

## On the Properties of Balancing and Lucas-Balancing $p$ -Numbers

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**ABSTRACT.** The main goal of this paper is to develop a new generalization of balancing and Lucas-balancing sequences namely balancing and Lucas-balancing  $p$ -numbers and derive several identities related to them. Some combinatorial forms of these numbers are also presented.

**Keywords:** Balancing  $p$ -numbers, Lucas-balancing  $p$ -numbers, Incomplete balancing  $p$ -numbers, Incomplete Lucas-balancing  $p$ -numbers.

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### 1. INTRODUCTION

A number sequence closely associated to the famous Fibonacci sequence is the balancing sequence. Behera and Panda [1] in 1999 defined a natural number  $n$  as a balancing number if it is the solution of a simple Diophantine equation  $1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r)$ , calling  $r$  as the balancer corresponding to  $n$ . In general if  $B_n$  denotes the  $n$ -th balancing number, then the balancing sequence is defined recursively as  $B_n = 6B_{n-1} - B_{n-2}$ , for  $n \geq 2$  with seeds  $B_0 = 0$  and  $B_1 = 1$ . The sequence companion to balancing sequence is the Lucas-balancing sequence whose recurrence relation is given by  $C_n = 6C_{n-1} - C_{n-2}$ , for  $n \geq 2$  with seeds  $C_0 = 1$  and  $C_1 = 3$ , where  $C_n$  denotes the  $n$ -th Lucas-balancing number. It is known that the ratio of two adjacent

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balancing numbers  $B_n$  and Lucas-balancing numbers  $C_n$  tends to a definite proportion  $3 + \sqrt{8}$  as  $n \rightarrow \infty$ . This number  $\lambda_1 = 3 + \sqrt{8}$  and its conjugate  $\lambda_2 = 3 - \sqrt{8}$  are indeed the roots of the characteristic equation  $x^2 - 6x + 1 = 0$ . Binet's formulas are well-known in the theory of the balancing numbers, these formulas allow all balancing numbers  $B_n$  and Lucas-balancing numbers  $C_n$  to be represented by the roots of the characteristic equation as

$$B_n = \frac{\lambda_1^n - \lambda_2^n}{2\sqrt{8}} \quad (1.1)$$

and

$$C_n = \frac{\lambda_1^n + \lambda_2^n}{2}. \quad (1.2)$$

The theory of balancing numbers is broadly studied by many authors, the interested readers may see [1, 3, 4, 6, 11] for a detail review. The combinatorial forms for balancing and Lucas-balancing numbers were almost studied by Patel et al. [7]. They defined incomplete balancing and Lucas-balancing numbers as

$$B_n(k) = \sum_{j=0}^k (-1)^j \binom{n-1-j}{j} 6^{n-2j-1}; \quad 0 \leq k \leq \tilde{n}, \quad \tilde{n} = \lfloor \frac{n-1}{2} \rfloor$$

and

$$C_n(k) = 3 \sum_{j=0}^k (-1)^j \frac{n}{n-j} \binom{n-j}{j} 6^{n-2j-1}; \quad 0 \leq k \leq \hat{n}, \quad \hat{n} = \lfloor \frac{n}{2} \rfloor.$$

Balancing and Lucas-balancing sequences are generalized in many ways. For details, one can see for example [2, 5, 6, 7, 9].

In this note, we generalize balancing and Lucas-balancing sequences by introducing balancing and Lucas-balancing  $p$ -numbers and deduce some of their properties. Further, we also present some of the combinatorial forms of these number sequences.

## 2. BALANCING AND LUCAS-BALANCING $p$ -NUMBERS

In this section we introduce balancing and Lucas-balancing  $p$ -numbers and establish some of their properties.

