. Also the subgraphs induced by $< c_1, c_2 >$ is the union of 2 paths P_n and the subgraph induced by $< c_3, c_4 >$ is the path P_n. Thus any pair of the colors in the set C will never induce a bichromatic cycle in the graph G. So the above said coloring is acyclic. Also the coloring is minimum, since G contains the subgraph K_4.
Hence $a(P_3 \boxtimes P_n) = 4$ for $n \geq 2$. □

Illustrate the above theorem with an example

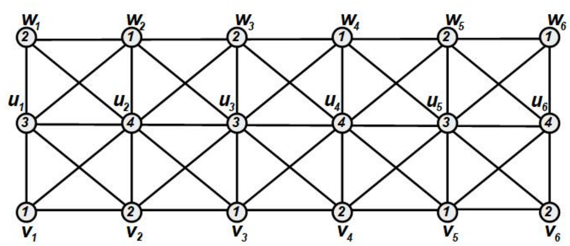

FIGURE 1. Acyclic Coloring of $P_3 \boxtimes P_6$.

Lemma 2.4. *The acyclic chromatic number* $a(P_4 \boxtimes P_4) = 5$.

Proof. Let $G = P_4 \boxtimes P_4$. This graph resembles a square, as depicted in Figure 2. We consider three cases according to the degree of the vertices in G. Let $v_i \in G$
Case 1. If $deg(v_i) = 3$, then these vertices are labeled with colors c_1, c_2, c_3, c_4 in anti-clock wise direction (from left bottom corner vertex).
Case 2. If $deg(v_i) = 5$, then these vertices are labeled with colors $c_4, c_5, c_1, c_5, c_2, c_5, c_3, c_5$ in anti-clock wise direction (from left bottom).
Case 3. If $deg(v_i) = 8$, then these vertices are assigned with colors c_4, c_1, c_2, c_3 such that the coloring is proper.

Next we prove that this coloring does not induce a bichromatic cycle. Consider the vertices which are adjacent to the vertices of degree 3 in G. Since these vertices are colored with distinct colors, it is not possible to form a bichromatic

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cycle through the degree 3 vertices in G. Remove these 4 vertices from G to form a subgraph G'. In G', the subgraph induced by $\prec c_i, c_5 \succ, i \leq 4$ always constitute disjoint union of 2 paths P_3 and hence they never form a bichromatic cycle through c_5. A new subgraph G'' is formed from G' by removing the vertices which are colored by the color c_5. This process is explained in the Figure 2. Also in $G^{''}$, the adjacent vertices of the remaining four vertices with degree 5 in G, which are mentioned in case 2 are colored with distinct colors. Therefore, we cannot find a bichromatic cycle passing through these vertices. Finally in $G^{''}$, the subgraph induced by the vertices of degree 8 in G, which are mentioned in case 3 form a complete graph K_4. Hence the coloring is acyclic. Next to prove that the coloring described above is minimum. Assume that $a(P_4 \boxtimes P_4) = 4$. By Theorem 2.3, $a(P_3 \boxtimes P_4) = 4$. The acyclic 4-coloring of $P_3 \boxtimes P_4$ is unique, up to permutation of colors and any 4-coloring of our required graph $P_4 \boxtimes P_4$ leaves only two colors on the top row which are also used in the second row. There are only two ways to color this top row, each of which will produce a bichromatic cycle. Thus $a(P_4 \boxtimes P_4) \geq 5$. Hence the lemma follows.

The color pattern described above can be exhibited in the form of a square matrix of order 4, $P_{4,4} = \begin{bmatrix} 4 & 5 & 2 & 3 \\ 3 & 1 & 4 & 5 \\ 5 & 2 & 3 & 1 \\ 1 & 4 & 5 & 2 \end{bmatrix}$, we call it as a generating matrix.

