Iranian Journal of Mathematical Sciences and Informatics
Vol. 17, No. 2 (2022), pp 289-305
DOI: 10.52547/ijmsi.17.2.289

Transmission of Cholera Disease with Laplacian and Triangular Parameters

Mehmet Merdan[a], Zafer Bekiryazici[b*], Tulay Kesemen[c], Tahir Khaniyev[d]

[a]Department of Mathematical Engineering, Gumushane University, Gumushane, Turkey
[b]Department of Mathematics, Recep Tayyip Erdogan University, Rize, Turkey
[c]Department of Mathematics, Karadeniz Technical University, Trabzon, Turkey
[d]Department of Industrial Engineering, TOBB University of Economics and Technology, Ankara, Turkey

E-mail: mmerdan@gumushane.edu.tr
E-mail: zafer.bekiryazici@erdogan.edu.tr
E-mail: tkesemen@gmail.com
E-mail: tahirkhaniyev@etu.edu.tr

Abstract. A mathematical model has been introduced for the transmission dynamics of cholera disease by GQ Sun et al. recently. In this study, we add Laplacian and Triangular random effects to this model and analyze the variation of results for both cases. The expectations and coefficients of variation are compared for the random models and the results are used to comment on the differences and similarities between the effects of these probability distributions. The randomness of the model itself is also investigated through comparison of the random and deterministic outcomes.

Keywords: Random differential equation, Simulation, Cholera, Laplace distribution, Triangular distribution.

*Corresponding Author

Received 11 December 2018; Accepted 24 April 2020

[DOI: 10.52547/ijmsi.17.2.289]

2000 Mathematics subject classification: 34F05, 92D30.

1. Introduction

Cholera is a significant health problem which causes thousands of deaths every year especially in developing countries. It is caused by the bacterium Vibrio cholera, which infects the small intestine. Cholera infected patients usually experience little to no symptoms. However, if untreated, the infection can lead to severe diarrhea, vomiting and even death [3, 4, 5]. Cholera is a water and food-borne disease where the ingestion of contaminated water and food are the primary causes of infection. This fact makes Cholera a leading health issue in especially many African countries lacking in infrastructure. World Health Organization (WHO) has reported more than 172000 infections in 2015 alone, leading to more than 1300 deaths in 42 countries from all parts of the world including Europe and Americas [2, 5]. Recurring Cholera epidemics show that the disease is still a global threat.

Cholera disease has also been studied by using many mathematical models. Numerous compartmental models analyzing the transmission dynamics, model stabilities and reproduction numbers of various Cholera cases around the world can be found in the literature [6, 7, 8, 9]. It can be seen that most of these modeling studies have been performed on a deterministic basis. This means that the numerical analysis of the equation systems in these deterministic compartmental models are done by using values for the parameters, initial conditions and etc. which are not random. However, it is known that some components of these models for cholera disease are random in their real life behavior. For instance, the compartmental model in [8] uses a value of 0.015 per day for disease induced death rate, whereas a survey of cholera related death rates in Americas shows that this value can range from 0.0009 to 0.14 depending on the patient's age, country and various other factors [10]. Hence, a deterministic analysis has its drawbacks for an accurate modeling of the real life dynamics of cholera.

In this study, the parameters of a deterministic mathematical model for the transmission dynamics of cholera will be added random effects to analyze the variation of results for Laplacian and Triangular distributions. The referred study of GQ Sun et al. uses a compartmental model to analyze the reproduction number, equilibrium states and stability of a cholera model via deterministic parameters. We will be transforming these parameters to random variables with Laplace and Triangular distributions to compare the randomness of the results from simulations. The motivation of our study is the previous studies of the authors where models of avian-influenza and bacterial resistance were analyzed under random effects [1, 12]. Random effects added to the parameters of a model play the role of noise terms for these parameters. Such random models with randomized parameters give the compartmental models the ability

[DOI: 10.52547/ijmsi.17.2.289]

to efficiently represent the real life randomness of the disease behavior described by these parameters. Several modeling investigations containing statistical and stability analyses can be found in the literature for analyzing various real life phenomena [21], [22].

The study outline can be given as follows. The second part includes the presentation of the deterministic model used in this study. The random models with Laplacian and Triangular random effects are introduced in the third section. The fourth section includes the results for these random models. The comparison of the results for these two random models is given in the last part.

2. A Deterministic Model of Cholera Transmission

GQ Sun et al. have used a compartmental model to mathematically investigate various properties of cholera disease in China [11]. The model is a modification of the SIR type with an additional compartment for the bacteria.

$$\begin{aligned}
\frac{dS}{dt} &= \mu N - \left(\beta_e S \frac{B}{k+B} + \beta_h SI\right) - \mu S - \nu S, \qquad (2.1)\\
\frac{dI}{dt} &= \beta_e S \frac{B}{k+B} + \beta_h SI - (\gamma + \mu) I,\\
\frac{dR}{dt} &= \gamma I - \mu R + \nu S,\\
\frac{dB}{dt} &= \xi I - \delta B - cB.
\end{aligned}$$

Equation system (2.1) consists of four deterministic ordinary differential equations with nonlinearities where the variables S, I and R describe the population sizes of the susceptible, infected and recovered humans, respectively. The variable B describes the concentration of vibrios in contaminated water. The variable t is interpreted as the number of years in the referred study [11].

Par.	Description	Value (Unit)
μ	Natural birth or death rate	0.0066 $(year^{-1})$
k	Concentration of Vibrio Cholerae in environment	500 $(\frac{cells}{mL})$
N	Human number in China	1.36×10^9 $(Unitless)$
β_e	Environment to human transmission rate	$Estimated$ $(year^{-1})$
β_h	Human to human transmission rate	$Estimated$ $(year^{-1})$
ν	Vaccination rate	$Estimated$ $(year^{-1})$
γ	Recovery rate	0.2 $(days^{-1})$
ξ	Rate of human contribution to human cholera	10 $(cells/(mL \times days))$
δ	Decay rate of vibrios	$\frac{1}{30}$ $(days^{-1})$
c	Disinfection rate	4 $(year^{-1})$

Table 1. Parameters of the deterministic model.

