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The Zariski Topology on $\mathbf{Cl.Spec_g(M)}$ as a Spectral Space

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Abstract. : Let G be a group, R be a G-graded commutative ring with identity, M be a unitary graded R-module, $Spec_g(R)$ be the set of graded prime ideals of R, and $Cl.Spec_g(M)$ be the set of all graded classical prime submodules of M. In this paper among other things, the author studied the Zariski topology on both and $Cl.Spec_g(M)$, and investigate some properties of the Zariski topology on $Cl.Spec_g(M)$ and some conditions under which the graded classical prime spectrum of M is a spectral for its Zariski topology.

Keywords: Graded classical prime submodule, Graded classical prime spectrum, Zariski topology.

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1. Introduction

The graded prime ideals were introduced and studied in [35, 37, 38]. The graded prime submodules of a graded module over a graded commutative ring were given in [8, 11, 12, 13, 32] as a generalization of graded prime ideals of a graded ring. The graded classical prime submodules of a graded module over a graded commutative ring were introduced in [19] and studied in [3, 6, 7, 8]. The Zariski topology on the spectrum of prime ideals for a ring is one of the main tools in Algebraic Geometry (see [14, 24]). In the literature, there are different generalizations of the Zariski topology over ring to module, (see [1, 2, 9, 10, 15, 16, 20, 26, 27, 29, 33]). Also the Zariski topology on the graded

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prime spectrum of graded prime ideals for a graded ring in [35, 36, 37] is generalized in different ways on the spectrum of graded prime submodules of graded modules over graded commutative rings as in [18, 34] and on other type of graded submodules as in [4]. In [6, 7, 19] the authors introduced and studied some topologies on the spectrum of graded classical prime submodules of a graded module over graded commutative rings.

Our main purpose is to study more properties of the Zariski topology on the graded classical prime submodules of a graded module over a graded commutative rings, where there is a wide variety of applications of graded algebras in geometry and physics, (for example see [39, Introduction]). In the sequel, in this article we investigate the topological properties of this topology and we add more results about the relationship between algebraic properties of topological graded modules and topological properties of the Zariski topology on the graded classical prime spectrum of them.

In Section 3, we introduce the Zariski topology on the set of graded prime ideals $Spec_g(R)$, in such a way that the Zariski topology was introduced in [35, 37], indeed in Theorem 3.9 we show that the Zariski topology on $Spec_g(R)$ is a spectral space. In Section 4, we study some new properties of the Zariski topology on $Cl.Spec_g(M)$. Graded modules whose Zariski topology has T_0-space property, the injectivety of the natural map ψ, the topological properties on the graded classical prime spectrum of graded modules such as connectedness property are studied, and several characterizations of such graded modules are given. We also show in Theorem 4.12, that if the natural map ψ is surjective, then the quasi-compact open sets of $Cl.Spec_g(M)$ are closed under finite intersection and form an open base. In Section 5, we study the irreducible closed subsets of the Zariski topology on $Cl.Spec_g(M)$ and their generic points. Also we obtain theorems related to the irreducible components of $Cl.Specg(M)$ and the combinatorial dimension of the graded prime classical spectrum, as in Theorem 5.9 and Corollary 5.10. We show in Theorem 5.15 that for any graded R-module M with surjective natural map $\psi : Cl.Spec_g(M) \to Spec_g(\overline{R})$ which is given by $\psi(P) = (P :_R M)$ for every $P \in Cl.Spec_g(M)$, where $\overline{R} = R/Ann(M)$, the set of all irreducible components of $Cl.Spec_g(M)$ is of the form $\Phi := \{\mathbb{V}^g(IM) \mid I$ is a minimal element of $V^g_R(Ann(M))$ with respect to inclusion $\}$. In Section 6, we present in Theorem 6.11, the conditions under which a graded module is a spectral space. In particular, we show that every g-Cl.Top R-module M with surjective natural map ψ is a spectral space, every graded classical weak multiplication R-module M with surjective natural map ψ is also a spectral space, and moreover, if $Im(\psi)$ is closed with surjective natural map ψ, then $Cl.Spec_g(M)$ is a spectral space.

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2. Preliminaries

Throughout this paper all rings are commutative with identity and all modules are unitary. Before we state some results let us introduce some notation and terminology. We refer to [21, 30, 31] for these basic properties and more information on graded rings and modules.

Let G be a group and R be a commutative ring with identity 1_R. Then R is a G-graded ring if there exist additive subgroups R_g of R such that $R = \oplus_{g\in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. The nonzero elements of R_g are called to be homogeneous of degree g. If $x \in R$, then x can be written uniquely as ${}_{g\in G} x_g$, where x_g is the component of x in R_g. Moreover, $h(R) = \cup_{g\in G} R_g$. Let I be an ideal of R. Then I is called a graded ideal of R if $I = \oplus_{g\in G}(I \cap R_g)$. Thus, if $x \in I$, then $x =_{g\in G} x_g$ with $x_g \in I$.

Let R be a G-graded ring and M be a R-module. We say that M is a graded R-module if there exists a family of subgroups $\{M_g\}_{g\in G}$ of M such that $M = \oplus_{g\in G} M_g$ as abelian groups and $R_g M_h \subseteq M_{gh}$ for all $g, h \in G$. Here, $R_g M_h$ denotes the additive subgroup of M consisting of all finite sums of elements $r_g s_h$ with $r_g \in R_g$ and $s_h \in M_h$. Also, we write $h(M) = \cup_{g\in G} M_g$ and the elements of $h(M)$ are called to be homogeneous. Let $M = \oplus_{g\in G} M_g$ be a graded R-module and N a submodule of M. Then N is called a graded submodule of M if $N = \oplus_{g\in G} N_g$ where $N_g = N \cap M_g$ for $g \in G$. In this case, N_g is called the g-component of N.

Let R be a G-graded ring and M be a graded R-module. A proper graded ideal I of R is said to be a graded prime ideal if whenever $rs \in I$, we have $r \in I$ or $s \in I$, where $r, s \in h(R)$, (see [36]). Let $Spec_g(R)$ denote the set of all graded prime ideals of R. A proper graded submodule N of M is said to be a graded prime submodule if whenever $r \in h(R)$ and $m \in h(M)$ with $rm \in N$, then either $r \in (N :_R M) = \{r \in R : rM \subseteq N\}$ or $m \in N$, (see [36]). It is shown in [36, Proposition 2.7] that if N is a graded prime submodule of M, then $p := (N :_R M)$ is a graded prime ideal of R, and N is called a graded p-prime submodule. Let $Spec_g(M)$ denote the set of all graded prime submodules of M. A proper graded submodule N of M is called a graded classical prime submodule if whenever $r, s \in h(R)$ and $m \in h(M)$ with $rsm \in N$, then either $rm \in N$ or $sm \in N$, (see [8, 19]). Of course, every graded prime submodule is a graded classical prime submodule, but the converse is not true in general, (see [8, Example 2.3]). Let $Cl.Spec_g(M)$ denote the set of all graded classical prime submodules of M. Some graded R-modules have no graded classical prime submodules, such modules are called g-Cl.primeless, for example, the zero module is clearly g-Cl.primeless,(see [19, p. 162]).

Let R be a G-graded ring and M be a graded R-module. For each graded ideal I of R, the graded variety of I is the set $V_R^g(I) = \{p \in Spec_g(R) | I \subseteq p\}$. Then the set $\xi(R) = \{V_R^g(I) | I \text{ is a graded ideal of } R\}$ satisfies the axioms for the

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closed sets of a topology on $Spec_g(R)$, called the Zariski topology on $Spec_g(R)$, (see [35, 36, 37]). For any $r \in h(R)$, the set $GX_r = Spec_g(R) - V_R^g(rR)$ is open in $Spec_g(R)$ and the family $\digamma = \{GX_r \mid r \in h(R)\}$ form a base for the Zariski topology on $Spec_g(R)$, (see [37, Proposition 3.4]). Further more each GX_r is known to be a quasi-compact subset, (see [37, Proposition 3.8]).

In [19], $Cl.Spec_g(M)$ has endowed with quasi-Zariski topology. For each graded submodule N of M, let $\mathbb{V}_*^g(N) = \{P \in Cl.Specg(M)|N \subseteq P\}$. In this case, the set $\eta_*^g(M) = \{\mathbb{V}_*^g(N)|N$ is a graded submodule of $M\}$ contains the empty set and $Cl.Spec_g(M)$, and it is closed under arbitrary intersections, but it is not necessarily closed under finite unions. The graded R-module M is said to be a g-Cl.Top module if $\eta_*^g(M)$ is closed under finite unions. In this case $\eta_*^g(M)$ satisfies the axioms for the closed sets of a unique topology $\varrho_*^g(M)$ on $Cl.Spec_g(M)$. In this case, the topology $\tau_*^g(M)$ on $Cl.Spec_g(M)$ is called the quasi-Zariski topology.

In [7] another variety was defined for a graded submodule N of a graded R-module M. They define the variety of N to be $\mathbb{V}^g(N) = \{P \in Cl.Spec_g(M) : (P :_R M) \supseteq (N :_R M)\}$. Then the set $\eta^g(M) = \{\mathbb{V}^g(N)|N$ is a graded submodule of $M\}$ contains the empty set and $Cl.Spec_g(M)$ and it satisfies the axioms for the closed sets of a topology on $Cl.Spec_g(M)$. This topology is called the Zariski topology on $Cl.Spec_g(M)$ and denoted by ϱ^g. Also some properties were studied on this topology as T_1-space, spectral space.

We will study the Zariski topology in such a way that the Zariski topology was introduced in [7], note that the case that $Cl.Spec_g(M) = \phi$, is the trivial case and we will not discuss it, so throughout the rest of the paper we assume that $Cl.Spec_g(M) \neq \phi$. We first review some important remarks which will be needed at the next sections.

Remark 2.1. For a topological space W, we recall:
(i) W is quasi compact if it satisfies one of the following two equivalent conditions:
(a) Every collection of open subsets whose union is W contains a finite subcollection whose union is W.
(b) Every collection of closed subsets whose intersection is empty contains a finite subcollection whose intersection is empty, (see [23, Definition 2.135]).
(ii) W is said to be irreducible if W is not the union of two proper closed subsets. For $W' \subseteq W$, W' is irreducible if it is irreducible as a space with the relative topology. This is equivalent to say that, if X, Y are closed subsets of W such that $W' \subseteq X \cup Y$, then $W' \subseteq X$ or $W' \subseteq Y$, (see [17, Ch. II, p. 119]).
(iii) A maximal irreducible subset of W is called an irreducible component of W. It is well known that every irreducible component of W is closed in W, and W is the union of its irreducible components, (see [17, Ch. II, p. 119]).
(iv) Let A and B be subsets of W such that $A \subseteq B \subseteq W$, where B is closed

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in W and equipped with the relative topology. Then A is an irreducible closed subset of B if and only if A is an irreducible closed subset of W. The proof is straightforward from (ii) and the fact that A is closed in B if and only if A is closed in W.
(v) W is said to be connected if it is not the union $W = W_0 \cup W_1$ of two disjoint closed non-empty subsets W_0 and W_1, (see [23, Definition 2.105]).
(vi) Closed subspaces of quasi compact topological spaces are quasi compact, (see [23, Theorem 2.137]).
(vii) Let f be a continuous mapping from a topological space W to a topological space T:
(a) If W is a connected (resp. quasi compact) topological space, then $f(W)$ is a connected (resp. quasi compact) topological space, (see [23, Theorem 2.107 and Theorem 2.138]).
(b) For every irreducible subset E of W, $f(E)$ is an irreducible subset of T, (see [17, Ch. II]).