**Definition 2.1.** For any given non-negative integer  $p$ , the balancing  $p$ -sequence is recursively defined as

$$B_p(n) = 6B_p(n-1) - B_p(n-p-1), \quad (2.1)$$

with seeds

$$B_p(n) = 6^{n-1}; \text{ for } n = 1, 2, \dots, p+1 \text{ and } B_p(0) = 0. \quad (2.2)$$

For different values of  $p$  the recurrence relation (2.1) generates some interesting known sequences. For example, for the case  $p = 0$ , the recurrence relation (2.1) is reduced to the identity  $B_0(n) = 5B_0(n-1)$ , which generates the sequence of power of five, that is  $B_0(n) = \{5^0, 5^1, 5^2, 5^3, \dots\}$  for  $n = 1, 2, \dots$  with the given initials  $B_0(0) = 0$  and  $B_0(1) = 1$ .

For the case  $p = 1$ , the basic recurrence relation (2.1) takes the form  $B_1(n) = 6B_1(n-1) - B_1(n-2)$ , with the initials  $B_1(2) = 6^1 = 6$  and  $B_1(1) = 6^0 = 1$  and which generates the classical balancing sequence  $B_1(n) = B_n = \{1, 6, 35, 204, 1189, 6930, \dots\}$  for all  $n \in \mathbb{N}$ .

**Definition 2.2.** For any given non-negative integer  $p$ , Lucas-balancing  $p$ -numbers are defined by the following recurrence relation:

$$C_p(n) = 6C_p(n-1) - C_p(n-p-1), \quad (2.3)$$

with seeds

$$C_p(p+1) = 3 \left( 6^p - \frac{p+1}{6} \right) \text{ and } C_p(n) = 3 \cdot 6^{n-1}, \text{ for } n = 1, 2, \dots, p. \quad (2.4)$$

Notice that  $C_p(0) = \frac{p+1}{2}$ . Furthermore, for the initials  $C_1(1) = 3$  and  $C_1(2) = 17$ , the recurrence relation (2.3) generates the classical Lucas-balancing numbers  $C_n = C_1(n) = \{3, 17, 99, 577, \dots\}$ .

**Proposition 2.3.** For any particular positive integer  $p$  the sum of the balancing  $p$ -numbers  $B_p(n)$  for all non-negative integers  $n$  is

$$\sum_{i=0}^n B_p(i) = \frac{1}{4} \{ B_p(n+1) - \sum_{i=0}^{p-1} B_p(n-i) - B_p(p+1) + (6^p - 1) \}.$$

*Proof.* We will prove this by using the principle of mathematical induction on  $n$ . Clearly the result is true for  $n = 0, 1$  and  $2$ . Let us assume the statement is true for  $n = k$ , and is

$$\sum_{i=0}^k B_p(i) = \frac{1}{4} \{ B_p(k+1) - \sum_{i=0}^{p-1} B_p(k-i) - B_p(p+1) + 6^p - 1 \}.$$

Now  $\sum_{i=0}^{k+1} B_p(i)$  can be written as

$$\begin{aligned}
 \sum_{i=0}^{k+1} B_p(i) &= \sum_{i=0}^k B_p(i) + B_p(k+1) \\
 &= \frac{1}{4} \left\{ B_p(k+1) - \sum_{i=0}^{p-1} B_p(k-i) - B_p(p+1) + 6^p - 1 \right\} + B_p(k+1) \\
 &= \frac{1}{4} \left\{ 5B_p(k+1) - \sum_{i=0}^{p-1} B_p(k-i) - B_p(p+1) + 6^p - 1 \right\} \\
 &= \frac{1}{4} \left\{ 6B_p(k+1) - B_p(k-p+1) - B_p(k+1) - \sum_{i=0}^{p-2} B_p(k-i) \right. \\
 &\quad \left. - B_p(p+1) + 6^p - 1 \right\} \\
 &= \frac{1}{4} \left\{ 6B_p(k+1) - B_p(k-p+1) - \sum_{i=0}^{p-1} B_p(k+1-i) \right. \\
 &\quad \left. - B_p(p+1) + 6^p - 1 \right\} \\
 &= \frac{1}{4} \left\{ B_p(k+2) - \sum_{i=0}^{p-1} B_p(k+1-i) - B_p(p+1) + 6^p - 1 \right\},
 \end{aligned}$$

which proves the result.  $\square$

**Proposition 2.4.** For any particular positive integer  $p$  the sum of the Lucas-balancing  $p$ -numbers  $C_p(n)$  for all positive integer  $n$  is

$$\sum_{i=1}^n C_p(i) = \frac{1}{4} \left\{ C_p(n+1) - \sum_{i=0}^{p-1} C_p(n-i) - C_p(p+1) + 3(6^p - 1) \right\}.$$