[DOI: 10.52547/ijmsi.17.2.289]

The parameters of the model (2.1) have been listed in Table 2 with their descriptions. The numerical values of the parameters are needed for the numerical analysis of system (2.1), since the analytical solutions of such nonlinear systems are mostly too complex. It should be noted that the time variable t denotes the number of years in the investigated time interval. The values of β_e, β_h and ν have been estimated in the referred study as below:

$$\beta_e \simeq 2.6699 \times 10^{-6},\ \beta_h \simeq 5.3508 \times 10^{-9},\ \nu \simeq 0.31017.$$

These values of the parameters will be used in the simulations of the model along with the following initial values of the variables [11]:

$$S(0) = 1.36 \times 10^9,\ I(0) = 28,\ R(0) = 0,\ B(0) = 0.$$

The initial values have been determined to simulate the disease transmission in a population of 1.36×10^9 people along with additional 28 diseases patients. It is assumed that there are initially no recovered people and no bacteria in water.

3. Random Parameters for the Model

In this part the parameters of the deterministic model (2.1) will be added random effects (random noise) with Laplace (also referred to as 'classical Laplace') and symmetrical triangular distributions, respectively. The general Laplace distribution is a continuous probability distribution with an unbounded support. If a random variable X has the general Laplace distribution with parameters a and b, its probability density function (PDF) f is given by [15]:

$$f(x) = \frac{1}{2b} \exp\left(-\frac{|x-a|}{b}\right),\ x \in \mathbf{R}. \tag{3.1}$$

Here, a is used to determine the location while b is used for scaling the distribution. Laplace distribution has been previously used in speech processing [13] and image compression [14]. Unlike Laplace distribution, the general Triangular distribution is a continuous probability distribution with a bounded support. If a random variable Y has the general Triangular distribution with parameters c, d and p, its probability density function g is given by [16]:

$$g(y) = \begin{cases} \frac{2(y-c)}{(d-c)(p-c)} & c \le y \le p \\ \frac{2(d-y)}{(d-c)(d-p)} & p \le y \le d \end{cases}. \tag{3.2}$$

Here, c and d are the lower and upper bounds of the support, respectively ($c \le y \le d$). The parameter p, which is the peak of the density function, determines the shape of the distribution. Triangular distribution is frequently used in decision making for modeling the distribution of random variables with a bounded domain and a known maximum value, where this known maximum is used to define the mode of the triangular distribution.

[DOI: 10.52547/ijmsi.17.2.289]

Since the general Laplace distribution is symmetrical about a, we will be using a symmetrical general Triangular distribution with $p = \frac{(c+d)}{2}$. The comparison of the general Laplace distribution and symmetrical general Triangular distribution with specific parameters ($X \sim Laplace(a = 0, b = 0.2)$ and $Y \sim Triangle(c = -1, d = 1, p = 0)$) has been given in Figure 1. The variance of the general Laplace distribution for these parameters is $\frac{2}{25}$, while the variance of the general Triangular distribution for the parameters above is $\frac{1}{6}$.

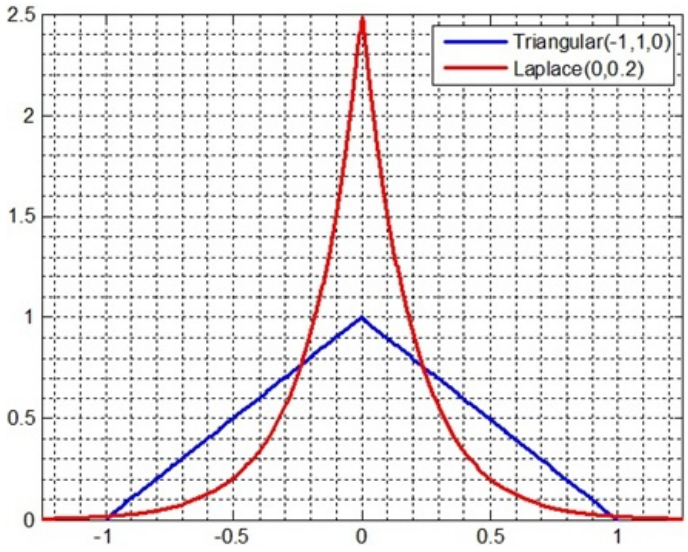

FIGURE 1. PDF of general Laplace and general symmetrical Triangular distributions for specific parameters.

The distribution function of the general Laplace distribution is given as

$$F(x) = \begin{cases} \frac{1}{2}\exp\left(\frac{x-a}{b}\right) & x \in (-\infty, a] \\ 1 - \frac{1}{2}\exp\left(-\frac{x-a}{b}\right) & x \in (a, \infty) \end{cases}.$$

We can use this distribution function to obtain the three sigma rule for general Laplace distribution. The probability of a general Laplace distributed random variable to assume values from an interval that is three standard deviations around its mean can be calculated as below. Since the standard deviation and the variation of a general Laplace distributed random variable are $E(X) = a$ and $Var(X) = 2b^2$ [19];

$$P(a - 3\sqrt{2b^2} < X < a + 3\sqrt{2b^2}) = F(a + 3\sqrt{2b^2}) - F(a - 3\sqrt{2b^2}).$$

[DOI: 10.52547/ijmsi.17.2.289]