Remark 2.2. Let W be a topological space and let x and y be points in W. We say that x and y can be separated if each lies in an open set which does not contain the other point. W is a T_1- space if any two distinct points in W can be separated. A topological space W is a T_1-space if and only if all points of W are closed in W, that is, given any x in W, the singleton set $\{x\}$ is a closed set, (see [28]).

Remark 2.3. A Spectral space is a topological space homomorphic to the prime spectrum of a commutative ring equipped with the Zariski topology. Spectral spaces have been characterized by Hochster [22, Proposition 4] as the topological spaces W which satisfy the following conditions:
(i) W is a T_0-space.
(ii) W is quasi-compact.
(iii) The quasi-compact open subsets of W are closed under finite intersection and form an open base.
(iv) Each irreducible closed subset of W has a generic point.

The following Lemma is known (see [25, Lemma 1.2 and Lemma 2.7]), but we write be it here for the sake of references.

Lemma 2.4. *Let R be a G-graded ring and M be a graded R-module. Then the the following hold:*
(1) If N is a graded submodule of M, then $(N :_R M) = \{r \in R : rM \subseteq N\}$ is a graded ideal of R.
(2) If N is a graded submodule of M, $r \in h(R)$, $x \in h(M)$ and I is a graded ideal of R, then Rx, IN and rN are graded submodules of M.
(3) If N and K are graded submodules of M, then $N + K$ and $N \cap K$ are also graded submodules of M.

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(4) If $\{N_i\}_{i\in I}$ is a collection of graded submodules of M, then $N = \cap_{i\in I} N_i$ is a graded submodule of M.

3. The Zariski topology on $Spec_g(R)$ as a spectral space

The Zariski topology on the prime spectrum of prime ideals were studied in [14, 24] and it was generalized to the Zariski topology on the graded prime spectrum of graded prime ideals of a graded ring in [35, 36, 37].

In this section, we observe $Spec_g(R)$ from the point of view of spectral topological spaces; we will follow Hochster's characterization closely, (see Remark 2.3).

The next theorem gives an important characterization of the Zariski topology on $Spec_g(R)$ and will be needed at the end of this section.

Theorem 3.1. *Let R be a G-graded ring. Then the quasi-compact open sets of $Spec_g(R)$ are closed under finite intersection and form an open base.*

Proof. It suffices to show that the intersection $U = U_1 \cap U_2$ of two quasi-compact open sets U_1 and U_2 of $Spec_g(R)$ is a quasi-compact set. Note that U is open; so U is a finite union of members of the open base $F = \{GX_r \mid r \in h(R)\}$. Put $U = \cup_{i=1}^{n} GX_{r_i}$ and let Γ be any open cover of U. Then Γ also covers each GX_{r_i} which is quasi-compact by [37, Proposition 3.8]. Hence, each GX_{r_i} has a finite subcover and so does U. The other part of the theorem is trivially true due to the existence of the open base B, (see [37, Proposition 3.4]). □

For a topological space W, if Y is a nonempty subset of W, then we let $\Im(Y)$ denote the intersection of the members of Y. Thus, if Y_1 and Y_2 are subsets of W, then $\Im(Y_1 \cup Y_2) = \Im(Y_1) \cap \Im(Y_2)$, and we will denote the closure of Y in W with respect to the Zariski topology by $Cl(Y)$. Let Y be a closed subset of W. An element $y \in Y$ is called a generic point of Y if $Y = Cl(\{y\})$.

Theorem 3.2. *Let R be a G-graded ring. Let Y be a subset of $V_R^g(I)$. Then $V_R^g(\Im(Y)) = Cl(Y)$. In particular $Cl(\{p\}\} = V_R^g(p)$ for any graded ideal p of R. Hence, Y is closed if and only if $V_R^g(\Im(Y)) = Y$.*

Proof. Since $Y \subseteq V_R^g(I)$, $I \subseteq J$, for every $J \in Y$. This implies $I \subseteq \cap_{J\in Y} J = \Im(Y)$, so $I \subseteq \Im(Y) \subseteq K$, for every $K \in V_R^g(\Im(Y))$. Thus $V_R^g(\Im(Y)) \subseteq V_R^g(I)$. Therefore $V_R^g(\Im(Y))$ is the smallest closed subset of $Spec_g(R)$ including Y, so $V_R^g(\Im(Y)) = Cl(Y)$. □

Corollary 3.3. *Let R be a G-graded ring. Then $V_R^g(p)$ is an irreducible closed subset of $Spec_g(R)$ for every graded prime ideal p of R.*

Proof. Both of a singleton subset and its closure in $Spec_g(R)$ are irreducible. Now, apply Theorem 3.2. □

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The following theorem characterizes the irreducible subset of $Spec_g(R)$ for a G-graded ring R.

Theorem 3.4. *Let R be a G-graded ring and Y be a subset of $Spec_g(R)$. Then, Y is an irreducible subset of $Spec_g(R)$ if and only if $\Im(Y)$ is a graded prime ideal of R.*

Proof. Suppose that Y is an irreducible subset of $Spec_g(R)$. Let I, J be graded ideals of R such that $I \cap J \subseteq \Im(Y)$ and suppose that $I \nsubseteq \Im(Y)$ and $J \nsubseteq \Im(Y)$, Then $\Im(Y) \nsubseteq V_R^g(I)$ and $\Im(Y) \nsubseteq V_R^g(J)$. Let $p \in Y$, then $I \cap J \subseteq \Im(Y) \subseteq p$. So, $p \in V_R^g(I \cap J) = V_R^g(I) \cup V_R^g(J)$. Therefore, $Y \subseteq V_R^g(I) \cup V_R^g(J)$ which is a contradiction to the irreducibility of Y. Therefore $I \subseteq \Im(Y)$ or $J \subseteq \Im(Y)$. Thus $\Im(Y)$ is a graded prime ideal by [37, Proposition 1.2]. Conversely suppose that $Y \subseteq Spec_g(R)$ such that $\Im(Y)$ is a graded prime ideal of R. Suppose that $Y \subseteq Y_1 \cup Y_2$, where Y_1, Y_2 are closed subset of $Spec_g(R)$, so there exist graded ideals I, J of R, such that $Y_1 = V_R^g(I)$ and $Y_2 = V_R^g(J)$. Hence $Y \subseteq V_R^g(I) \cup V_R^g(J) = V_R^g(I \cap J)$. So, $I \cap J \subseteq p$, for all p in Y. Thus $I \cap J \subseteq \Im(Y)$, but $\Im(Y)$ is graded prime, so by [37, Proposition 1.2] we have $I \subseteq \Im(Y)$ or $J \subseteq \Im(Y)$. This means that either $\Im(Y) \in V_R^g(I)$ or $\Im(Y) \in V_R^g(J)$. So, $Y \subseteq V_R^g(I) = Y_1$ or $Y \subseteq V_R^g(J) = Y_2$. Therefore, Y is irreducible by Remark 2.1(ii). □

Recall that a generic point of an irreducible closed subset Y of a topological space is unique if the topological space is a T_0-space, (see [22]).

Theorem 3.5. *Let R be a G-graded ring. Let Y be a nonempty subset of $Spec_g(R)$. Then, Y is an irreducible subset of $Spec_g(R)$ if and only if $Y = V_R^g(p)$ for some graded prime ideal p of R. Hence every nonempty irreducible closed subset of $Spec_g(R)$ has a generic point; namely $p := \Im(Y)$.*

Proof. It is clear that $Y = V_R^g(p)$ is an irreducible closed subset of $Spec_g(R)$ for any $p \in Spec_g(R)$ by Corollary 3.3. Conversely if Y is an irreducible closed subset of $Spec_g(R)$, then $Y = V_R^g(I)$ for some graded ideal I of R and $p := \Im(Y) = \Im(V_R^g(I))$ is a graded prime ideal of R by Theorem 3.4. Hence $Y = V_R^g(I) = V_R^g(\Im(V_R^g(I))) = V_R^g(p)$ as desired. □

Let R be a G-graded ring. Since $Spec_g(R)$ is a T_0-space by [37, Proposition 3.10], so by Theorem 3.5 every nonempty irreducible closed subset of $Spec_g(R)$ has a unique generic point; namely $p := \Im(Y)$.

Corollary 3.6. *Let R be a G-graded ring. The mapping $\theta : p \to V_R^g(p)$ is a surjection of $Spec_g(R)$ onto the set of irreducible closed subsets of $Spec_g(R)$.*

Proof. Follows directly from Theorem 3.5. □

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Let R be a G-graded ring. Then a graded prime I of R is called a graded minimal prime ideal of R, if for any graded prime ideal J of R such that $J \subseteq I$, we have $I = J$.

Theorem 3.7. *Let R be a G-graded ring. Then the irreducible components of $Spec_g(R)$ are the closed subset $V_R^g(p)$, where p is a graded minimal prime ideal of R.*

Proof. Let Y be an irreducible component of $Spec_g(R)$. By Remark 2.1(iii) and Theorem 3.5, the irreducible component of $Spec_g(R)$ is a maximal element of the set $\{V_R^g(p)$, where $p \in Spec_g(R)\}$. Thus $Y = V_R^g(p)$ for some $p \in Spec_g(R)$. Obviously, p is a graded minimal prime ideal, for if $q \in Spec_g(R)$ with $q \subseteq p$, then $V_R^g(p) \subseteq V_R^g(q)$. So $p = q$ due to the maximality of $V_R^g(p)$ and the property that if $V_R^g(I) = V_R^g(J)$, then $I = J$, for any $I, J \in Spec_g(R)$. □

The graded dimension, $dim_g(R)$ of R was defined in [12] as the supremum of all numbers n for which there exists a chain of graded prime ideals $p_0 \subseteq p_1 \subseteq ... \subseteq p_n$ in R, where $dim_g(R) = -1$ if $Spec_g(R) = \phi$ and $dim_g(R) = 0$ if every graded prime ideal is maximal. A proper graded ideal J of R is said to be a graded maximum prime ideal if whenever $I \subseteq J$, we have $I = J$, where $J \in Spec_g(R)$, (see [36]), we will denote the set of graded maximum prime ideals of R by $Max_g^p(R)$. In the next theorem we study the relation between the T_1-space property and the graded dimension of a graded ring R.

Theorem 3.8. *Let R be a G-graded ring. Then, $Spec_g(R)$ is a T_1-space if and only if $Spec_g(R) = Max_g^p(R)$ if and only if $dim_g(R) \leq 0$.*

Proof. First assume that $Spec_g(R)$ is T_1-space. If $Spec_g(R) = \phi$, then $dim_g(R) = -1$. Also, if $Spec_g(R)$ has one element, clearly $dim_g(R) = 0$. So we can assume that $Spec_g(R)$ has more than two elements. Then by [37, Proposition 3.13], $Spec_g(R)$ is a T_1-space if and only if every graded prime ideal of R is a graded maximal prime ideal if and only if $dim_g(R) = 0$. □

The next theorem gives an important result about a G-graded ring R, for which the Zariski topology on $Spec_g(R)$ is a spectral space. We remark that any closed subset of a spectral space is spectral for the induced topology.