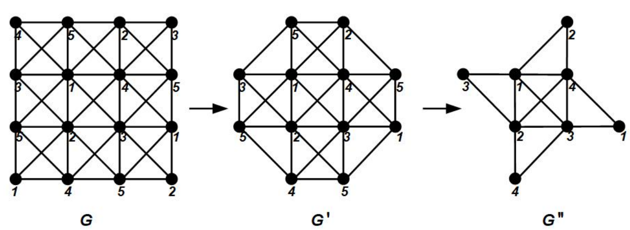

FIGURE 2. An acyclic coloring of $G = P_4 \boxtimes P_4$, the color c_i is marked as i.

□

Theorem 2.5. *For $m, n \geq 4$, the acyclic chromatic number $a(P_m \boxtimes P_n) = 5$.*

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Proof. Let $G = P_m \boxtimes P_n$. Then by Lemma 2.4, we have $a(P_4 \boxtimes P_4) = 5$. By using the generating matrix,

$$P_{4,4} = \begin{bmatrix} 4 & 5 & 2 & 3 \\ 3 & 1 & 4 & 5 \\ 5 & 2 & 3 & 1 \\ 1 & 4 & 5 & 2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \tag{2.1}$$

we can construct a matrix $P_{m,n} = [p_{ij}]_{m \times n}$ which represents the acyclic coloring of G. Define

$$p_{i,j} = \begin{cases} a_{(i \mod 5),(j \mod 5)} & \text{when both } i \not\equiv 0 \text{ and } j \not\equiv 0 \mod 5, \\ 5 & \text{when both } i \equiv 0 \text{ and } j \equiv 0 \mod 5, \\ a_{(-j \mod 5),(-j \mod 5)} & \text{when both } i \equiv 0 \text{ and } j \not\equiv 0 \mod 5, \\ a_{(-i \mod 5),5-(-i \mod 5)} & \text{when both } j \equiv 0 \text{ and } i \not\equiv 0 \mod 5, \end{cases} \tag{2.2}$$

where, the positive $x \mod k$ is taken, while considering the congruences.

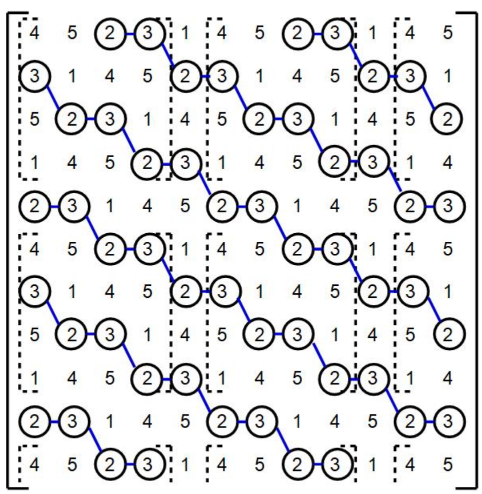

FIGURE 3. The matrix $P_{11,12}$ and the subgraph $\prec 2, 3 \succ$.

Here the minimality is obvious from Lemma 2.4. Now the coloring described in the matrix $P_{m,n}$ for $m, n \geq 4$ will never constitute a bichromatic cycle for any pair of colors, which can be explained as follows. By the above definition 2.2, in the matrix $P_{m,n}$ we can find full or partial blocks of the generating matrix $P_{4,4}$ separated by the rows R_i and the columns C_j, where $i, j \equiv 0 \mod 5$. In the matrix $P_{4,4}$, it can be noticed that the subgraphs induced by the opposite corner pair of colors namely $\prec 1, 3 \succ$ and $\prec 2, 4 \succ$ are union of a path P_4 and two points; while the subgraphs induced by all other pair of colors are union

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of two non-intersecting paths P_4 and P_3. Thus in $P_{m,n}$ the subgraphs induced by any two colors are either non-intersecting paths or union of non-intersecting paths and points. Therefore the coloring is acyclic. Hence $a(P_m \boxtimes P_n) = 5$, $m, n \geq 4$.
The matrix $P_{11,12}$ with different blocks and the subgraph induced by one pair $\prec 2, 3 \succ$ is illustrated in Figure 3. □

3. Acyclic chromatic number of strong product of trees

In this section, we determine exact value for the acyclic chromatic number of strong product of two trees, and bound for the strong product of a graph and a tree.