Using the distribution function of the general Laplace distribution and the parameters $a = 0,\ b = 0.2$, we get

$$F(a+3\sqrt{2b^2}) - F(a-3\sqrt{2b^2}) = F\left(\frac{3\sqrt{2}}{5}\right) - F\left(-\frac{3\sqrt{2}}{5}\right)$$
$$= 1 - \frac{1}{2}\exp\left(-5\frac{3\sqrt{2}}{5}\right) - \frac{1}{2}\exp\left(-5\frac{3\sqrt{2}}{5}\right) = 1 - \exp(-3\sqrt{2}) \simeq 0.9856.$$

Thus, the probability of a general Laplace distributed ($a = 0,\ b = 0.2$) random variable to be within the interval

$$(a-3\sqrt{2b^2}, a+3\sqrt{2b^2}) = \left(-\frac{3\sqrt{2}}{5}, \frac{3\sqrt{2}}{5}\right) = (-0.8485, 0.8485)$$

is about 98.5%. For the interval $(-1, 1)$, we can calculate the interval as

$$P(-1 < X < 1) = F(1) - F(-1)$$
$$= 1 - \frac{1}{2}\exp(-5) - \frac{1}{2}\exp(-5) = 1 - \exp(-5) \simeq 0.9933.$$

The general Triangular distribution has a bounded support and for the parameters $c = -1, d = 2, p = 0.5$, such a random variable gets all (100%) of its values from the interval $(-1, 1)$. Using the results above, we find that for a general Laplace distribution (which has unbounded support), for the parameters $a = 0, b = 0.2$, a random variable gets about 99.33% of its values from the interval $(-1, 1)$ (see Figure 1). This popular rule, known as the three sigma rule, states that for a normally distributed variable, about 99.73% of the values lie within three standard deviations of around the mean. Hence, by using appropriate parameters, we will be comparing the variations of the results for two continuous distributions with a bounded and an unbounded support, respectively. The appropriate parameters will guarantee that almost all of the possible values for the random effects will be drawn from the same interval for both distributions.

3.1. **The Model with General Laplacian Random Effects.** The parameters of the equation system (2.1), $\mu, k, N, \beta_e, \beta_h, \nu, \gamma, \xi, \delta, c$, will be transformed into random variables with general Laplace distribution to obtain a random model for Cholera transmission under Laplacian random effects. The random effects act as noise terms in each parameter to model the random behavior of these parameters. Old parameters are replaced with the new random set of

[DOI: 10.52547/ijmsi.17.2.289]

parameters $\mu^*, k^*, N^*, \beta_e^*, \beta_h^*, \nu^*, \gamma^*, \xi^*, \delta^*, c^*$ to obtain a set of random differential equations:

$$\begin{aligned}
\frac{dS}{dt} &= \mu^* N^* - \left(\beta_e^* S \frac{B}{k^* + B} + \beta_h^* SI\right) - \mu^* S - \nu^* S, \\
\frac{dI}{dt} &= \beta_e^* S \frac{B}{k^* + B} + \beta_h^* SI - (\gamma^* + \mu^*) I, \\
\frac{dR}{dt} &= \gamma^* I - \mu^* R + \nu^* S, \\
\frac{dB}{dt} &= \xi^* I - \delta^* B - c^* B.
\end{aligned} \tag{3.3}$$

Since the equations include random variables, the compartments of the model now become random variables as well, (S, I, R, B). However, we use the same notation for the compartments to underline the transition from the deterministic model to the random model by only updating the parameters. The initial values of system (3.3) are once again $S(0) = 1.36 \times 10^9,\ I(0) = 28,\ R(0) = 0,\ B(0) = 0$.

Random parameters with independent general Laplace distributions are given as $(a_i \in R,\ b_i \in (0, \infty),\ i = \overline{(1, 10)})$:

$$\begin{aligned}
&\mu^* \sim Laplace(a_1, b_1),\ k^* \sim Laplace(a_2, b_2),\ N^* \sim Laplace(a_3, b_3), \\
&\beta_e^* \sim Laplace(a_4, b_4),\ \beta_h^* \sim Laplace(a_5, b_5),\ \nu^* \sim Laplace(a_6, b_6), \\
&\gamma^* \sim Laplace(a_7, b_7),\ \xi^* \sim Laplace(a_8, b_8),\ \delta^* \sim Laplace(a_9, b_9), \\
&c^* \sim Laplace(a_{10}, b_{10}).
\end{aligned}$$

where $a_i, i = \overline{(1, 10)}$ and $b_i, i = \overline{(1, 10)}$ are the corresponding location and scale parameters of the distributions, respectively. The expected value and variation of a general Laplace distributed random variable, whose PDF is in the form of (3.1), is given as $E(X) = a$ and $Var(X) = 2b^2$. A general Laplace distributed random variable X can be denoted by using a standard Laplace distributed random variable U, which has the probability density function $h(u) = \frac{1}{2} e^{-|u|}, u \in \mathbf{R}$, as $X = a + bU$ $(E(U) = 0, Var(U) = 2)$. Thus,

$$\begin{aligned}
E(X) &= E(a + bU) = a + bE(U) = a, \\
Var(X) &= Var(a + bU) = Var(bU) = b^2 Var(U) = 2b^2.
\end{aligned}$$

The random parameters can thus be denoted as:

$$\begin{aligned}
&\mu^* = a_1 + b_1 U_1,\ k^* = a_2 + b_2 U_2,\ N^* = a_3 + b_3 U_3, \\
&\beta_e^* = a_4 + b_4 U_4,\ \beta_h^* = a_5 + b_5 U_5,\ \nu^* = a_6 + b_6 U_6, \\
&\gamma^* = a_7 + b_7 U_7,\ \xi^* = a_8 + b_8 U_8,\ \delta^* = a_9 + b_9 U_9,\ c^* = a_{10} + b_{10} U_{10}.
\end{aligned}$$

for random variables $U_i, i = \overline{(1, 10)}$ with independent standard Laplace distributions. The location and scaling coefficients of the random parameters are determined in the following way to obtain random variables with the expected