Theorem 3.9. *Let R be a G-graded ring. Then $Spec_g(R)$ is a spectral space.*

Proof. Every subset of $Spec_g(R)$ is quasi-compact by [37, Proposition 3.8]. Hence the quasi-compact open sets of $Spec_g(R)$ are closed under finite intersection and form an open basis by Theorem 3.1. Also by [37, Proposition 3.10], $Spec_g(R)$ is a T_0-space. Moreover, every irreducible closed subset of $Spec_g(R)$ has a generic point by Theorem 3.5. Therefore $Spec_g(R)$ is a spectral space by Remark 2.3. □

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4. Some topological properties of the Zariski topology on $Cl.Spec_g(M)$

In this section we review some preliminary results and study new properties about the Zariski topology on $Cl.Spec_g(M)$, which will be needed at next sections.

The assertions in the following proposition are straightforward to prove.

Proposition 4.1. *Let R be a G-graded ring, M be a graded R-module and N a graded R-submodule of M. Then we have the following:*
(i) $\mathbb{V}^g(N) = \cup_{p\in V^g_R((N:_R M))} Cl.Spec^p_g(M)$, where $Cl.Spec^p_g(M) = \{P \in Cl.Spec_g(M) \mid (P :_R M\} = p\}$.
(ii) Let Y be a subset of $Cl.Spec_g(M)$. Then $Y \subseteq \mathbb{V}^g(N)$ if and only if $(N :_R M) \subseteq (\Im(Y) :_R M)$.
(iii) If P is a graded classical prime submodule of M, then $(N :_R M) \subseteq (P :_R M)$ if and only if $\mathbb{V}^g(N) \supseteq \mathbb{V}^g(P)$; consequently, $(N :_R M) = (P :_R M)$ if and only if $\mathbb{V}^g(N) = \mathbb{V}^g(P)$.

Proposition 4.2. *Let R be a G-graded ring, M be a graded R-module, and $p \in Spec_g(R)$. Then we have the following statements:*
(i) If $P \in Cl.Spec^p_g(M)$, then $(P :_R M)M \in Cl.Spec^p_g(M)$.
(ii) If $\{P_\lambda\}_{\lambda\in\Lambda}$ is a family of graded classical prime submodules of M with $(P_\lambda :_R M) = p$ for each $\lambda \in \Lambda$, then $\cap_{\lambda\in\Lambda} P_\lambda \in Cl.Spec^p_g(M)$.

Proof. (i) Since $P \in Cl.Spec^p_g(M)$ we have $(P :_R M) = p$. Clearly, $p \subseteq (pM :_R M)$. On the other hand, $(pM :_R M)M \subseteq pM$. So that $(pM :_R M) \subseteq p$. Thus $p = (pM :_R M)$.
(ii) By the fact that if $\{N_\lambda\}_{\lambda\in\Lambda}$ is an arbitrary family of graded submodules in M, then $(\cap_{\lambda\in\Lambda} N_\lambda :_R M) = \cap_{\lambda\in\Lambda}(N_\lambda :_R M)$. □

Proposition 4.3. *Let R be a G-graded ring and M be a graded R-module with surjective natural map $\psi : Cl.Spec_g(M) \longrightarrow Spec_g(\overline{R})$, where $\overline{R} = R/Ann(M)$. Then, for every $p \in V^g_R(Ann(M))$ in $Spec_g(R)$, there exists a $P \in Cl.Spec_g(M)$ with $(P :_R M) = p$. Hence $Cl.Spec_g(M)$ and $Cl.Spec^p_g(M)$ are nonempty for every graded prime ideal $p \in V^g_R(Ann(M))$.*

Proof. Suppose that $p \in V^g_R(Ann(M))$, thus $Ann(M) \subset p$. Since the natural map ψ is surjective, there exists $P \in Cl.Spec_g(M)$ such that $\psi(P) = \overline{p}$, where $\overline{p} = \overline{(P :_R M)}$. Thus $(P :_R M) = p$. □

For a G-graded ring R, a graded ideal I of R is called a graded radical ideal if $I = Gr(I)$, where $Gr(I) = \Im(V^g_R(I))$, (see [37]).

Proposition 4.4. *Let R be a G-graded ring and M be a graded R-module with surjective natural map $\psi : Cl.Spec_g(M) \longrightarrow Spec_g(\overline{R})$, where $\overline{R} = R/Ann(M)$. Then we have the following statements:*

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(i) Let I be a graded radical ideal of R. Then $(IM :_R M) = I$ if and only if $Ann(M) \subseteq I$.
(ii) $pM \in Cl.Spec^p_g(M)$ for every $p \in V^g_R(Ann(M)) \cap Max_g(R)$.

Proof. (i) The necessity is clear. For sufficiency, we note that $Ann(M) \subseteq I = \cap_i p_i$, where p_i runs through $V^g_R(I)$ since I is a graded radical ideal. On the other hand, M has surjective natural map ψ and $p_i \in V^g_R(Ann(M))$ so by Proposition 4.3, there exists a graded classical prime submodule P_i such that $(P_i :_R M) = p_i$. Now, we obtain that $I \subseteq (IM :_R M) = ((\cap_i p_i)M :_R M) \subseteq \cap_i(p_i M :_R M) = \cap_i p_i = I$. Thus $(IM :_R M) = I$.
(ii) Follows from part (i) and [12, Proposition 2.4(i)], since every graded prime submodule is a graded classical prime submodule. □

Let M be a graded R-module. A graded classical prime submodule P of M is called a graded maximal classical prime submodule of M whenever $P \subseteq Q$, where Q is a graded classical prime submodule of M, implies that $P = Q$, (see [6]). We will denote the set of graded maximal classical prime submodules of M by $Max^{cl}_g(M)$. The following theorem is one of the important theorems in this article, where we characterize the injectivety of the natural map ψ.

Theorem 4.5. *Let R be a G-graded ring and M be a graded R-module with surjective natural map $\psi : Cl.Spec_g(M) \longrightarrow Spec_g(\overline{R})$, where $\overline{R} = R/Ann(M)$. Suppose that $dim_g(R) = 0$. Then the following are equivalent:*
(i) ψ is injective.
(ii) $\mid Cl.Spec^p_g(M) \mid \leq 1$ for every $p \in Cl.Spec_g(M)$.
(iii) $Cl.Spec_g(M)$ is a T_0-Space.
(iv) For every $p \in V^g_R(Ann(M))$, $Cl.Spec^p_g(M) \subseteq Max^{cl}_g(M)$.
(v) For every $p \in V^g_R(Ann(M))$, $P \in Cl.Spec^p_g(M) \Rightarrow P = pM$.
(vi) $Max^{cl}_g(M) = \{pM \mid p \in V^g_R(Ann(M)), pM \neq M\}$.

Proof. (i)⇔(ii)⇔(iii) By [7, Theorem 4.10].
(i)⇒(iv) For every $p \in V^g_R(Ann(M))$ consider $Cl.Spec^p_g(M)$. Let $P \in Cl.Spec^p_g(M)$, then $(P :_R M) = p$ implies $pM \subseteq P$. Suppose that there exist a graded classical prime submodule Q of M such that $P \subseteq Q$, then $(P :_R M) = p \subseteq (Q :_R M)$. But p is a graded maximal ideal of R by Theorem 3.8, so we have $(P :_R M) = p = (Q :_R M)$. Since ψ is injective, by [7, Theorem 4.10] we have $P = Q$, so P is a graded maximal classical prime submodule. Thus $Cl.Spec^p_g(M) \subseteq Max^{cl}_g(M)$.
(iv)⇒(i) Let $P, Q \in Cl.Spec_g(M)$ such that $\psi(P) = \psi(Q)$. So $(P :_R M) = (Q :_R M) = p \in Spec_g(R)$. Then $p \subseteq (pM :_R M) \subseteq (P :_R M) = p$, thus $(pM :_R M) = p \in V^g_R(Ann(M))$. Now by Proposition 4.4(ii) we get $pM \in Cl.Spec^p_g(M) \subseteq Max^{cl}_g(M)$, but $pM \subseteq P$ and $pM \subseteq Q$, so $pM = P = Q$. Therefore ψ is injective.

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(iv)⇒(v) Let $p \in V_R^g(Ann(M))$ and $P \in Cl.Spec_g^p(M)$. Then we have $p = (pM :_R M) = (P :_R M)$, and by Proposition 4.4(ii), we get $pM \in Cl.Spec_g^p(M)$. Since $pM \subseteq P$ we have $pM = P$.
(v)⇒(vi) Set $T := \{pM \mid p \in V_R^g(Ann(M)), pM \neq M\}$. Let $Q \in Max_g^{cl}(M)$. Then $(Q :_R M) = q \supseteq V_R^g(Ann(M))$. Thus $Q \in Cl.Spec_g^q(M)$ and by assumption $Q = qM$. Thus $Q \in T$. To prove the converse assume $q \in V_R^g(Ann(M))$ and $qM \neq M$. Let K be a graded classical prime submodule of M such that $qM \subseteq K \subset M$. Now by Theorem 3.8, we have $(qM :_R M) = (K :_R M) = q$. Hence by assumption we have $K = qM$, that is $qM \in Max_g^{cl}(M)$.
(vi)⇒(iv) Let $p \in V_R^g(Ann(M))$ and $P \in Cl.Spec_g^p(M)$. Thus $(P :_R M) = p \Rightarrow pM \subseteq P$, but by our assumption $pM \in Max_g^{cl}(M) \Rightarrow pM = P$. Therefore $P \in Max_g^{cl}(M)$. □

Now we introduce the definition of the classical weak multiplication module and then we study the relationship between this algebraic property and the injectivety of the natural map $\psi : Cl.Spec_g(M) \longrightarrow Spec_g(\overline{R})$, where $\overline{R} = R/Ann(M)$.

Definition 4.6. Let R be a G-graded ring. A graded R-module M is called graded classical weak multiplication if $Cl.Spec_g(M) = \phi$ or for every graded classical prime submodule P of M, we have $P = IM$ for some graded ideal I of R.

One can easily show that if M is a graded classical weak multiplication module, then $P = (P :_R M)M$ for every graded classical prime submodule P of M. It is clear that if P is a graded classical prime submodule of M, then $p := (P :_R M)$ is a graded prime ideal of R, (see [8, Lemma 3.1]); and P is called a graded p-classical prime submodule.

Theorem 4.7. *Let R be a G-graded ring and M be a graded R-module with natural map $\psi : Cl.Spec_g(M) \longrightarrow Spec_g(\overline{R})$, where $\overline{R} = R/Ann(M)$. Then the following are equivalent:*
(i) ψ is injective.
(ii) M is graded classical weak multiplication R-module.
(iii) $|Cl.Spec_g^p(M)| \leq 1$ for every graded prime ideal p of R.