*Proof.* The proof has similar approach to the above.  $\square$

As the limit of the ratio of two adjacent balancing and Lucas-balancing  $p$ -numbers  $B_p(n)$  and  $C_p(n)$  respectively tends to a definite proportion, we have

$$\lim_{n \rightarrow \infty} \frac{B_p(n)}{B_p(n-1)} = x.$$

Which imply by recurrence formula that

$$\begin{aligned}
 \frac{B_p(n)}{B_p(n-1)} &= \frac{6B_p(n-1) - B_p(n-p-1)}{B_p(n-1)} \\
 &= 6 - \frac{1}{\frac{B_p(n-1)}{B_p(n-p-1)}}.
 \end{aligned}$$

It follows that

$$\frac{B_p(n)}{B_p(n-1)} = 6 - \frac{1}{\frac{B_p(n-1)B_p(n-2) \cdots B_p(n-p)}{B_p(n-2)B_p(n-3) \cdots B_p(n-p-1)}}.$$

Taking  $\lim_{n \rightarrow \infty}$  on both sides, we get the result

$$x^{p+1} - 6x^p + 1 = 0. \quad (2.5)$$

The result (2.5) is the algebraic equation of  $(p+1)$ -th degree and has  $(p+1)$  roots namely be  $x_1, x_2, x_3, \dots, x_{p+1}$ . Now we examine the equation (2.5) for different values of  $p$ . By taking  $p = 0$ , (2.5) is the trivial equation  $x = 5$ , and for  $p = 1$ , (2.5) becomes  $x^2 - 6x + 1 = 0$ . After solving this equation, we get two defined roots  $\lambda_1$  and  $\lambda_2$ , and has Binet's formulas (1.1) and (1.2).

Now we derive the Binet's formula for  $B_p(n)$  and  $C_p(n)$ . Let  $x_1, x_2, \dots, x_p, x_{p+1}$  be roots of the polynomial equation  $x^{p+1} - 6x^p + 1 = 0$ , then the Binet's formulas for balancing and Lucas-balancing  $p$ -numbers with  $p > 0$ , are of the forms

$$B_p(n) = k_1 x_1^n + k_2 x_2^n + \dots + k_{p+1} x_{p+1}^n \quad (2.6)$$

and

$$C_p(n) = a_1 x_1^n + a_2 x_2^n + \dots + a_{p+1} x_{p+1}^n, \quad (2.7)$$

respectively, where  $k_1, k_2, \dots, k_{p+1}$  and  $a_1, a_2, \dots, a_{p+1}$  are coefficient constants.

By considering the balancing  $p$ -numbers given by the recurrence relation (2.1) and by using (2.2) and (2.6), we will get a set of following results.

$$\begin{aligned} B_p(0) &= k_1 + k_2 + \dots + k_{p+1} = 0; \\ B_p(1) &= k_1 x_1 + k_2 x_2 + \dots + k_{p+1} x_{p+1} = 1; \\ B_p(2) &= k_1 x_1^2 + k_2 x_2^2 + \dots + k_{p+1} x_{p+1}^2 = 6; \\ &\dots\dots\dots \\ B_p(p) &= k_1 x_1^p + k_2 x_2^p + \dots + k_{p+1} x_{p+1}^p = 6^{p-1}. \end{aligned} \quad (2.8)$$

Similarly by considering (2.3) and by using (2.4) and (2.7), we get

$$\begin{aligned} C_p(0) &= a_1 + a_2 + \dots + a_{p+1} = \frac{p+1}{2}; \\ C_p(1) &= a_1 x_1 + a_2 x_2 + \dots + a_{p+1} x_{p+1} = 3; \\ C_p(2) &= a_1 x_1^2 + a_2 x_2^2 + \dots + a_{p+1} x_{p+1}^2 = 18; \\ &\dots\dots\dots \\ C_p(p) &= a_1 x_1^p + a_2 x_2^p + \dots + a_{p+1} x_{p+1}^p = 3 \cdot 6^{p-1}. \end{aligned} \quad (2.9)$$

Solving the above sets of equations, we get the approximate values of all constants  $k_1, k_2, \dots, k_{p+1}$  and  $a_1, a_2, \dots, a_{p+1}$ .