[DOI: 10.52547/ijmsi.17.2.289]

values as listed in Table 2 and standard deviations that are around 5% of the same values. In particular, for the parameter k;

$$E(k^*) = 500,\ Var(k^*) = \left(\left(\frac{5}{100}\right) \times 500\right)^2 = 625;\ k^* \sim Laplace(a_2, b_2)$$
$$\Rightarrow E(k^*) = a_2 = 500,\ Var(k^*) = 2b_2^2 = 625.$$

Thus, $(a_2, b_2) = (500, 25\frac{\sqrt{2}}{2})$ for the random parameter k^*. Similarly for rest of the random parameters;

$$\mu^* = 0.0066 + 2.3335 \times 10^{-4} U_1,\ k^* = 500 + 17.6777 U_2,$$
$$N^* = 1.36 \times 10^9 + 4.8083 \times 10^7 U_3,\ \beta_e^* = 2.6699 \times 10^{-6} + 9.4395 \times 10^{-8} U_4,$$
$$\beta_h^* = 5.3508 \times 10^{-9} + 1.8918 \times 10^{-10} U_5,\ \nu^* = 0.31017 + 0.0110 U_6,$$
$$\gamma^* = 0.2 + 0.0071 U_7,\ \xi^* = 10 + 0.3536 U_8,$$
$$\delta^* = \frac{1}{30} + 0.0012 U_9,\ c^* = 4 + 0.1414 U_{10}.$$

These random variables are written in (3.3) to obtain the random model:

$$\begin{aligned}
\frac{dS}{dt} =& (0.0066 + 2.3335 \times 10^{-4} U_1)(1.36 \times 10^9 + 4.8083 \times 10^7 U_3) \\
& - ((2.6699 \times 10^{-6} + 9.4395 \times 10^{-8} U_4) S \frac{B}{(500 + 17.6777 U_2) + B} \\
& + (5.3508 \times 10^{-9} + 1.8918 \times 10^{-10} U_5) SI) \\
& - (0.0066 + 2.3335 \times 10^{-4} U_1) S - (0.31017 + 0.0110 U_6) S, \\
\frac{dI}{dt} =& (2.6699 \times 10^{-6} + 9.4395 \times 10^{-8} U_4) S \frac{B}{(500 + 17.6777 U_2) + B} \\
& + (5.3508 \times 10^{-9} + 1.8918 \times 10^{-10} U_5) SI - ((0.2 + 0.0071 U_7) \\
& + (0.0066 + 2.3335 \times 10^{-4} U_1)) I, \\
\frac{dR}{dt} =& (0.2 + 0.0071 U_7) I - (0.0066 + 2.3335 \times 10^{-4} U_1) R \\
& + (0.31017 + 0.0110 U_6) S, \\
\frac{dB}{dt} =& (10 + 0.3536 U_8) I - \left(\frac{1}{30} + 0.0012 U_9\right) B \\
& - (4 + 0.1414 U_{10}) B.
\end{aligned} \tag{3.4}$$

Once again, the same initial values are used for the random model (3.4). The random model can be simulated in MATLAB by generating the independent random variables $U_i, i = \overline{(1, 10)}$ with standard Laplace distribution through exponentially distributed random numbers [20].

3.2. **The Model with General Triangular Random Effects.** Similarly, a random model of the form (3.3) will be created for Cholera transmission, this time using symmetrical triangularly distributed random effects. It is

[DOI: 10.52547/ijmsi.17.2.289]

known that a random variable Y with symmetrical triangular distribution on the interval $(-1, 1)$ can be denoted as the difference of two independent random variables Y_{i_1}, Y_{i_2} with standard uniform distribution: $Y = Y_{i_1} - Y_{i_2}$ [17, 18]. The expectation and variance of such a random variable would be $E(Y) = 0,\ Var(Y) = \frac{1}{6}$ respectively. If a random variable Z is defined as $Z = m + nY$ for the random variable defined above, its expectation and variance would be (parameters m and n determine the location and scales of each random parameter):

$$E(Z) = E(m + nY) = m + nE(Y) = m,$$
$$Var(Z) = Var(m + nY) = Var(nY) = n^2 Var(Y) = \frac{n^2}{6}.$$

Thus, for introducing around 5% random noise (standard deviation) to the deterministic parameters, the coefficients m and n must be calculated for each of these parameters. In particular, for the parameter k^* (where $Y \sim Triangular(-1, 1, 0)$);

$$E(k^*) = 500,\ Var(k^*) = \left(\left(\frac{5}{100}\right) \times 500\right)^2 = 625;\ k^* = m_2 + n_2 Y$$
$$\Rightarrow E(k^*) = m_2 = 500,\ Var(k^*) = \frac{n^2}{6} = 625.$$

Thus, $(m_2, n_2) = (500, 25\sqrt{6})$ for the random parameter k^*. Hence, the rest of the parameters are defined as

$$\mu^* = 0.0066 + 8.0833 \times 10^{-4} Y_1,\ k^* = 500 + 61.2372 Y_2,$$
$$N^* = 1.36 \times 10^9 + 1.6657 \times 10^8 Y_3,\ \beta_e^* = 2.6699 \times 10^{-6} + 3.2699 \times 10^{-7} Y_4,$$
$$\beta_h^* = 5.3508 \times 10^{-9} + 6.5534 \times 10^{-10} Y_5,\ \nu^* = 0.31017 + 0.0380 Y_6,$$
$$\gamma^* = 0.2 + 0.0245 Y_7,\ \xi^* = 10 + 1.2247 Y_8,$$
$$\delta^* = \frac{1}{30} + 0.0041 Y_9,\ c^* = 4 + 0.4899 Y_{10}.$$

where $Y_i, i = \overline{(1, 10)}$ are defined as $Y_i = Y_{i_1} - Y_{i_2}$ for independent random variables Y_{i_1}, Y_{i_2} with standard uniform distribution. Using these random effects,