Proof. (i)⇒(ii) Let P be a graded p-classical prime submodule of M. By Proposition 4.2(i), $(P :_R M)M \in Cl.Spec_g^p(M)$. Combining this fact with [7, Theorem 4.10], we obtain that $P = (P :_R M)M$. Thus M is graded classical weak multiplication.
(ii)⇒(iii) The case $Cl.Spec_g^p(M) = \phi$ is trivially true. Let $Q_1, Q_2 \in Cl.Spec_g^p(M)$ for some graded prime ideal p of R, with $(Q_1 :_R M) = (Q_2 :_R M)$. Therefore $Q_1 = (Q_1 :_R M)M = (Q_2 :_R M)M = Q_2$.
(iii)⇔(i) By [7, Theorem 4.10]. □

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A graded submodule of R-graded module M is said to be a graded classical semiprime submodule of M, if it is an intersection of graded classical prime submodules of M, (see [19]). A graded R-module M is called a fully graded classical semiprime module, if all the graded submodules of M are graded classical semiprime submodules.

Theorem 4.8. *Let R be a G-graded ring and M be a graded R-module with natural map $\psi : Cl.Spec_g(M) \longrightarrow Spec_g(\overline{R})$, where $\overline{R} = R/Ann(M)$. Then the following are equivalent:*
(i) ψ is injective.
(ii) For any graded submodules N_1, N_2 of M, if $\mathbb{V}^g(N_1) = \mathbb{V}^g(N_2)$, then $N_1 = N_2$.
(iii) M is a fully graded classical semiprime R-module.

Proof. (i)$\Leftrightarrow$(ii) By [7, Theorem 4.10].
(ii)$\Rightarrow$(iii) Assume N be a proper graded submodule of M. We claim that $\mathbb{V}^g(N) \neq \phi$, for if not, then $\mathbb{V}^g(N) = \mathbb{V}^g(M) = \phi$ and so $N = M$, a contradiction. By [7, Lemma 3.3 (vi)] $\mathbb{V}^g(N) = \mathbb{V}^g(\Im(\mathbb{V}^g(N)))$ and so by our hypothesis $N = \Im(\mathbb{V}^g(N))$. It follows that N is an intersection of graded classical prime submodules.
(iii)$\Rightarrow$(ii) It is clear that a graded submodule N of M is an intersection of graded classical prime submodules if and only if $N = \cap_{P\in\mathbb{V}^g(N)} P$. Thus the proof is completed. $\square$

From the definition of the Zariski topology on $Cl.Spec_g(M)$ for every graded R-module M, it is evident that the topological space $Cl.Spec_g(M)$ is closely related to $Spec_g(\overline{R})$, where $\overline{R} = R/Ann(M)$, particularly under the natural map $\psi : Cl.Spec_g(M) \longrightarrow Spec_g(\overline{R})$. The surjectivity of the natural map of $Cl.Spec_g(M)$ is particularly important in studying properties of the Zariski topology on $Cl.Spec_g(M)$. The next theorem can be obtained by [7, Theorem 3.7], [7, Theorem 3.8] and [7, Theorem 3.9].

Theorem 4.9. *Let R be a G-graded ring and M be a graded R-module with surjective natural map $\psi : Cl.Spec_g(M) \longrightarrow Spec_g(\overline{R})$, where $\overline{R} = R/Ann(M)$. Then ψ is bijective if and only if ψ is homeomorphic.*

Theorem 4.10. *Let R be a G-graded ring, M and M' be graded R-modules, and f be an epimorphism of M to M'. Then the mapping $\nu : P' \to f^{-1}(P')$ from $Cl.Spec_g(M')$ to $Cl.Spec_g(M)$ is continuous.*

Proof. For any $P' \in Cl.Spec_g(M')$ and any closed set $\mathbb{V}^g(N)$ in $Cl.Spec_g(M)$, where N is a graded submodule M, we have $P' \in \nu^{-1}(\mathbb{V}^g(N)) = \nu^{-1}(\mathbb{V}^g_*((N :_R M)M))$ if and only if $\nu(P') = f^{-1}(P') \supseteq (N :_R M)M$ if and only if $P' \supseteq f((N :_R M)M) = (N :_R M)M'$ if and only if $P' \in \mathbb{V}^g_*((N :_R M)M') =$

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$\mathbb{V}^g((N:M)M')$. Hence $\nu^{-1}(\mathbb{V}^g(N)) = \mathbb{V}^g((N:_R M)M')$, so ν is continuous. $\square$

Theorem 4.11. *Let R be a G-graded ring and M be a graded R-module with the surjective natural map $\psi : Cl.Spec_g(M) \longrightarrow Spec_g(\overline{R})$, where $\overline{R} = R/Ann(M)$. Then we have the following statements:*
(i) $Cl.Spec_g(M)$ is connected if and only if $Spec_g(\overline{R})$ is connected.
(ii) If the intersection of all graded classical prime submodules of M is equal to zero, then $Cl.Spec_g(M)$ is connected.
(iii) If Y is a subset of $Cl.Spec_g(M)$ such that $(0) \in Y$, Then Y is dense in $Cl.Spec_g(M)$.

Proof. (i) The first direction follows from that ψ is surjective and by [6, Theorem 3.7] ψ is a continuous map of the connected space $Cl.Spec_g(M)$ onto $Spec_g(\overline{R})$. Conversely assume that $Spec_g(\overline{R})$ is connected. If $Cl.Spec_g(M)$ is disconnected, then $Cl.Spec_g(M)$ must contain a nonempty proper subset Y that is both open and closed. Accordingly, $\psi(Y)$ is a nonempty subset of $Spec_g(\overline{R})$ that is both open and closed by [7, Theorem 4.8]. To complete the proof, it suffices to show that $\psi(Y)$ is a proper subset of $Spec_g(\overline{R})$ so that $Spec_g(\overline{R})$ is disconnected, a contradiction. Since Y is open, $Y = Cl.Spec_g(M) - \mathbb{V}^g(N)$ for some graded submodule N of M whence $\psi(Y) = Spec_g(\overline{R}) - V^g_{\overline{R}}((N:_R M))$ by [7, Theorem 4.8] again. Therefore, if $\psi(Y) = Spec_g(\overline{R})$, then $V^g_{\overline{R}}((N:_R M)) = \phi$ so that $\overline{(N:_R M)} = \overline{R}$, i.e., $N = M$. It follows that $Y = Cl.Spec_g(M) - \mathbb{V}^g(N) = Cl.Spec_g(M) - \mathbb{V}^g(M) = Cl.Spec_g(M)$ which is impossible since Y is a proper subset of $Cl.Spec_g(M)$. Thus $\psi(Y)$ is a proper subset of $Spec_g(\overline{R})$.
(ii) By [7, Theorem 4.4], we have $Cl(Cl.Spec_g(M)) = \mathbb{V}^g(\Im(Cl.Spec_g(M))) = \mathbb{V}^g(0) = Cl.Spec_g(M)$. Therefore, $Cl.Spec_g(M)$ is connected by [23, Theorem 23.4].
(iii) Is clear by [7, Theorem 4.4] and [7, Theorem 4.5(i)]. $\square$

The following theorems provides important characterizations about the quasi-compact open sets of the Zariski topology on $Cl.Spec_g(M)$.

In Theorem 3.1, we show that for a G-graded ring R, the quasi-compact open sets of $Spec_g(R)$ are closed under finite intersection and form an open base. The next theorem is a generalization of this fact for topological graded modules.

Let $r \in h(R)$, we define $GX_r^{cl} = Cl.Spec_g(M) - \mathbb{V}^g(rM)$. Then every GX_r^{cl} is an open set of $Cl.Spec_g(M)$, $GX_0^{cl} = \phi$, and $GX_1^{cl} = Cl.Spec_g(M)$, (see [7, p. 7]).

Theorem 4.12. *Let R be a G-graded ring and M be a graded R-module with surjective natural map $\psi : Cl.Spec_g(M) \to Spec_g(\overline{R})$, where $\overline{R} = R/Ann(M)$,*

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of $Cl.Spec_g(M)$. Then the quasi-compact open sets of $Cl.Spec_g(M)$ are closed under finite intersection and form an open base.

Proof. It suffices to show that the intersection $D = D_1 \cap D_2$ of two quasi-compact open sets D_1 and D_2 of $Cl.Spec_g(M)$ is a quasi-compact set. Each D_j, $j = 1$ or 2, is a finite union of members of the open base $B = \{GX_r^{cl} \mid r \in R\}$, hence so is D due to [7, Proposition 4.2]. Put $D = \cup_{i=1}^{n} GX_{r_i}^{cl}$ and let ζ be any open cover of D. Then ζ also covers each $GX_{r_i}^{cl}$ which is quasi-compact by [7, Theorem 4.3]. Hence, each $GX_{r_i}^{cl}$ has a finite subcover of ζ and so does D. The other part of the theorem is trivially true due to the existence of the open base B. $\square$

Theorem 4.13. *Let R be a G-graded ring, M be a graded R-module and N be a graded submodule of M. Let $T(N) := \{L \mid L \subseteq N$ and L is a finitely generated submodule of $M\}$. Then we have $\mathbb{V}^g(N) = \cap_{L\in T(N)}\mathbb{V}^g(L)$, and hence $\mathbb{U}^g(N) = Cl.Spec_g(M) - \mathbb{V}^g(N) = \cup_{L\in T(N)}\ \mathbb{U}^g(L)$.*

Proof. Let $P \in \mathbb{V}^g(N)$. If $L \in T(N)$, then $(L :_R M) \subseteq (N :_R M) \subseteq (P :_R M)$. Hence $P \in \mathbb{V}^g(L)$, thus $\mathbb{V}^g(N) \subseteq \cap_{L\in T(N)}\mathbb{V}^g(L)$. Now suppose $P \in \mathbb{V}^g(L)$ for every $L \in T(N)$. If $x \in N$, then $xR \in T(N)$, and hence $P \in \mathbb{V}^g(xR)$. Hence $x \in xR \subseteq P$. Thus $N \subseteq P$. $\square$

Theorem 4.14. *Let R be a G-graded ring and M be a graded R-module. Then every quasi-compact open subset of $Cl.Spec_g(M)$ is of the form $\mathbb{U}^g(N)$ for some finitely generated graded submodule N of M.*

Proof. Let N be a finitely generated R-submodule of M. Suppose $\mathbb{U}^g(N) = Cl.Spec_g(M) - \mathbb{V}^g(N)$ is a quasi-compact open subset of $Cl.Spec_g(M)$. By Theorem 4.13, we have $\mathbb{U}^g(N) = \cup_{L\in T(N)}\mathbb{U}^g(L)$. Since $\mathbb{U}^g(N)$ is quasi-compact by [6, Theorem 4.3], every open covering of $\mathbb{U}^g(N)$ has a finite subcovering, thus $\mathbb{U}^g(N) = \mathbb{U}^g(L_1) \cup ... \cup \mathbb{U}^g(L_n) = \mathbb{U}^g(\sum_{i=1}^{n} L_i)$. This completes the proof. $\square$