Lemma 3.1. *Let $G = T_1 \boxtimes T_2$, where T_i is a tree and $diam(T_i) \geq 3$ for $i = 1, 2$. Then $a(G) = 5$.*

Proof. **Case 1**. Assume that T_1 and T_2 are paths and $diam(T_i) \geq 3$. Then by Theorem 2.5 $a(G) = 5$.
Case 2. Suppose that T_1 and T_2 are trees with branches and sub-branches and diameters of T_1 and T_2 be m and n respectively. Then the largest paths in T_1 and T_2 will be P_{m+1} and P_{n+1} respectively. Let $c, v_1, v_2, v_3, \cdots, v_{m-1}, d$ and $a, u_1, u_2, u_3, \cdots, u_{n-1}, b$ be vertices of the largest path in T_1 and T_2 respectively, where a, b, c and d are pendant vertices.
The maximum possible length of any branch at the internal vertices v_r and v_{m-r} will be of at most r, where $r \in \{1, 2, 3, \cdots, \lfloor \frac{m}{2} \rfloor\}$. Also in each branch at the vertex v_r, the maximum length of the sub-branch at $v_{r,s}$ and $v_{m-r,s}$ will be of length at most $r - s$, for $r = 1, 2, 3, \cdots, \lfloor \frac{m}{2} \rfloor$ and $s \leq r$. Similarly for the tree T_2 the maximum possible length of any branch at the internal vertices u_i and u_{n-i} will be of at most i, where $i \in \{1, 2, 3, \cdots, \lfloor \frac{n}{2} \rfloor\}$. Also, in each branch at the vertex u_i, the maximum length of the sub-branch at $u_{i,j}$ and $u_{n-i,j}$ will be of length at most $i - j$, for $i = 1, 2, 3, \cdots, \lfloor \frac{n}{2} \rfloor$ and $j \leq i$. This can be extended to any number of sub-branches of T_1 and T_2. By Theorem 2.5, $a(P_{m+1} \boxtimes P_{n+1}) = 5$ and $P_{m+1} \boxtimes P_{n+1}$ is a subgraph of G, which gives $a(G) \geq 5$.
By equation (2.2) of Theorem 2.5, we construct a matrix $P_{m+1,n+1}$ that represents the acyclic coloring of G.

$$P_{m+1,n+1} = \begin{bmatrix} 4 & 5 & 2 & 3 & 1 & 4 & \cdots & p_{1,n+1} \\ 3 & 1 & 4 & 5 & 2 & 3 & \cdots & p_{2,n+1} \\ 5 & 2 & 3 & 1 & 4 & 5 & \cdots & p_{3,n+1} \\ 1 & 4 & 5 & 2 & 3 & 1 & \cdots & p_{4,n+1} \\ 2 & 3 & 1 & 4 & 5 & 2 & \cdots & p_{6,n+1} \\ 4 & 5 & 2 & 3 & 1 & 4 & \cdots & p_{7,n+1} \\ 3 & 1 & 4 & 5 & 2 & 3 & \cdots & p_{8,n+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ p_{m+1,1} & p_{m+1,2} & p_{m+1,3} & p_{m+1,4} & p_{m+1,5} & p_{m+1,6} & \cdots & p_{m+1,n+1} \end{bmatrix}$$

Subcase 1. Assume that the tree T_1 is a path and T_2 is a tree with branches and sub-branches. Consider the arbitrary branch at the internal vertex u_x of the tree T_2 having maximum length x, for $x = 1, 2, 3, \cdots, \lceil \frac{n}{2} \rceil$. Then we get a king's subgraph $P_{m+1} \boxtimes P_{x+1}$ of $P_{m+1} \boxtimes P_{n+1}$ at the $(x+1)^{th}$ branch of the original graph $P_{m+1} \boxtimes P_{n+1}$. For $1 \leq j \leq x+1$, the vertices of this subgraph which lies in the j^{th} vertical columns are assigned by the $(j+x)^{th}$ column colors of the matrix $P_{m+1,n+1}$, such that the coloring of the subgraph coincide with the coloring of $P_{m+1} \boxtimes P_{n+1}$. As the acyclic coloring is hereditary, we can conclude that $(G) = 5$.