[DOI: 10.52547/ijmsi.17.2.289]

the new random model becomes

$$\begin{aligned}
\frac{dS}{dt} =&(0.0066 + 8.0833 \times 10^{-4}Y_1)(1.36 \times 10^9 + 1.6657 \times 10^8 Y_3) \\
&- ((2.6699 \times 10^{-6} + 3.2699 \times 10^{-7}Y_4)S\frac{B}{(500 + 61.2372Y_2) + B} \\
&+ (5.3508 \times 10^{-9} + 6.5534 \times 10^{-10}Y_5)SI) \\
&- (0.0066 + 8.0833 \times 10^{-4}Y_1)S - (0.31017 + 0.0380Y_6)S, \\
\frac{dI}{dt} =&(2.6699 \times 10^{-6} + 3.2699 \times 10^{-7}Y_4)S\frac{B}{(500 + 61.2372Y_2) + B} \\
&+ (5.3508 \times 10^{-9} + 6.5534 \times 10^{-10}Y_5)SI - ((0.2 + 0.0245Y_7) \\
&+ (0.0066 + 8.0833 \times 10^{-4}Y_1))I, \\
\frac{dR}{dt} =&(0.2 + 0.0245Y_7)I - (0.0066 + 8.0833 \times 10^{-4}Y_1)R \\
&+ (0.31017 + 0.0380Y_6)S, \\
\frac{dB}{dt} =&(10 + 1.2247Y_8)I - \left(\frac{1}{30} + 0.0041Y_9\right)B \\
&- (4 + 0.4899Y_{10})B.
\end{aligned} \tag{3.5}$$

Using the same initial conditions, system (3.5) can easily be simulated in MATLAB by generating uniform random variables $Y_{i_j}, i = \overline{(1, 10)}, j = 1, 2$.

4. Results

The deterministic results of the equation system (2.1) is obtained in MATLAB using the built-in lower order schemes as follows (Figure 2).

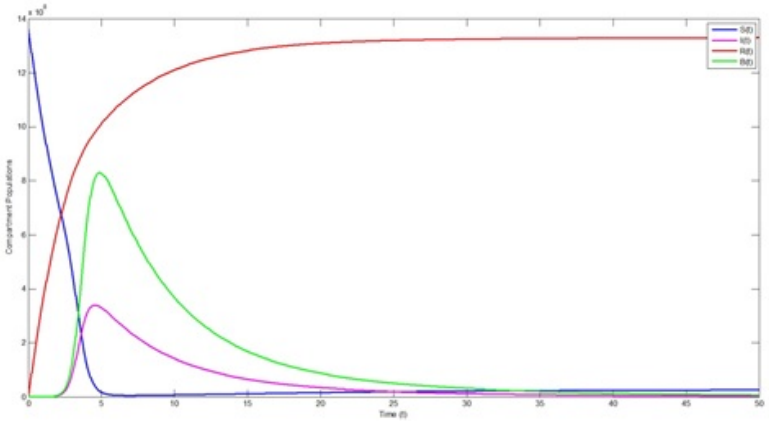

Figure 2. Deterministic results for the compartments of model (2.1).

[DOI: 10.52547/ijmsi.17.2.289]

The extremum points for the compartments are as follows: $S(t)$ gets its maximum value 1.36×10^9 at $t = 0$ and its minimum value 5.541×10^6 at $t = 6.905$. $I(t)$ gets its maximum value 3.401×10^8 at $t = 4.459$ and its minimum value 28 at $t = 0$. $R(t)$ gets its maximum value 1.33×10^9 at $t = 50$ and its minimum value 0 at $t = 0$. $B(t)$ gets its maximum value 8.304×10^8 at $t = 4.897$ and its minimum value 0 at $t = 0$. The deterministic results show that the population of susceptible people decreases until year 7 and then maintains a similar level until it ends the process at the value of 2.672×10^7, while the population of recovered people increases throughout the examined interval. The infected population and the concentration of vibrios show similar behavior, both reaching their peak levels before year 5 and then decreasing afterwards.

4.1. **Variation of the Results for Laplacian Random Effects.** The random model (3.4) is simulated in MATLAB more than 10^5 times to obtain the following expectations and variation coefficients for the random compartments (Figure 3, 4).

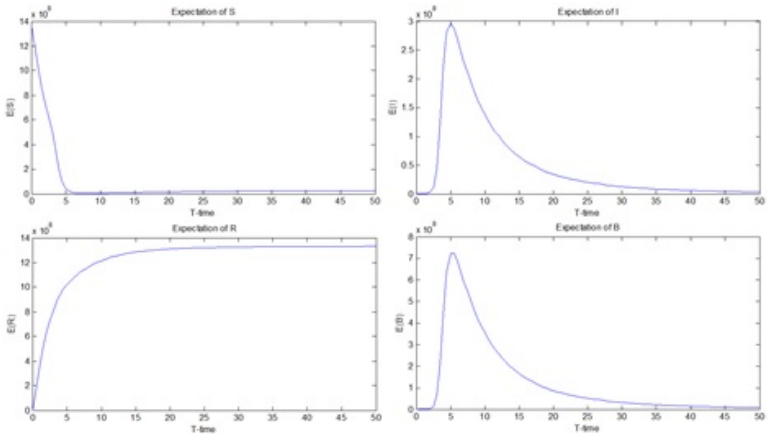

FIGURE 3. Expected values of the compartments of model (3.4).

The results show that the compartments show similar behavior under Laplacian random effects in the parameters. $S(t)$ gets its maximum value 1.36×10^9 at $t = 0$ and its minimum value 6.555×10^6 at $t = 8$ ($S(t) = 2.691 \times 10^7$ at $t = 50$). $I(t)$ gets its maximum value 2.968×10^8 at $t = 5$ and its minimum value 28 at $t = 0$. $R(t)$ gets its maximum value 1.33×10^9 at $t = 50$ and its minimum value 0 at $t = 0$. $B(t)$ gets its maximum value 7.247×10^8 at $t = 5.5$ and its minimum value 0 at $t = 0$.