Theorem 4.15. *Let R be a G-graded ring and M be a graded R-module with surjective natural map $\psi : Cl.Spec_g(M) \to Spec_g(\overline{R})$, where $\overline{R} = R/Ann(M)$. $Cl.Spec_g(M)$ is a quasi-compact topological space if and only if $Spec_g(\overline{R})$ is a quasi-compact topological space.*

Proof. Suppose that $Cl.Spec_g(M)$ is a quasi compact topological space. Then it follows from [7, Theorem 3.7] and Remark 2.1(vii) that $Spec_g(\overline{R})$ is quasi compact. To show the converse, let $\{\mathbb{V}^g(N_\alpha) : \alpha \in \Lambda\}$ be a family of closed subsets of $Cl.Spec_g(M)$ such that $\cap_{\alpha\in\Lambda}\mathbb{V}^g(N_\alpha) = \phi$, where N_α is a graded submodule of M for every $\alpha \in \Lambda$. Then $\{\psi(\mathbb{V}^g(N_\alpha)) : \alpha \in \Lambda\}$ is a family of closed subset of $Spec_g(\overline{R})$ because ψ is closed by [7, Theorem 3.9]. Since ψ is surjective, it is easy to see that $\cap_{\alpha\in\Lambda}\psi(\mathbb{V}^g(N_\alpha)) = \phi$. As $Spec_g(\overline{R})$ is quasi compact by Theorem 3.8, there exists a finite subset $\Gamma$of Λ such that $\cap_{\alpha\in\Gamma}\psi(\mathbb{V}^g(N_\alpha)) = \phi$. This implies that $\cap_{\alpha\in\Gamma}\mathbb{V}^g(N_\alpha) = \phi$, and hence $Cl.Spec_g(M)$ is quasi compact. $\square$

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5. Irreducible closed subsets and irreducible components of the Zariski topology on $Cl.Spec_g(M)$

In this section, we study the irreducibility of a subset of the Zariski topology on $Cl.Spec_g(M)$, and their generic points.

The next theorem shows that the irreducible subsets of the topological space $Cl.Spec_g(M)$ have a close relationship to the graded classical prime submodules of the graded R-module M.

Theorem 5.1. *Let R be a G-graded ring, M be a graded R-module and Y be a subset of $Cl.Spec_g(M)$. If $\Im(Y)$ is a graded classical prime submodule of M, then Y is irreducible. Conversely, if Y is irreducible, then $H := \{(P :_R M) \mid P \in Y\}$ is an irreducible subset of $Spec_g(R)$, i.e., $\Im(H) = (\Im(Y) :_R M)$ is a graded prime ideal of R.*

Proof. By [7, Corollary 4.6], $\mathbb{V}^g(\Im(Y)) = Cl(Y)$ is irreducible whence Y is irreducible. Conversely, if Y is irreducible, then the image $\psi(Y) = Y'$ of Y under the natural map ψ of $Cl.Spec_g(M)$ is an irreducible subset of $Spec_g(\overline{R})$ because ψ is continuous by [7, Theorem 3.7]. Consequently, we have that $\Im(Y') = (\Im(Y) :_R M)$ is a graded prime ideal of R by Theorem 3.4. Therefore, $\Im(H) = (\Im(Y) :_R M)$ is a graded prime ideal of R so that H is an irreducible subset of $Spec_g(R)$. □

Corollary 5.2. *Let R be a G-graded ring and M be a graded R-module. Then $Cl.Spec_g(M)$ is an irreducible space if and only if $\Im(Cl.Spec_g(M))$ is a graded classical prime of M. In particular, if $(0) \in Cl.Spec_g(M)$, then $Cl.Spec_g(M)$ is an irreducible space.*

Proof. This follows from Theorem 5.1 and the fact that $Cl.Spec_g(M) = \mathbb{V}^g(\Im(Cl.Spec_g(M)))$. □

Corollary 5.3. *Let R be a G-graded ring, M be a graded R-module and Y be an irreducible closed subset of $Cl.Spec_g(M)$. Then $Y = \mathbb{V}^g((\Im(Y) :_R M)M)$.*

Proof. By [7], $Y = \mathbb{V}^g(N)$ for some graded submodule N of M. By Theorem 5.1, $p := (\Im(Y) :_R M)$ is a graded prime ideal of R. Due to [7, Lemma 3.3(vi)] and [7, Lemma 3.6(iii)], it follows that $\mathbb{V}^g(pM) = \mathbb{V}^g((\Im(\mathbb{V}^g(N)) :_R M)M) = \mathbb{V}^g(\Im(\mathbb{V}^g(N))) = \mathbb{V}^g(N) = Y$. □

In Theorem 3.4, we see that in a G-graded ring R, a subset Y of $Spec_g(R)$ is irreducible if and only if $\Im(Y)$ is a graded prime ideal of R. The next theorem is a generalization of this fact to topological graded modules, and in the same time it is a modification for [7, Theorem 4.7].

Theorem 5.4. *Let R be a G-graded ring, M be a graded R-module and Y be a subset of $Cl.Spec_g(M)$. Y is irreducible if and only if $\Im(Y)$ is a graded classical prime submodule. Hence $(\Im(Y) :_R M)$ is a graded prime ideal of R.*

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Proof. If $\Im(Y)$ is a graded classical prime submodule, then by [7, Theorem 4.7], Y is an irreducible. Conversely, assume that Y is irreducible. Then it is clear that $\Im(Y) = \cap_{P\in Y} P$ is a graded submodule of M and $Y \subseteq \mathbb{V}^g(\Im(Y))$. Let I, J be graded ideals of R and K be a graded submodule of M such that $IJK \subseteq \Im(Y)$. Then by [7, Theorem 3.4], $Y \subseteq \mathbb{V}^g(\Im(Y)) \subseteq \mathbb{V}^g(IJK) = \mathbb{V}^g(IK) \cup \mathbb{V}^g(JK)$. Since Y is irreducible we get that $Y \subseteq \mathbb{V}^g(IK)$ or $Y \subseteq \mathbb{V}^g(JK)$, thus $\Im(Y) \supseteq \Im(\mathbb{V}^g(IK)) \supseteq IK$ or $\Im(Y) \supseteq \Im(\mathbb{V}^g(JK)) \supseteq JK$. Therefore by [8, Theorem 2.1 and Lemma 3.1], $\Im(Y)$ is a graded classical prime submodule and $(\Im(Y) :_R M)$ is a graded prime ideal of R. □

In the next theorem, we show that the irreducibility of $Cl.Spec_g(M)$, which is a topological property, implies that $Gr(Ann(M))$ is a graded prime ideal of R, which is an algebraic property.

Theorem 5.5. *Let R be a G-graded ring and M be a graded R-module with surjective natural map $\psi : Cl.Spec_g(M) \to Spec_g(\overline{R})$, where $\overline{R} = R/Ann(M)$. Suppose that $dim_g(R) = 0$. Then following statements are equivalent:*
(i) $Cl.Spec_g(M)$ is an irreducible space.
(ii) $Spec_g(\overline{R})$ is an irreducible space.
(iii) $V^g_R(Ann(M))$ is an irreducible space.
(iv) $Gr(Ann(M))$ is a graded prime ideal of R.
(v) $Cl.Spec_g(M) = \mathbb{V}^g(IM)$ for some $I \in V^g_R(Ann(M))$.

Proof. (i) ⇒ (ii) By [7, Theorem 3.7], the natural map ψ is continuous and by assumption ψ is surjective. Hence $Im(\psi) = Spec_g(\overline{R})$ is also irreducible.
(ii) ⇒ (iii) It is well-known that the mapping $\mu : Spec_g(\overline{R}) \to Spec_g(R)$ given by $J/Ann(M) \to J$ is a graded R-homeomorphism. This implies that $V^g_R(Ann(M))$ is an irreducible space.
(iii) ⇒ (iv) By Theorem 3.4, $\Im(V^g_R(Ann(M))) = Gr(Ann(M))$ is a prime ideal of R.
(iv) ⇒ (v) By Proposition 4.4(ii), $Gr(Ann(M))M$ is a graded classical prime submodule of M. Now, let $P \in Cl.Spec_g(M)$. Then $Gr(Ann(M)) \subseteq (P :_R M)$, and so $Gr(Ann(M))M \subseteq P$. Therefore $Cl.Spec_g(M) = V^g_R(Gr(Ann(M)))M$, where $Gr(Ann(M)) \in V^g_R(Ann(M))$.
(v) ⇒ (i) By Proposition 4.4(ii), IM is a graded classical prime submodule of M. By [7, Corollary 4.6], $\mathbb{V}^g(IM) = Cl.Spec_g(M)$ is irreducible. □

Proposition 5.6. *Let R be a G-graded ring and M be a graded R-module. Then we have the following:*
(i) If $Y = \{P_i \mid i \in \Lambda\}$ is a non-empty family of graded classical prime submodules of M, which is linearly ordered by inclusion, then Y is irreducible subset in $Cl.Spec_g(M)$.
(ii) Assume that the natural map $\psi : Cl.Spec_g(M) \to Spec_g(\overline{R})$, where $\overline{R} =$

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$R/Ann(M)$ is surjective and $Cl.Spec^p_g(M) \neq \phi$ for some $p \in Spec_g(R)$. If p is a graded maximal ideal of R, then $Cl.Spec^p_g(M)$ is an irreducible closed subset of $Cl.Spec_g(M)$.

Proof. (i) Let $\Im(Y) = \cap_{i\in\Lambda} P_i = P$. P is a graded submodule of M. Suppose that $rsm \in P$ but $sm \notin P$ where $r, s \in h(R)$ and $m \in h(M)$. Then $sm \notin P_i$ for some $i \in \Lambda$. Since P_i is a graded classical prime submodule, we get $rm \in P_i$. Let j be any element of Λ such that $j \neq i$. Since Y is linearly ordered by inclusion, we have either$P_i \subseteq P_j$ or $P_j \subseteq P_i$. If $P_i \subseteq P_j$, then we obtain $rm \in P_i \subseteq P_j$. If $P_j \subseteq P_i$, then since $sm \notin P_i$ and P_j is a graded classical prime submodule, we have $rm \in P_j$. Hence $rm \in P$ and $\Im(Y)$ is a graded classical prime submodule, so Y is irreducible on $Cl.Spec_g(M)$ by [7, Proposition 4.7].
(ii) Since p is a graded maximal ideal of R, so by Proposition 4.4(ii), $pM \in Cl.Spec_g(M)$. Now using [7, Corollary 4.6], it suffices to show that $Cl.Spec^p_g(M) = \mathbb{V}^g(pM)$ for the graded maximal ideal p. Let $Q \in \mathbb{V}^g(pM)$, that is, $(Q :_R M) \supseteq (pM :_R M) \supseteq p$. Since p is a graded maximal ideal, $(Q :_R M) = p$. So, $Q \in Cl.Spec^p_g(M)$. Conversely, let $P \in Cl.Spec^p_g(M)$. Then $(P :_R M) = p \subseteq (pM :_R M)$ and because of maximality of p, we obtain $p = (pM :_R M)$ and so $P \in \mathbb{V}^g(pM)$. Therefore $Cl.Spec^p_g(M) = \mathbb{V}^g(pM)$ so $Cl.Spec^p_g(M)$ is closed. □

In [7, Theorem 4.9], we have seen that every graded classical prime submodule P of a graded module M is a generic point of the irreducible closed subset $\mathbb{V}^g(P)$ of $Cl.Spec_g(M)$.