For the case  $p = 1$ , the characteristic equation  $x^{p+1} - 6x^p + 1 = 0$  is  $x^2 - 6x + 1 = 0$ , which implies the roots  $x_1 = \lambda_1 = 3 + \sqrt{8}$  and  $x_2 = \lambda_2 = \frac{1}{\lambda_1} = 3 - \sqrt{8}$ . Hence for  $p = 1$ , equation (2.6) becomes the Binet's formula for balancing 1-number and is

$$B_1(n) = k_1 x_1^n + k_2 x_2^n = k_1 (3 + \sqrt{8})^n + k_2 (3 - \sqrt{8})^n \quad (2.10)$$

To find out the values of  $k_1$  and  $k_2$ , use equation (2.8) and get  $k_1 = \frac{1}{2\sqrt{8}}$  and  $k_2 = \frac{-1}{2\sqrt{8}}$ . Hence by manipulating  $k_1$  and  $k_2$  in (2.10), we get the desired Binet's formula (1.1).

In a similar way we find the Binet's formula for Lucas-balancing 1-numbers  $C_1(n)$ , equation (2.7) implies

$$C_1(n) = a_1x_1^n + a_2x_2^n = a_1(3 + \sqrt{8})^n + a_2(3 - \sqrt{8})^n. \quad (2.11)$$

To find out the values of  $a_1$  and  $a_2$ , use equation (2.9) and get  $a_1 = \frac{1}{2}$  and  $a_2 = \frac{1}{2}$ . Hence by manipulating  $a_1$  and  $a_2$  in (2.11), we get the desired Binet's formula (1.2).

For  $p = 2$ , from the algebraic equation  $x^{p+1} - 6x^p + 1 = 0$  we get  $x^3 - 6x^2 + 1 = 0$ , which gives  $x_1 = -0.39543$ ,  $x_2 = 0.42347$  and  $x_3 = 5.9720$ . Again for  $p = 2$ , the Binet's formula (2.6) and equation (2.8) become

$$B_2(n) = k_1x_1^n + k_2x_2^n + k_3x_3^n \quad (2.12)$$

and

$$\begin{aligned} k_1 + k_2 + k_3 &= 0; \\ k_1x_1 + k_2x_2 + k_3x_3 &= 1; \\ k_1x_1^2 + k_2x_2^2 + k_3x_3^2 &= 6, \end{aligned}$$

respectively, and solving this system of equations, we get  $k_1 = -0.0758435$ ,  $k_2 = -0.0931908$  and  $k_3 = 0.169034$ .

Finally, (2.12) can be written as

$$\begin{aligned} B_2(n) &= (-0.0758435)(-0.39543)^n + (-0.0931908)(0.42347)^n \\ &\quad + (0.169034)(5.9720)^n, \end{aligned}$$

which is the Binet's formula for balancing 2-numbers for any integers  $n = 0, \pm 1, \pm 2, \pm 3, \dots$

Similarly we can calculate the Binet's formula for the Lucas-balancing 2-numbers. Put  $p = 2$  in the algebraic equation  $x^{p+1} - 6x^p + 1 = 0$ , we get the desired equation  $x^3 - 6x^2 + 1 = 0$ , which acquire same roots  $x_1 = -0.39543$ ,  $x_2 = 0.42347$  and  $x_3 = 5.9720$ . Again by using  $p = 2$ , the Binet's formula (2.7) and equation (2.9) become

$$C_2(n) = a_1x_1^n + a_2x_2^n + a_3x_3^n \quad (2.13)$$

and

$$\begin{aligned} a_1 + a_2 + a_3 &= \frac{3}{2}; \\ a_1x_1 + a_2x_2 + a_3x_3 &= 3; \\ a_1x_1^2 + a_2x_2^2 + a_3x_3^2 &= 18, \end{aligned}$$

respectively and solving this system of equations, we get  $a_1 = 0.499979$ ,  $a_2 = 0.500028$  and  $a_3 = 0.499993$ .