The coefficients of variation (CV) for each of the compartments under Laplacian random effects are shown in Figure 4.

[DOI: 10.52547/ijmsi.17.2.289]

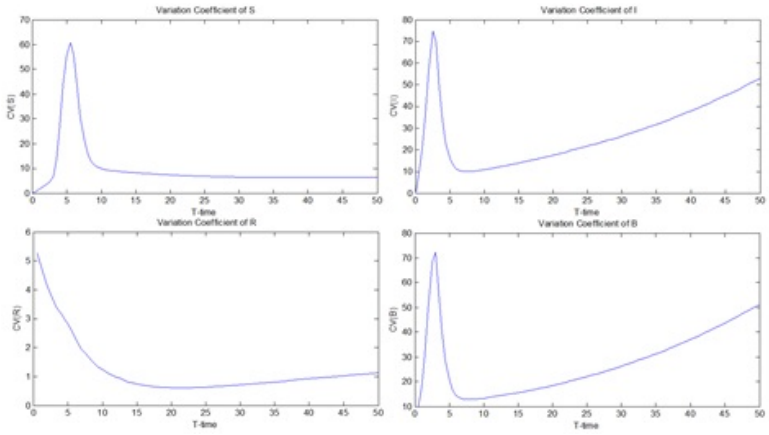

FIGURE 4. Variation coefficients of the compartments of model (3.4).

The coefficient of variation is obtained by the formula $\frac{100\times std.dev.}{expectation}$. CV for the susceptible population changes between 0% to 60.85% ($t = 5.5$), meaning that the compartment $S(t)$ can show values that vary up to almost 60% from its expectation. CV for the other compartments vary within the following intervals: 0% to 74.71% ($t = 2.5$) for $I(t)$, 0.6098% to 5.271% ($t = 0.5$) for $R(t)$ and 10.13% to 72.73% ($t = 3$) for $B(t)$. The results show that all of the compartments, except $R(t)$, can produce results that vary almost up to 75% from their expectations under only 5% random noise in their parameters. The population of recovered people has a relatively smaller CV, meaning that it does not behave as randomly compared to other compartments.

4.2. **Variation of the Results for Triangular Random Effects.** Model (3.5) with triangular random effects is simulated in MATLAB ($N > 10^5$ times) to obtain the following expectations and variation coefficients for the compartments (Figure 5, 6).

The results for the extremum points of the expectations for the triangularly random effects are as follows: $S(t)$ gets its maximum value 1.36×10^9 at $t = 0$ and its minimum value 6.706×10^6 at $t = 8$ ($S(t) = 2.65 \times 10^7$ at $t = 50$). $I(t)$ gets its maximum value 2.863×10^8 at $t = 5$ and its minimum value 28 at $t = 0$. $R(t)$ gets its maximum value 1.328×10^9 at $t = 50$ and its minimum value 0 at $t = 0$. $B(t)$ gets its maximum value 7.138×10^8 at $t = 5.5$ and its minimum value 0 at $t = 0$. Once again, the results for the random model 3.5 are similar to both the deterministic and the (Laplacian) random models.

The coefficients of variation for the compartments of the random model (3.5) are obtained as follows: CV for the susceptible population changes between 0% to 64.12% ($t = 5.5$), meaning that the compartment $S(t)$ can show values that

[DOI: 10.52547/ijmsi.17.2.289]

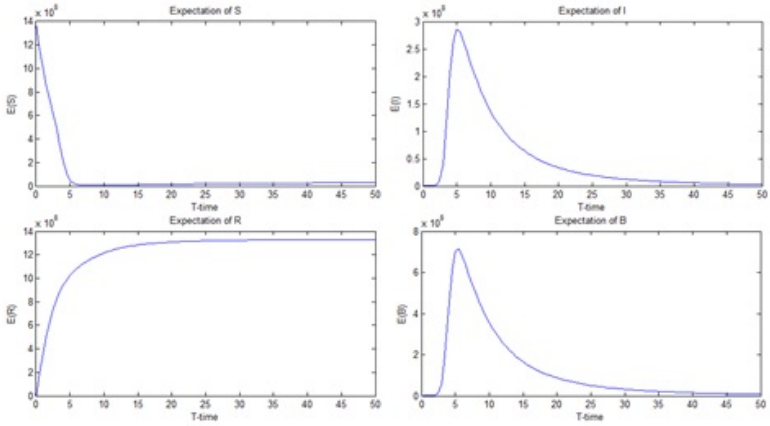

FIGURE 5. Expected values of the compartments of model (3.5).

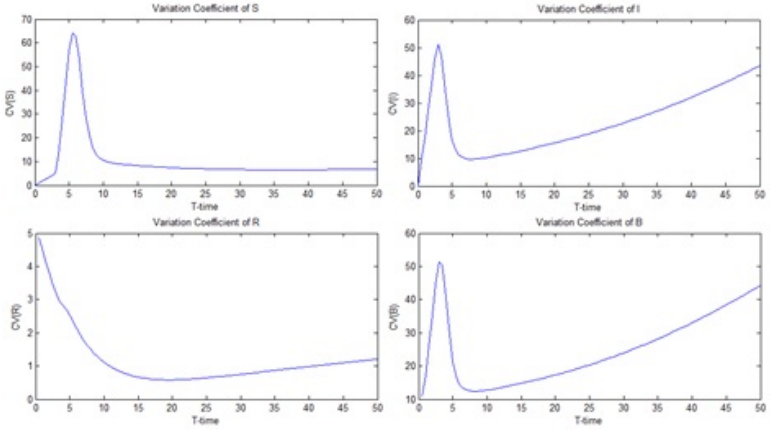

FIGURE 6. Variation coefficients of the compartments of model (3.5).

vary up to almost 64% from its expectation. CV for the other compartments vary within the following intervals: 0% to 51.33% ($t = 3$) for $I(t)$, 0.584% to 4.859% ($t = 0.5$) for $R(t)$ and 10.91% to 51.33% ($t = 3$) for $B(t)$. The compartment $R(t)$ shows significantly smaller randomness compared to the other compartments under triangularly distributed random effects too.