The next results are a good application of the Zariski topology on modules. Indeed, the next theorem show that there is a correspondence between the irreducible closed subsets of $Cl.Spec_g(M)$ and the graded classical prime submodules of the graded R-module M.

Theorem 5.7. *Let R be a G-graded ring, M be a graded R-module with surjective natural map $\psi : Cl.Spec_g(M) \to Spec_g(\overline{R})$, where $\overline{R} = R/Ann(M)$, and Y be a subset of $Cl.Spec_g(M)$. Then Y is an irreducible closed subset of $Cl.Spec_g(M)$ if and only if $Y = \mathbb{V}^g(P)$ for some $P \in Cl.Spec_g(M)$. Thus every irreducible closed subset of $Cl.Spec_g(M)$ has a generic point.*

Proof. Clearly, $Y = \mathbb{V}^g(P)$ is an irreducible closed subset of $Cl.Spec_g(M)$ for any $P \in Cl.Spec_g(M)$ by [6, Corollary 4.6]. Conversely, if Y is an irreducible closed subset of $Cl.Spec_g(M)$, then $Y = \mathbb{V}^g(N)$ for some graded submodule N of M such that $(\Im(\mathbb{V}^g(N)) :_R M) = (\Im(Y) :_R M) = p$ is a graded prime ideal of R by Theorem 5.4. Since ψ is surjective, there must exist a graded classical prime submodule $P \in Cl.Spec_g(M)$ such that $(P :_R M) = p$. Now, we have that $Y = \mathbb{V}^g(N) = \mathbb{V}^g(P)$, for $p = (\Im(\mathbb{V}^g(N)) :_R M) = (P :_R M)$

implies $\mathbb{V}^g(\Im(\mathbb{V}^g(N))) = \mathbb{V}^g(P)$ by [7, Lemma 3.6(i)], whence $\mathbb{V}^g(N) = \mathbb{V}^g(P)$ as $\mathbb{V}^g(N)$ is closed. Thus P is a generic point of Y. $\square$

A generic point P of an irreducible closed subset $Y = \mathbb{V}^g(P)$ may not be unique. However, all generic points P of Y have the same residual $(P :_R M)$. More exactly, $\mathbb{V}^g(P) = \mathbb{V}^g(Q)$ if and only if $(P :_R M) = (Q :_R M)$.

Lemma 5.8. *Let R be a G-graded ring, M be a graded R-module with surjective natural $\psi : Cl.Spec_g(M) \to Spec_g(\overline{R})$, where $\overline{R} = R/Ann(M)$, and N be a graded submodule M. Let $Cl.Spec_g(M)$ be equipped with the Zariski topology and Y be a nonempty subset of the closed set $\mathbb{V}^g(N)$. Then Y is an irreducible closed subset of $\mathbb{V}^g(N)$ if and only if $Y = \mathbb{V}^g(P)$ for some $P \in \mathbb{V}^g(N)$.*

Proof. Since $Y \subseteq \mathbb{V}^g(P) \subseteq Cl.Spec_g(M)$ and $\mathbb{V}^g(P)$ is closed in $Cl.Specg(M)$, by applying Remark 2.1(iv) we have that Y is an irreducible closed subset of $\mathbb{V}^g(N)$ if and only if Y is an irreducible closed subset of $Cl.Spec_g(M)$ if and only if $Y = \mathbb{V}^g(P)$ for some $P \in Cl.Spec_g(M)$ by Theorem 5.7 if and only if $Y = \mathbb{V}^g(P)$ for some $P \in \mathbb{V}^g(N)$ as $P \in \mathbb{V}^g(P) \subseteq \mathbb{V}^g(N)$. $\square$

Let W be a topological space. We consider strictly decreasing chain $Z_0 \supseteq Z_1 \supseteq ... \supseteq Z_r$ of length r of irreducible closed subsets Z_i of W. The supremum of the lengths, taken over all such chains, is called the combinatorial dimension of W and denoted by $dim(W)$. For the empty set, ϕ, the combinatorial dimension of ϕ is defined to be -1.

Theorem 5.9. *Let R be a G-graded ring and M be a graded R-module with surjective natural map $\psi : Cl.Spec_g(M) \to Spec_g(\overline{R})$, where $\overline{R} = R/Ann(M)$. Then the following hold:*
(i) $dim(Cl.Spec_g(M)) = dim(Spec_g(\overline{R}))$.
(ii) $Cl.Spec_g(M)$ is an irreducible topological space if and only if $Spec_g(\overline{R})$ is an irreducible topological space.

Proof. (i) Let $Z_0 \supset Z_1 \supset ... \supset Z_s$ be a descending chain of irreducible closed subsets of $Cl.Spec_g(M)$. Then by Theorem 5.7, for $i(1 \le i \le s)$, there exists a graded submodule P_i of M such that $(P_i :_R M)$ is a graded prime ideal of R and $Z_i = \mathbb{V}^g(P_i)$. By Theorem 3.5, there exists a one to one correspondence between the graded prime ideals of R and the irreducible closed subsets of $Spec_g(R)$, it follows that $V^g_{\overline{R}}(\overline{(P_0 :_R M)}) \supset V^g_{\overline{R}}(\overline{(P_1 :_R M)}) \supset ... \supset V^g_{\overline{R}}(\overline{(P_s :_R M)})$ is a descending chain of irreducible closed subsets of $Spec_g(\overline{R})$ by Theorem 3.4. Hence $dim(Cl.Spec_g(M)) \le dim(Spec_g(\overline{R}))$. Now let $W_0 \supset W_1 \supset ... \supset W_t$ be a descending chain of irreducible closed subsets of $Spec_g(R)$. By Theorem 3.4, for each $j(1 \le j \le t)$, there exists a graded prime ideal q_i of R such that $W_j = V^g_{\overline{R}}(q_j)$. This yields that $q_0 \supset q_1 \supset ... \supset q_t$ is an ascending chain of graded prime ideal of R. Since M has a surjective natural map ψ, by Proposition 4.3,

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for every p_j $i(1 \leq i \leq t)$, there exists a graded submodule Q_j of M such that $q_j = (Q_j :_R M)$. Hence by Theorem 5.7, $\mathbb{V}^g(Q_0) \supset \mathbb{V}^g(Q_1) \supset ... \supset \mathbb{V}^g(Q_t)$ is a descending chain of irreducible closed subsets of $Cl.Spec_g(M)$. It follows that $dim(Cl.Spec_g(M)) \geq dim(Spec_g(\overline{R}))$ and the proof is completed.
(ii) We have a similar argument as in part (i) of Theorem 4.11. □

Corollary 5.10. *Let R be a G-graded ring and M be a graded R-module with surjective natural map $\psi : Cl.Spec_g(M) \to Spec_g(\overline{R})$, where $\overline{R} = R/Ann(M)$, such that $Cl.Spec_g(M)$ has combinatorial dimension zero. Then:*
(i) Every irreducible closed subset of $Cl.Spec(M)$ is an irreducible component of $Cl.Spec_g(M)$.
(ii) For every $p \in V^g_R(Ann(M))$ and for every graded p-classical prime submodule P of M, $Cl.Spec^p_g(M) = \{p$-graded classical prime submodules of $M\} = \mathbb{V}^g(P)$.

Proof. (i) Is clear because $dim(Cl.Spec_g(M)) = 0$.
(ii) Since by Theorem 5.9(iii) $dim(Cl.Spec_g(M)) = 0 = dim(Spec(\overline{R}))$, every irreducible closed subset of $Spec_g(\overline{R})$ is an irreducible component, that is $Spec_g(\overline{R}) = \{$ graded maximal ideals of $\overline{R}\}$. Hence $p = (P :_R M)$ is a graded maximal ideal of R. If $Q \in Cl.Spec_g(M)$, then $(Q :_R M)$ is also a graded maximal ideal. Therefore, $Q \in \mathbb{V}^g(P)$ if and only if $(Q :_R M) = (P :_R M) = p$ if and only if $Q \in Cl.Spec^p_g(M)$. Thus $\mathbb{V}^g(P) = Cl.Spec^p_g(M)$. □

Theorem 5.11. *Let R be a G-graded ring and M be a graded R-module with surjective natural map $\psi : Cl.Spec_g(M) \to Spec_g(\overline{R})$, where $\overline{R} = R/Ann(M)$. Then for any $P \in Cl.Spec_g(M)$ we have the following statements:*
(i) The correspondence $\rho : P \to \mathbb{V}^g(P)$ is a surjection of $Cl.Spec_g(M)$ onto the set of irreducible closed subsets of $Cl.Spec_g(M)$.
(ii) The correspondence $\varphi : \mathbb{V}^g(P) \to \overline{(P :_R M)}$ is a bijection of the set of irreducible components of $Cl.Spec_g(M)$ onto the set of graded minimal prime ideals of $\overline{R}$.

Proof. (i) Follows from Theorem 5.7.
(ii) Since each irreducible component of $Cl.Spec_g(M)$ is a maximal element of the set $\{\mathbb{V}^g(Q) \mid Q \in Cl.Spec_g(M)\}$, by Theorem 5.7. Now we get the result by applying [7, Theorem 4.5(ii)]. □

Let R be a G-graded ring and I be a graded ideal of R. If $V^g_R(I)$ has at least one minimal member with respect to inclusion, then every minimal member in this form is called a graded minimal prime divisors of I, (see [36, Corollary 2.3]).

Corollary 5.12. *Let R be a G-graded ring, M be a graded R-module with surjective natural map $\psi : Cl.Spec_g(M) \to Spec_g(\overline{R})$, where $\overline{R} = R/Ann(M)$,*

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and N be a graded submodule of M. Then we have the following statements:
(i) The mapping $\rho^* : P \to \mathbb{V}^g(P)$ is a surjection of $\mathbb{V}^g(N)$ onto the set of irreducible closed subsets of $\mathbb{V}^g(N)$.
(ii) The mapping $\varphi^* : \mathbb{V}^g(P) \to \overline{(P :_R M)}$ is a bijection of the set of irreducible components of $\mathbb{V}^g(N)$ onto the set of graded minimal prime divisors of $\overline{(N :_R M)}$ in $\overline{R} = R/Ann(M)$.