Finally, (2.13) can be written as

$$C_2(n) = (0.499979)(-0.39543)^n + (0.500028)(0.42347)^n + (0.499993)(5.9720)^n,$$

which is the Binet's formula for the Lucas-balancing 2-numbers for any integers  $n = 0, \pm 1, \pm 2, \pm 3, \dots$ .

In this way we can find out the Binet's formulas for all remaining balancing and Lucas-balancing  $p$ -numbers for occurrence of  $p = 3, 4, \dots$ . In general the Binet's formulas for balancing and Lucas-balancing  $p$ -numbers are of the form given by (2.6) and (2.7) in which the coefficients  $k_1, k_2, \dots, k_{p+1}$  and  $a_1, a_2, \dots, a_{p+1}$  can be calculated by using the equations (2.8) and (2.9).

Before going to prove the following theorem it is better to discuss one more thing, that is, if  $x_1, x_2, x_3, \dots, x_{p+1}$  are roots of the characteristic equation  $x^{p+1} - 6x^p + 1 = 0$ , then these roots can be written in balancing and Lucas-balancing  $p$ -numbers as in form:

$$x_k^n = 6x_k^{n-1} - x_k^{n-p-1} = x_k(6x_k^{n-2} - x_k^{n-p-2}) = x_k \cdot x_k^{n-1}, \quad (2.14)$$

for all integer values  $n$  and  $k = 1, 2, 3, \dots, p+1$ .

**Theorem 2.5.** *For any given positive integers  $p(p > 0)$ , balancing  $p$ -numbers can be written for  $(n = 0, \pm 1, \pm 2, \pm 3, \dots)$  in the form:*

$$B_p(n) = k_1x_1^n + k_2x_2^n + \dots + k_{p+1}x_{p+1}^n, \quad (2.15)$$

where  $k_1, k_2, \dots, k_{p+1}$  are coefficient constants and  $x_1, x_2, \dots, x_{p+1}$  are roots of the polynomial equation  $x^{p+1} - 6x^p + 1 = 0$ .

*Proof.* We can easily find out the first  $p$ -terms for  $n = 0, 1, 2, \dots, p$  of the balancing  $p$ -numbers by using (2.6), (2.8) and algebraic equation  $x^{p+1} - 6x^p + 1 = 0$ . Now our seek is to prove  $B_p(n) = k_1x_1^n + k_2x_2^n + \dots + k_{p+1}x_{p+1}^n$  for remaining positive integers. For the case  $n = p+1$ , we have

$$\begin{aligned} B_p(p+1) &= k_1x_1^{p+1} + k_2x_2^{p+1} + \dots + k_{p+1}x_{p+1}^{p+1} \\ &= 6[k_1x_1^p + k_2x_2^p + \dots + k_{p+1}x_{p+1}^p] - [k_1x_1^0 + k_2x_2^0 + \dots \\ &\quad + k_{p+1}x_{p+1}^0]. \end{aligned}$$

Therefore according to (2.8), we have

$$B_p(p+1) = 6B_p(p) - B_p(0),$$

which is the basic recurrence relation (2.1) for  $n = p+1$ .

Similarly it is easy to prove that equation (2.15) is true for all remaining positive values from  $n = p+2$ .

Finally, we have to prove equation (2.15) is true for all negative values of  $n$ . For the case  $n = -1$ :

$$B_p(-1) = k_1x_1^{-1} + k_2x_2^{-1} + \dots + k_{p+1}x_{p+1}^{-1}. \quad (2.16)$$

Let write (2.14) in the form:

$$x_k^{n-p-1} = 6 \cdot x_k^{n-1} - x_k^n. \quad (2.17)$$

By putting  $n = p$  in (2.17), we get

$$x_k^{-1} = 6 \cdot x_k^{p-1} - x_k^p. \quad (2.18)$$

Apply (2.18) in (2.16), we get

$$B_p(-1) = 6[k_1x_1^{p-1} + k_2x_2^{p-1} + \cdots + k_{p+1}x_{p+1}^{p-1}] - [k_1x_1^p + k_2x_2^p + \cdots + k_{p+1}x_{p+1}^p]. \quad (2.19)$$

Using (2.8), expression (2.19) will become

$$B_p(-1) = 6B_p(p-1) - B_p(p) = 0,$$

which is the balancing  $p$ -number  $B_p(-1) = 0$ .