5. Conclusions

In this study, the recent Cholera transmission model of GQ Sun et al. was analyzed under Laplacian and symmetrical triangular random effects. The deterministic model (2.1) and the random models (3.4) and (3.5) all indicate

[DOI: 10.52547/ijmsi.17.2.289]

that the susceptible population decreases during the first years (until $t \sim 7-8$, t indicates years) of the process and then remain at almost the same level. The infected population and the vibrio concentration both increase in the first years of the process (~ 5 years) and then decrease to low levels. The recovered population compartment starts at 0 and increase throughout the 50 years of examination until almost all of the population becomes recovered.

Although the behavior of $S(t), I(t), R(t)$ and $B(t)$ are similar in all models (Figures 2, 3, 5), there are still some small differences between the results for the compartments. The differences between the deterministic and random models show the effects of the random behavior of the parameters on the results. The differences between the random models with Laplacian and symmetrical Triangular random effects show the effects of the distributions on the results. These differences can be seen on the table of extremum values (Table 5). It can be seen that there is a noticeable difference between the maximum points of $R(t), B(t)$ and the minimum points of $S(t)$ for the models with random noise in the parameters. Susceptible population gets a higher minimum under random effects in a longer time, while $I(t)$ and $B(t)$ get a lower peak value.

	Deterministic	Laplacian	Triangular
$S(t)$	$(1.36 \times 10^9, 0)$	$(1.36 \times 10^9, 0)$	$(1.36 \times 10^9, 0)$
$I(t)$	$(3.401 \times 10^8, 4.459)$	$(2.968 \times 10^8, 5)$	$(2.863 \times 10^8, 5)$
$R(t)$	$(1.330 \times 10^9, 50)$	$(1.330 \times 10^9, 50)$	$(1.328 \times 10^9, 50)$
$B(t)$	$(8.304 \times 10^8, 4.897)$	$(7.247 \times 10^8, 5.5)$	$(7.138 \times 10^8, 5.5)$

TABLE 2. Maximum values for models (2.1), (3.4) and (3.5) (with their corresponding times).

	Deterministic	Laplacian	Triangular
$S(t)$	$(5.541 \times 10^6, 6.905)$	$(6.555 \times 10^6, 8)$	$(6.706 \times 10^6, 8)$
$I(t)$	$(28, 0)$	$(28, 0)$	$(28, 0)$
$R(t)$	$(0, 0)$	$(0, 0)$	$(0, 0)$
$B(t)$	$(0, 0)$	$(0, 0)$	$(0, 0)$

TABLE 3. Minimum values for models (2.1), (3.4) and (3.5) (with their corresponding times).

The random behavior of the compartments can also be investigated through the comparison of the coefficients of variation. All of the parameters were added random effects (for both distributions) which had $\sim 5\%$ standard deviations of their corresponding expected values, meaning that the coefficients of variation for $\mu^*, k^*, N^*, \beta_e^*, \beta_h^*, \nu^*, \gamma^*, \xi^*, \delta^*, c^*$ were 5% too (for instance, $E(k^*) = 500$,

[DOI: 10.52547/ijmsi.17.2.289]

	Laplacian Effects	Triangular Effects
$S(t)$	60.85%	64.12%
$I(t)$	74.71%	51.33%
$R(t)$	5.271%	4.859%
$B(t)$	72.73%	51.33%

TABLE 4. Maximum values of the CV for models (3.4) & (3.5).

while $std(k^*) = 25$). The coefficients of variation for the random compartments can be compared as below (Table 5).

Coefficients of variation for $S(t), R(t)$ are similar in both models. However, this is not the case for the compartments $I(t), B(t)$ which have significantly larger CV for Laplacian random effects. Hence, although CV for $S(t)$ is slightly larger for Triangular effects, it can be said that the model acts more randomly under Laplacian random effects. In part 3 it has been shown that for specific parameters, $> 99\%$ of the values of a Laplacian random variable can be adjusted to remain in the bounded support of a Triangular random variable (Figure 1). Yet this does not change the fact that Laplace distribution has an unbounded support. This may be the reason behind a larger CV for the Laplacian effects case. Another point that should be noted is that the compartments $S(t), I(t)$ and $B(t)$ show more than 50% randomness for the random parameters with 5% CV, whereas the compartment $R(t)$ of recovered people show at most $\sim 5\%$ randomness. This means that the behavior of the recovered population remains stable under random effects with both distributions.

Random model of Cholera transmission provides a confidence interval for the compartments of the system at any time t, whereas the deterministic model can only provide a numerical value. Hence, random modeling of the disease enables a more accurate modeling of the real life phenomena by using a straightforward replacement of the parameters. In particular, for $t = 50$, we see that $S(t) = 2.672 \times 10^7$, meaning that the deterministic model predicts that there will be 2.672×10^7 people left in China that are susceptible to Cholera under the conditions assumed by the model. The random model (3.4) gives us an expectation of $S(t)$ at $t = 50$ as 2.691×10^7, with a variance of 2.978×10^{12}. Using the three sigma rule for Laplacian random variables and the results from the simulations, we find that there is a 98.56% chance that the number of susceptible people in China will remain within the interval $[2.173 \times 10^7, 3.209 \times 10^7]$ for $t = 50$. This approach gives a better insight to the the disease by taking into account the randomness of the disease dynamics in real life.