Proof. (i) Follows directly from Theorem 5.11.
(ii) Applying Lemma 5.8 and Theorem 5.11 implies that φ^* is a well defined injective mapping. We show that φ^* is surjective. Let $\overline{p}$ be a graded minimal prime divisor of $\overline{(N :_R M)}$ in $\overline{R}$ and let p be the graded prime ideal of R such that $p/Ann(M) = \overline{p}$. Then $p \supseteq (N :_R M) \supseteq Ann(M)$. Since M has a surjective natural map ψ, thus by Proposition 4.3, there exists a graded p-classical prime submodule $P \in Cl.Spec_g(M)$. Now $(P :_R M) = p \supseteq (N :_R M)$ implies that $P \in \mathbb{V}^g(N)$, and so $\mathbb{V}^g(P) \subseteq \mathbb{V}^g(N)$. Thus $\mathbb{V}^g(P)$ is an irreducible closed subset of $\mathbb{V}^g(N)$ by Lemma 5.8. Note that the minimality of $\overline{p} \in V^g_{\overline{R}}(\overline{(N :_R M)})$ implies the maximality of $\mathbb{V}^g(P)$ among all irreducible closed subsets $\mathbb{V}^g(Q)$ of $\mathbb{V}^g(N)$ where $Q \in \mathbb{V}^g(N)$, as $\overline{(Q :_R M)} \supseteq \overline{(N :_R M)}$. Therefore, $\mathbb{V}^g(P)$ is an irreducible component of $\mathbb{V}^g(N)$. This proves that φ^* is surjective. □

Let R be a G-graded ring and M be a graded R-module. A graded submodule P of M is called a graded minimal classical prime submodule of M if whenever $Q \subseteq P$, where Q is a graded classical prime submodule of M, implies that $P = Q$.

Theorem 5.13. *Let R be a G-graded ring, M be a graded R-module with bijective natural map $\psi : Cl.Spec_g(M) \to Spec_g(\overline{R})$, where $\overline{R} = R/Ann(M)$, and Y be a subset of $Cl.Spec_g(M)$, then the correspondence $\pi : \mathbb{V}^g(P) \to P$ is a bijection of the set of irreducible components of $Cl.Spec_g(M)$ onto the set of minimal elements of $Cl.Spec_g(M)$ with respect to inclusion.*

Proof. Let Y be an irreducible component of $Cl.Spec_g(M)$. Since each irreducible component of $Cl.Spec_g(M)$ is a maximal element of the set $\{\mathbb{V}^g(P) \mid P \in Cl.Spec_g(M)\}$ by Theorem 5.7, we have $Y = \mathbb{V}^g(P)$ for some $P \in Cl.Spec_g(M)$. Obviously, P is a minimal element of $Cl.Spec_g(M)$, for if $Q \in Cl.Spec_g(M)$ with $Q \subseteq P$, then $\mathbb{V}^g(P) \subseteq \mathbb{V}^g(Q)$. So $P = Q$ due to the maximality of $\mathbb{V}^g(P)$ and [7, Theorem 4.10]. Let P be a minimal element of $Cl.Spec_g(M)$ with $\mathbb{V}^g(P) \subseteq \mathbb{V}^g(K)$ for some $K \in Cl.Spec_g(M)$. Then $P \in \mathbb{V}^g(K)$ whence $(K :_R M)M \subseteq P$. By Proposition 4.2(i), $(K :_R M)M$ belongs to $Cl.Spec_g^{(K:_RM)}(M)$. Hence, $P = (K :_R M)M$ due to the minimality of P. By [7, Lemma 3.6(iii)], $\mathbb{V}^g(K) = \mathbb{V}^g((K :_R M)M) = \mathbb{V}^g(P)$. This implies that $\mathbb{V}^g(P)$ is an irreducible component of $Cl.Spec_g(M)$, as desired. □

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Corollary 5.14. *Let R be a G-graded ring, M be a graded R-module with bijective natural map $\psi : Cl.Spec_g(M) \to Spec_g(\overline{R})$, where $\overline{R} = R/Ann(M)$, and N be a graded submodule of M. Then the mapping $\pi^* : \mathbb{V}^g(P) \to P$ is a bijection of the set of irreducible components of $\mathbb{V}^g(N)$ onto the set of graded minimal classical prime submodule of N.*

Proof. Directly from Theorem 5.13. □

Theorem 5.15. *Let R be a G-graded ring, M be a graded R-module with surjective natural map $\psi : Cl.Spec_g(M) \to Spec_g(\overline{R})$, where $\overline{R} = R/Ann(M)$, and Y be a subset of $Cl.Spec_g(M)$. Then the set of all irreducible components of $Cl.Spec_g(M)$ is of the form $\Phi := \{\mathbb{V}^g(IM) \mid I$ is a minimal element of $V_R^g(Ann(M))$ with respect to inclusion$\}$.*

Proof. Let Y be an irreducible component of $Cl.Spec_g(M)$. By [7, Theorem 4.7], $Y = \mathbb{V}^g(P)$ for some $P \in Cl.Spec_g(M)$. Hence, $Y = \mathbb{V}^g(P) = \mathbb{V}^g((P :_R M)M)$ by [7, Lemma 3.6(iii)]. So, we have $p := (P :_R M) \in V_R^g(Ann(M))$. We must show that p is a minimal element of $V_R^g(Ann(M))$ with respect to inclusion. To see this let $q \in V_R^g(Ann(M))$ and $q \subseteq p$. Then $q/Ann(M) = \overline{q} \in Spec_g(\overline{R})$, and because M has surjective natural map ψ, by Proposition 4.3 there exists an element $Q \in Cl.Spec_g(M)$ such that $(Q :_R M) = q$. So, $Y = \mathbb{V}^g(P) \subseteq \mathbb{V}^g(Q)$. Hence, $Y = \mathbb{V}^g(P) = \mathbb{V}^g(Q)$ due to the maximality of $\mathbb{V}^g(P)$. By [7, Theorem 4.5(ii)], we have that $p = q$. For the reverse containment, let $Y = \mathbb{V}^g(IM)$ for some minimal element I in $V_R^g(Ann(M))$. Since M has surjective natural map ψ, then by Proposition 4.3, there exists an element $K \in Cl.Spec_g(M)$ such that $(K :_R M) = I$. So using [7, Lemma 3.6(iii)] we have $Y = \mathbb{V}^g(IM) = \mathbb{V}^g((K :_R M)M) = \mathbb{V}^g(K)$, and so Y is irreducible by Theorem 5.7. Suppose that $Y = \mathbb{V}^g(K) \subseteq \mathbb{V}^g(Q)$, where Q is an element of $Cl.Spec_g(M)$. Since $K \in \mathbb{V}^g(Q)$ and I is minimal, it follows that $(K :_R M) = (Q :_R M)$. Now, by [7, Lemma 3.6(iii)], $Y = \mathbb{V}^g(K) = \mathbb{V}^g((K :_R M)M) = \mathbb{V}^g((Q :_R M)M) = \mathbb{V}^g(Q)$. □

6. The Zariski topology on $Cl.Spec_g(M)$ as a spectral space

Unlike the case of $Spec_g(R)$, $Cl.Spec_g(M)$ is not a T_0-space in general. We consider now some conditions for which $Cl.Spec_g(M)$ has the T_0, T_1, and T_2-space properties and we give a characterization for the graded classical prime spectrum of a graded modules over a graded commutative ring.

The classical prime dimension of a graded R-module M, $dim_g^{cl}(M)$ was defined in [5], as $dim_g^{cl}(M) = \sup\{ht(P) \mid P \in Cl.Spec_g(M)\}$, where $ht(P)$ is the greatest non-negative integer n such that there exists a chain of graded classical prime submodules of M, $P_0 \subset P_1 \subset ..... \subset P_n = P$, and $ht(P) = \infty$ if no such n exists. The next theorem is a generalization of Theorem 3.8, for topological graded modules which is studying the relation between the T_1-space property and the graded classical prime dimension of a graded R-module M.

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Theorem 6.1. *Let R be a G-graded ring and M be a graded R-module. Then $Cl.Spec_g(M)$ is a T_1-space if and only if $Cl.Spec_g(M) = Max_g^{cl}(M)$ if and only if $dim_g^{cl}(M) \leq 0$.*

Proof. By [7, Theorem 4.13] $Cl.Spec_g(M)$ is a T_1-space if and only if every graded classical prime submodule is a graded maximal. Therefore $Cl.Spec_g(M)$ is a T_1-space if and only if $Cl.Spec_g(M) = Max_g^{cl}(M)$ if and only if $dim_g^{cl}(M) \leq 0$. □

Suppose that W is a topological space. Let x and y be points in W. We say that x and y can be separated by neighborhoods if there exists a neighborhood U of x and a neighborhood V of y such that U and V are disjoint ($U \cap V = \phi$). W is a T_2-spaces if any two distinct points of W can be separated by neighborhoods, (see [28]).

It is well-known that if W is a finite space, then W is a T_1-space if and only if W is the discrete space. The cofinite topology is the smallest topology satisfying the T_1 axiom; i.e., it is the smallest topology for which every singleton set is closed. In fact, an arbitrary topology on W satisfies the T_1 axiom if and only if it contains the cofinite topology. If W is not finite, then this topology is not T_2, since no two open sets in this topology are disjoint, (See [28]). Thus the next corollary can be obtained by Remark 2.2, [7, Theorem 4.14], and Theorem 6.1.

Corollary 6.2. *Let R be a G-graded ring and M be a graded R-module such that $Cl.Spec_g(M)$ is finite. Then the following statements are equivalent:*
(i) $Cl.Spec_g(M)$ is a T_1-space.
(ii) $Cl.Spec_g(M)$ is the cofinite topology.
(iii) $Cl.Spec_g(M)$ is discrete.
(iv) $dim_g^{cl}(M) \leq 0$.

In the next proposition, we show that if the topological space $Cl.Spec_g(M)$ is a T_1-space, then we can obtain some properties of the graded classical prime submodules of M.

Theorem 6.3. *Let R be a G-graded ring, M be a graded R-module, Y be a subset of $Cl.Spec_g(M)$ and $P \in Cl.Spec_g^p(M)$.*
(i) If $\{P\}$ is closed in $Cl.Spec_g(M)$, then P is a maximal element of $Cl.Spec_g(M)$ and $Cl.Spec_g^p(M) = \{P\}$.
(ii) $Cl.Spec_g(M)$ is a T_1-space if and only if $Cl.Spec_g(M)$ is a T_0-space and for every element $Q \in Cl.Spec_g(M)$, $(Q :_R M)$ is a maximal element in $\{(L :_R M) \mid L \in Cl.Spec_g(M)\}$.
(iii) If $Cl.Spec_g(M)$ is a T_1-space, then $Cl.Spec_g(M)$ is a T_0-space and every graded classical prime submodule is a maximal element of $Cl.Spec_g(M)$. The converse is also true, when M is finitely generated.