Subcase 2. In the case of both T_1 and T_2 having branches and sub-branches, by the same argument of Subcase 1, we have corresponding to every branch at the internal vertex v_y of T_1, we get king's subgraph $P_{y+1} \boxtimes P_{n+1}$ of $P_{m+1} \boxtimes P_{n+1}$. This is true for any branch or sub-branch of T_1 or T_2. Moreover, since a tree is an acyclic graph and the coloring assigned to the vertices of king's subgraph are submatrices of the matrix $P_{m+1,n+1}$, the coloring will never induce bichromatic cycles in G. Thus we can color G with 5 colors acyclically.
Hence in all cases $a(G) = 5$. □

Remark 3.2. Let $G = P_m \boxtimes T$ and $diam(T) \geq 1$. Then $a(G) = 4$, $m \in \{2, 3\}$.

Proposition 3.3. *Let $G = G_1 \boxtimes G_2$ where G_1 and G_2 are two complete graphs with maximum degree Δ_i for $i = 1, 2$. Then $a(G) = \Delta(G) + 1$.*

Proof. Let $m, n \geq 1$ be the number of vertices of G_1 and G_2 respectively. Then the graph G will have mn vertices and $\Delta_1 = m - 1$ and $\Delta_2 = n - 1$. By Proposition 2.1(i), we have $\Delta(G) = (m-1+1)(n-1+1) - 1 = mn - 1$. Thus we get G is complete, and $a(G) = mn = \Delta(G) + 1$. □

Corollary 3.4. *Let $G = G_1 \boxtimes G_2 \boxtimes \cdots \boxtimes G_n$, where G_i are complete graphs with maximum degree Δ_i for $i = 1, 2, \cdots, n$. Then $a(G) = \Pi_{i=1}^{n}(\Delta_i + 1)$.*

Proof. By Propositions 2.1(i) and 3.3, we have $a(G_1 \boxtimes G_2) = (\Delta_1 + 1)(\Delta_2 + 1)$. Since the strong product of two complete graphs are complete, G will be a complete graph. Hence by extending the above result we can conclude that $a(G) = \Pi_{i=1}^{n}(\Delta_i + 1)$. □

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Theorem 3.5. *Let $G = K_n \boxtimes T$, where $diam(T) \geq 1$. Then $a(G) = 2n$.*

Proof. In $K_n \boxtimes P_2$ there are 2 copies of K_n, say $K_n^{(1)}$ and $K_n^{(2)}$ and we can find an edge from each vertex of $K_n^{(1)}$ to all other vertices of $K_n^{(2)}$ or vice versa. By Proposition 2.1(i), $\Delta(K_n \boxtimes P_2) = (n-1+1)(1+1) - 1 = 2n-1$. Thus by Proposition 3.3 $a(K_n \boxtimes P_2) = 2n$.

Let c be an acyclic coloring of $K_n \boxtimes P_2$ using the color set $C = \{1, 2, 3, \cdots, 2n\}$. Let us take $C = C_1 \cup C_2 = \{1, 2, 3, \cdots, n\} \cup \{n+1, n+2, n+3, \cdots, 2n\}$ and the colors of the sets C_1 and C_2 are assigned to $K_n^{(1)}$ and $K_n^{(2)}$ respectively. Here the subgraph induced by $\prec i, j \succ$ is a path P_2, for all $1 \leq i, j \leq 2n$. The coloring is represented in Figure 4.

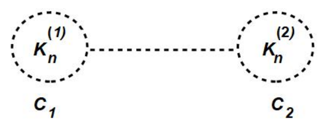

FIGURE 4. Representation of $K_n \boxtimes P_2$ and its coloring.

The acyclic coloring c can be extended to the product $K_n \boxtimes P_n$ such that the adjacent K_n's are assigned with different color set as explained in Figure 5. Here the subgraph induced by $\prec i, j \succ$ is the union of paths P_2, if both i and j belong to the same color set C_1 or C_2, otherwise it will be a path P_n. Thus $a(K_n \boxtimes P_n) = 2n$.