The transmission model of GQ Sun et al. has been analyzed with random effects with Laplacian and symmetrical triangularly distribution in the parameters. The results show that the susceptible human population, infected human population and vibrio concentration can produce distinct behaviors that vary

[DOI: 10.52547/ijmsi.17.2.289]

more than 50% from their expectations. It was also seen that the population of recovered humans was not affected very much from the noise term in the parameters for both distributions. The comparison of the variation coefficients shows that the randomness in the behavior of the compartments is greater under Laplacian effects which can be linked with the fact that the range of a Laplacian random variable is infinite. Modeling of infectious diseases through the use of random effect (noise) terms in the parameters to analyze the random behavior of the compartments can is a method which is applicable to a wide range of events. Various distributions could be used with different scaling to accurately model the random nature of the event using real life data which would also the accuracy of the models. Results for the random models could be analyzed for other numerical characteristics of the systems such as variance, skewness, kurtosis and etc. where a deeper statistical analysis is needed.

Acknowledgments

The authors would like to thank the referee for useful and helpful comments and suggestions.

References

1. M. Merdan, Z. Bekiryazici, T. Kesemen, T. Khaniyev, Comparison of Stochastic and Random Models for Bacterial Resistance, *Advances in Difference Equations*, **2017**(133), (2017).
2. World Health Organization, Cholera, 2015, *Weekly Epidemiological Report*, **91**(38), (2016), 433-440.
3. J. P. Tian, J. Wang, Global Stability for Cholera Epidemic Models, *Mathematical biosciences*, **232**(1), (2011), 31-41.
4. A. Mwasa, J. M. Tchuenche, Mathematical Analysis of a Cholera Model with Public Health Interventions, *Biosystems*, **105**(3), (2011), 190-200.
5. L. Mari, E. Bertuzzo, L. Righetto, R. Casagrandi, M. Gatto, I. Rodriguez-Iturbe, A. Rinaldo, Modelling Cholera Epidemics: the Role of Waterways, Human Mobility and Sanitation, *Journal of the Royal Society Interface*, **9**, (2012), 376-388.
6. Z. Mukandavire, S. Liao, J. Wang, H. Gaff, D. L. Smith, J. G. Morris, Estimating the Reproductive Numbers for the 2008-2009 Cholera Outbreaks in Zimbabwe, *Proceedings of the National Academy of Sciences*, **108**(21), (2011), 8767-8772.
7. Z. Shuai, P. Van den Driessche, Global Dynamics of Cholera Models with Differential Infectivity, *Mathematical biosciences*, **234**(2), (2011), 118-126.
8. S. D. Hove-Musekwa, F. Nyabadza, C. Chiyaka, P. Das, A. Tripathi, Z. Mukandavire, Modelling and Analysis of the Effects of Malnutrition in the Spread of Cholera, *Mathematical and computer modelling*, **53**(9), (2011), 1583-1595.
9. E. Bertuzzo, R. Casagrandi, M. Gatto, I. Rodriguez-Iturbe, A. Rinaldo, On Spatially Explicit Models of Cholera Epidemics, *Journal of the Royal Society Interface*, **7**, (2010), 321-333.
10. J. Sepulveda, H. Gomez-Dantes, M. Bronfman, Cholera in the Americas: an Overview, *Infection*, **20**(5), (1992), 243-248.

[DOI: 10.52547/ijmsi.17.2.289]

11. G-Q. Sun, J-H. Xie, S-H. Huang, Z. Jin, M-T. Li, L. Liu, Transmission Dynamics of Cholera: Mathematical Modeling and Control Strategies, *Communications in Nonlinear Science and Numerical Simulation*, **45**, (2017), 235-244.
12. M. Merdan, T. Khaniyev, On the Behavior of Solutions under the Stochastic-Effect of Avian-Human Influenza Epidemic Model, *International Journal of Biotechnology and Biochemistry*, **4**, (2008), 75-100.
13. T. Eltoft, T. Kim, T. W. Lee, On the Multivariate Laplace Distribution, *IEEE Signal Processing Letters*, **13**(5), (2006), 300-303.
14. J. Minguillon, J. Pujol, JPEG Standard Uniform Quantization Error Modeling with Applications to Sequential and Progressive Operation Modes, *Journal of Electronic Imaging*, **10**(2), (2001), 475-485.
15. S. Kotz, T. Kozubowski, K. Podgorski, *The Laplace Distribution and Generalizations: a Revisit with Applications to Communications, Economics, Engineering, and Finance*, Springer Science & Business Media, 2001.
16. C. Forbes, M. Evans, N. Hastings, B. Peacock, *Statistical Distributions*, John Wiley & Sons, 2011.
17. M. Merdan, Z. Bekiryazici, T. Kesemen, T. Khaniyev, Deterministic Stability and Random Behavior of a Hepatitis C Model, *PloS One*, **12**(7), (2017), e0181571.
18. S. Kotz, J. R. van Dorp, *Beyond Beta: Other Continuous Families of Distributions with Bounded Support and Applications*, World Scientific Press, Singapore, 2004.
19. E. W. Grafarend, *Linear and Nonlinear Models: Fixed Effects, Random Effects, and Mixed Models*, de Gruyter, 2006.
20. Z. Bekiryazici, M. Merdan, T. Kesemen, M. Najmuldeen, Mathematical Modeling of Dengue Disease under Random Effects, *Math. Sci. Appl. E-Notes*, **4**(2), (2016), 58-70.
21. A. Ashyani, H. Mohammadinejad, O. RabieiMotlagh, Stability Analysis of Mathematical Model of Virus Therapy for Cancer, *Iranian Journal of Mathematical Sciences and Informatics*, **11**(2), (2016), 97-110.
22. F. Koushki, Modeling Dynamic Production Systems with Network Structure, *Iranian Journal of Mathematical Sciences and Informatics*, **12**(1), (2017), 13-26.

[DOI: 10.52547/ijmsi.17.2.289]