(iv) Let $(0) \in Cl.Spec_g(M)$*. Then* $Cl.Spec_g(M)$ *is a* T_1*-space if and only if* (0) *is the only graded classical prime submodule of* M*.*

Proof. (i) Let $Q \in Cl.Spec_g(M)$ such that $P \subseteq Q$. Then $(P :_R M) \subseteq (Q :_R M)$. Now by [7, Theorem 4.4] we have $Q \in \mathbb{V}^g(P) = Cl(\{P\}) = \{P\}$. Hence, $Q = P$, and so P is a maximal element of $Cl.Spec_g(M)$.
(ii) The result follows from Remark 2.2, [7, Theorem 4.10] and [7, Theorem 4.12].
(iii) Trivially, $Cl.Spec_g(M)$ is a T_0-space and every singleton subset of $Cl.Spec_g(M)$ is closed. Every graded classical prime submodule is a maximal element of $Cl.Spec_g(M)$ by [7, Theorem 4.13]. Now, we suppose that M is finitely generated. Thus, every graded classical prime submodule is maximal. Let $Q \in Cl.Spec_g(M)$ such that $Q \in Cl(\{P\}) = \mathbb{V}^g(P)$. Since P is maximal, $(P :_R M) = (Q :_R M)$. By [7, Theorem 4.10], $Q = P$. Hence, every singleton subset of $Cl.Spec_g(M)$ is closed. So, $Cl.Spec_g(M)$ is a T_1-space.
(iv) Use part (i). □

Theorem 6.4. *Let* R *be a* G*-graded ring and* M *be a graded classical weak multiplication* R*-module. Then* $Cl.Spec_g(M)$ *is a* T_1*-space if and only if it is a* T_2*-space.*

Proof. Assume that $Cl.Spec_g(M)$ is a T_2 -space. Then it is a T_1-space. Conversely assume that $Cl.Spec_g(M)$ is a T_1 -space. If $| Cl.Spec_g(M) |= 1$ or $| Cl.Spec_g(M) |= 2$, then $Cl.Spec_g(M)$ is a T_2-space. Now assume that $| Cl.Spec_g(M) |> 2$. Then we can take three distinct elements in $Cl.Spec_g(M)$, say P_1, P_2, and P_3. Since M is graded classical weak multiplication, $\mathbb{V}^g(P_1P_3) = \{P_1, P_3\} = Cl.Spec_g(M) - \mathbb{V}^g(P_2)$, $\mathbb{V}^g(P_2P_3) = \{P_2, P_3\} = Cl.Spec_g(M) - \mathbb{V}^g(P_1)$ and $\mathbb{V}^g(P_2) = \{P_2\} = Cl.Spec_g(M) - \mathbb{V}^g(P_1P_3)$ are open sets in $Cl.Spec_g(M)$. This implies that $P_1 \in \mathbb{V}^g(P_1P_3)$ and $P_2 \in \mathbb{V}^g(P_2)$. Moreover, $\mathbb{V}^g(P_1P_3) \cap \mathbb{V}^g(P_2) = \phi$. □

In the sequel, we present conditions under which the Zariski topology on $Cl.Spec_g(M)$ is a spectral space. We recall that any closed subset of a spectral space is spectral for the induced topology.

Theorem 6.5. *Let* R *be a* G*-graded ring and* M *be a graded* R*-module with bijective natural map* ψ*:* $Cl.Spec_g(M) \longrightarrow Spec_g(\overline{R})$*, where* $\overline{R} = R/Ann(M)$*. Then* $Cl.Spec_g(M)$ *is a spectral space.*

Proof. Since ψ is surjective then, by [7, Theorem 4.3] the open set GX_r^{cl} in $Cl.Spec_g(M)$ for each $r \in h(R)$ is quasi-compact, and by Theorem 4.12 and [7, Theorem 4.2] the quasi-compact open sets of $Cl.Spec_g(M)$ are closed under finite intersection and form an open base, and by [7, Proposition 4.9] and Theorem 5.7 each irreducible closed subset has a generic point, and since ψ is injective then by [7, Theorem 4.10], $Cl.Spec_g(M)$ is a T_0-space. It follows from Remark 2.3, $Cl.Spec_g(M)$ is a spectral space. □

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Theorem 6.6. *Let R be a G-graded ring and M be a graded R-module. If the natural map ψ: $Cl.Spec_g(M) \longrightarrow Spec_g(\overline{R})$, where $\overline{R} = R/Ann(M)$, is surjective, then the following statements are equivalent:*
(i) $Cl.Spec_g(M)$ is a spectral space.
(ii) $Cl.Spec_g(M)$ is a T_0-space.
(iii) $\mid Cl.Spec_g^p(M) \mid \leq 1$ for every $p \in Spec_g(R)$.
(iv) ψ is injective.
(v) $Cl.Spec_g(M)$ is homeomorphic to $Spec_g(\overline{R})$ under ψ.

Proof. (i)⇒(ii) Is trivial
(ii)⇒(i) is by [7, Theorem 4.10] and Theorem 6.5.
(ii)⇔(iii)⇔(iv) The equivalence is due to [7, Theorem 4.10].
(iv)⇔(v) Follows from Theorem 4.9. □

Although the surjectivity of the natural map of $Cl.Spec_g(M)$ is an important condition for $Cl.Spec_g(M)$ to be spectral, it is not a necessary condition. Some $Cl.Spec_g(M)$ is a spectral space without being surjective.

Theorem 6.7. *Let R be a G-graded ring and M be a graded R-module with natural map ψ: $Cl.Spec_g(M) \longrightarrow Spec_g(\overline{R})$, where $\overline{R} = R/Ann(M)$, such that the image $Im(\psi)$ of ψ is a closed subset of $Spec_g(\overline{R})$. Then $Cl.Spec_g(M)$ is a spectral space if and only if ψ is injective.*

Proof. By Theorem 3.9, $Spec_g(\overline{R})$ is a spectral space, so since $Im(\psi)$ is a closed subset of the spectral space $Spec_g(\overline{R})$, $Im(\psi)$ is spectral for the induced topology. Assume that ψ is injective. Then the bijection $\psi : Cl.Spec_g(M) \longrightarrow Im(\psi)$ is continuous by [7, Theorem 3.7]. To show that ψ is also closed, let N be a graded submodule of M. Then $Y = Im(\psi) \cap V^g_{\overline{R}}((\overline{N :_R M}))$ is a closed subset of $Im(\psi)$ and $\psi^{-1}(Y) = \psi^{-1}(Im(\psi) \cap V^g_{\overline{R}}((\overline{N :_R M}))) = \psi^{-1}(Im(\psi)) \cap \psi^{-1}(V^g_{\overline{R}}((\overline{N :_R M}))) = Cl.Spec_g(M) \cap \psi^{-1}(V^g_{\overline{R}}((\overline{N :_R M}))) = \mathbb{V}^g((N :_R M)M) = \mathbb{V}^g(N)$ by [7, Theorem 3.7] and [7, Lemma 3.6(iii)]. Since $\psi : Cl.Spec_g(M) \longrightarrow Im(\psi)$ is surjective, $\psi(\mathbb{V}^g(N)) = \psi(\psi^{-1}(Y)) = Y$, a closed subset of Y'. Now, we can conclude that $\psi : Cl.Spec_g(M) \longrightarrow Im(\psi)$ is a homeomorphism, whence $Cl.Spec_g(M)$ is a spectral space. Conversely, if $Cl.Spec_g(M)$ is a spectral space, then $Cl.Spec_g(M)$ is a a T_0-space so that ψ is injective due to [7, Theorem 4.10]. □

Theorem 6.8. *Let R be a G-graded ring and M be a graded R-module with surjective natural map ψ: $Cl.Spec_g(M) \longrightarrow Spec_g(\overline{R})$, where $\overline{R} = R/Ann(M)$. Suppose that $dim_g(R) = 0$. The following statements are equivalent:*
(i) $Cl.Spec_g(M)$ is a spectral space.
(ii) ψ is injective and $Im(\psi)$ is closed.
(iii) ψ is injective and for any $q \in V^g_R(Ann(M))$, we have $\cap_{p \in Im(\psi)} p \subseteq q \Rightarrow qM \neq M$.

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Proof. (i)$\Leftrightarrow$(ii) By Theorem 6.7.
(ii)$\Leftrightarrow$(iii) $Im(\psi)$ is closed if and only if $Im(\psi) \supseteq Cl(Im(\psi)) = V_R^g(\cap_{p\in Im(\psi)}p)$ by Theorem 3.2. But by Theorem 4.5, Theorem 6.1, we have $Im(\psi) = \{p|p \in V_R^g(Ann(M)),\ pM \neq M\}$. Therefore $Im(\psi)$ is closed if and only if for any $q \in V_R^g(Ann(M))$, we have $\cap_{p\in Im(\psi)}p \subseteq q \Rightarrow qM \neq M$. □

As a corollary to [7, Lemma 3.6(iii)], we have that $\eta^g(M) \subseteq \eta_*^g(M)$ which will be used for the next results.

Theorem 6.9. *Let R be a G-graded ring and M be a g-Cl.Top R-module. Then the quasi-Zariski topology $\varrho_*^g(M)$ on $Cl.Spec_g(M)$ is finer than the Zariski topology $\varrho^g(M)$. That is $\varrho^g(M) \leq \varrho_*^g(M)$.*

Theorem 6.10. *Let R be a G-graded ring and M be a g-Cl.Top R-module. Then $Cl.Spec_g(M)$ is a T_0-space for both the Zariski topology and the quasi-Zariski topology. Hence if M is a g-Cl.Top R-module, then M has an injective natural map ψ.*

Proof. Let $P_1, P_2 \in Cl.Spec_g(M)$. Then by [6, Corollary 2.2(i)], $Cl(\{P_1\}) = Cl(\{P_2\})$ if and only if $P_1 = P_2$. Now by the fact that a topological space is a T_0-space if and only if the closures of distinct points are distinct, we conclude that for any g-Cl.Top R-module M, $Cl.Spec_g(M)$ is a T_0-space. Since $\varrho^g(M) \leq \varrho_*^g(M)$, by Theorem 6.9, $Cl.Spec_g(M)$ is a T_0-space too. □

The next theorem is an important result about a graded R-module M for which $Cl.Spec_g(M)$ is a spectral topological space. The result is obtained by combining Remark 2.1, Remark 2.3, [7, Theorem 4.10], Theorem 4.7, Theorem 4.8, Theorem 6.5, Theorem 6.6, Theorem 6.7, Theorem 6.8 and Theorem 6.10.

Theorem 6.11. *Let R be a G-graded ring and M be a graded R-module. Then $Cl.Spec_g(M)$ is a spectral topological space in each of the following cases:*
(i) The natural map $\psi : Cl.Spec_g(M) \to Spec_g(\overline{R})$, where $\overline{R} = R/Ann(M)$, is surjective and one of the following conditions is satisfied:
(1) $Cl.Spec_g(M)$ is a T_0 space.
(2) ψ is injective.
(3)$\mid Cl.Spec_g^p(M) \mid \leq 1$ for every $p \in Spec_g(R)$.
(4) For any graded submodule N_1, N_2 of M, if $\mathbb{V}^g(N_1) = \mathbb{V}^g(N_2)$, then $N_1 = N_2$.
(5) M is a fully graded classical semiprime submodule R-module.
(6) M is a graded classical weak multiplication R-module.
(7) M is a g-Cl.Top R-module.
(8) $Cl.Spec_g(M) = Max_g^{cl}(M)$.
(9) $dim_g^{cl}(M) \leq 0$.
(10) $Cl.Spec_g(M)$ is homeomorphic to $Spec_g(\overline{R})$ under ψ.

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(ii) $Cl.Spec_g(M)$ *is a finite space and one of the parts (1)-(10) in part (i) is satisfied.*
(iii) $Im(\psi)$ *is closed and one of the parts (1)-(10) in part (i) is satisfied.*
(iv) If $dim_g(R) = 0$ *and for any* $q \in V_R^g(Ann(M))$*, we have* $\cap_{p \in Im(\psi)} p \subseteq q \Rightarrow qM \neq M$ *and one of the parts (1)-(10) in part (i) is satisfied.*

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