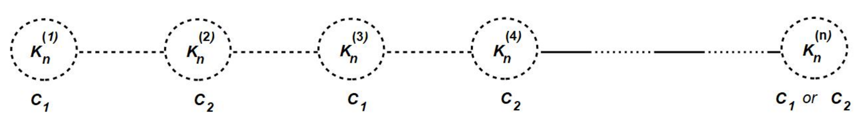

FIGURE 5. Representation of $K_n \boxtimes P_n$ and its coloring.

We can extend the coloring c to $K_n \boxtimes T$ by preserving its acyclicity. Because the tree T is a connected acyclic graph, so in any proper coloring of $K_n \boxtimes T$ using c, no adjacent K_n's will be assigned with same color set C_1 or C_2. Thus the subgraph induced by any two colors will be always a forest. Hence $a(K_n \boxtimes T) = 2n$.

A tree T and an acyclic coloring of $K_n \boxtimes T$ is explained in Figure 6.

□

From Proposition 3.3 and Theorem 3.5, the Remark 3.6 is obtained.

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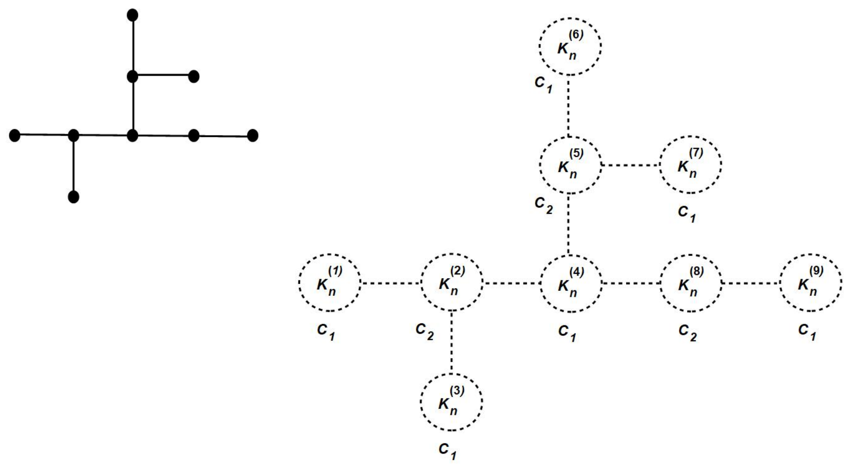

FIGURE 6. A tree T and a representation of an acyclic coloring of $K_n \boxtimes T$.

Remark 3.6. Let $G = K_n \boxtimes H$, where the graph H is non-empty and $diam(H) \geq 1$. Then $2n \leq a(G) \leq \Delta(G) + 1$.

Theorem 3.7. *Let $G = H \boxtimes T$, where H is a non-empty graph other than a tree having n vertices and T be a tree other than path. If Δ_1 and Δ_2 are respectively the maximum degrees of H and T with $\Delta_1 \geq \lceil \frac{n}{2} \rceil$, then $a(G) \leq \Delta(G) - 3$.*

Proof. By Theorem 3.5, we have $a(G) \leq 2n$. Since T is a tree other than a path, $\Delta_2 \geq 3$. Now by Proposition 2.1(i), we have $\Delta(G) \geq (\lceil \frac{n}{2} \rceil + 1)(3+1) - 1$. If n is even, we get $\Delta(G) \geq (\frac{n}{2} + 1)(4) - 1 = 2n + 3$. That is $\Delta(G) - 3 \geq 2n$. If n is odd, we get $\Delta(G) \geq (\frac{n+1}{2} + 1)(4) - 1 = 2n + 5$. That is $\Delta(G) - 3 \geq 2n + 2 > 2n$.
Thus we get $a(G) \leq \Delta(G) - 3$. □

The determination of the exact value for the acyclic chromatic number of strong product of three or more graphs is a tedious job. However, we compute the exact value of $a(P_m \boxtimes P_2 \boxtimes P_2)$ in the following Theorem.

Theorem 3.8. *The acyclic chromatic number, $a(P_m \boxtimes P_2 \boxtimes P_2) = 8$, for $m \geq 2$.*

Proof. The proof is by induction on m. For $m = 2$, the graph $G = P_2 \boxtimes P_2 \boxtimes P_2$ is isomorphic to the complete graph K_8. Thus $a(G) = 8$. Now before moving to the next step of induction, for $m = 2$, define an acyclic coloring c of G by using the colors $c_1, c_2, \cdots, c_8$ as follows.
Let $v_i \in V(G)$, $1 \leq i \leq 8$ and $V = \{v_1, v_2, \cdots, v_8\}$ be the vertex set of G which are marked in anti-clockwise direction in the same order as they appear

in V. Then the coloring given by the function $c(v_i) = c_i$, $1 \leq i \leq 8$ gives an acyclic coloring of G. Next assume that the result is true for $m = k$. That is, an acyclic coloring of $G = P_k \boxtimes P_2 \boxtimes P_2$ is a map $c : V(G) \rightarrow \{c_1, \cdots, c_8\}$ such that

$$c(v_{4r+i}) = \begin{cases} c_i & r = 0, 2, 4 \cdots \\ c_{4+i} & r = 1, 3, 5, \cdots \end{cases} \quad 1 \leq i \leq 4 \tag{3.1}$$

for $r = 0, 1, 2, 3, 4, \cdots, k-1$.
It can be noted that in this coloring the subgraphs induced by

$$\prec c_i, c_j \succ = \begin{cases} \left\lceil \frac{k}{2} \right\rceil \text{copies of } P_2 & \text{for } i, j = 1, 2, 3, 4, \\ \left\lfloor \frac{k}{2} \right\rfloor \text{copies of } P_2 & \text{for } i, j = 5, 6, 7, 8, \\ P_k & \text{for } i = 1, 2, 3, 4 \text{ and } j = 5, 6, 7, 8, \end{cases}$$

forms a forest.

Next to prove the result holds for $m = k + 1$. According to equation (3.1), we can extend the coloring to $G = P_{k+1} \boxtimes P_2 \boxtimes P_2$, with $r = 0, 1, 2, 3, 4, \cdots, k$. It gives the induced subgraphs

$$\prec c_i, c_j \succ = \begin{cases} \left\lceil \frac{k+1}{2} \right\rceil \text{copies of } P_2 & \text{for } i, j = 1, 2, 3, 4, \\ \left\lfloor \frac{k+1}{2} \right\rfloor \text{copies of } P_2 & \text{for } i, j = 5, 6, 7, 8, \\ P_{k+1} & \text{for } i = 1, 2, 3, 4 \text{ and } j = 5, 6, 7, 8, \end{cases}$$

which again forms a forest. Thus the result is true for $m = k + 1$.
Hence by method of mathematical induction, $a(P_m \boxtimes P_2 \boxtimes P_2) = 8, m \geq 2$.
An acyclic coloring of $P_2 \boxtimes P_2 \boxtimes P_2$ is illustrated in Figure 7.

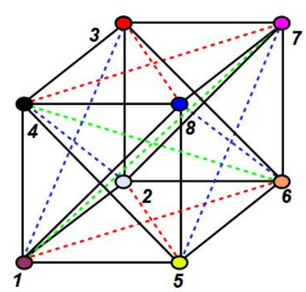

FIGURE 7. An acyclic coloring of $P_2 \boxtimes P_2 \boxtimes P_2$.

□

Observation 3.1. (i) The acyclic chromatic number, $a(P_m \boxtimes P_2 \boxtimes P_3) = 10$, for $m \geq 3$. (ii) The acyclic chromatic number, $a(P_m \boxtimes P_2 \boxtimes P_n) \leq 2(n + 2)$, for $m, n \geq 3$.

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4. Conclusions

In this paper, the acyclic chromatic number of the strong product of paths, trees, and graphs are studied. The exact value of the strong product of paths and trees are derived. In other cases, bounds are obtained. Some observations are made on the upper bound for the strong product of three paths.

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