Similarly, for negative values of  $n = -2, -3, -4, \dots$ , we will get all balancing  $p$ -numbers. Hence the equation (2.15) is true for all  $n = 0, \pm 1, \pm 2, \pm 3, \dots$ . This completes the proof.  $\square$

Using a similar approach to Theorem 2.5, we can also prove the following theorem for Lucas-balancing  $p$ -numbers

**Theorem 2.6.** *For any given positive integers  $p(p > 0)$ , Lucas-balancing  $p$ -numbers can be written for  $(n = 0, \pm 1, \pm 2, \pm 3, \dots)$  in the form:*

$$C_p(n) = a_1x_1^n + a_2x_2^n + \cdots + a_{p+1}x_{p+1}^n,$$

where  $a_1, a_2, \dots, a_{p+1}$  are coefficient constants and  $x_1, x_2, \dots, x_{p+1}$  are roots of the polynomial equation  $x^{p+1} - 6x^p + 1 = 0$ .

### 3. INCOMPLETE BALANCING AND LUCAS-BALANCING $p$ -NUMBERS

In this section we introduce incomplete balancing and Lucas-balancing  $p$ -numbers and present some of their properties.

**Definition 3.1.** The incomplete balancing  $p$ -numbers denoted by  $B_p^k(n)$  are defined by

$$B_p^k(n) = \sum_{j=0}^k (-1)^j \binom{n-1-pj}{j} 6^{n-(p+1)j-1}, \quad \left( n = 1, 2, 3, \dots; 0 \leq k \leq \left\lfloor \frac{n-1}{p+1} \right\rfloor \right). \quad (3.1)$$

In a similar manner incomplete Lucas-balancing  $p$ -numbers can also be defined as follows:

**Definition 3.2.** The incomplete Lucas-balancing  $p$ -numbers denoted by  $C_p^k(n)$  are defined by

$$C_p^k(n) = 3 \sum_{j=0}^k (-1)^j \frac{n}{n-pj} \binom{n-pj}{j} 6^{n-(p+1)j-1}, \quad (3.2)$$

$$\left( n = 1, 2, 3, \dots; 0 \leq k \leq \left\lfloor \frac{n}{p+1} \right\rfloor \right).$$

Notice that  $B_1^{\lfloor \frac{n-1}{2} \rfloor}(n) = B_n$ ,  $C_1^{\lfloor \frac{n}{2} \rfloor}(n) = C_n$  and  $B_1^k(n) = B_n(k)$ ,  $C_1^k(n) = C_n(k)$ .

Some cases based on definitions 3.1 and 3.2 are

$$B_p^0 = 6^{n-1}; \text{ for all } n \geq 1,$$

$$B_p^1(n) = 6^{n-1} - 6^{n-p-2}(n-p-1); \text{ for all } n \geq p+2,$$

$$B_p^2(n) = 6^{n-1} - (n-1-p)6^{n-p-2} + 3(n-2p-1)(n-2p-2)6^{n-2p-4};$$

for all  $n \geq 2p+1$ ,

$$B_p^{\lfloor \frac{n-1}{p+1} \rfloor}(n) = B_p(n); \text{ for all } n \geq 1,$$

$$C_p^0(n) = 3 \cdot 6^{n-1}; \text{ for all } n \geq 1,$$

$$C_p^1(n) = 3[6^{n-1} - n6^{n-p-2}]; \text{ for all } n \geq p+1,$$

$$C_p^2(n) = 3[6^{n-1} - n6^{n-p-2} + 3n(n-2p-1)6^{n-2p-4}]; \text{ for all } n \geq 2p+2$$

and

$$C_p^{\lfloor \frac{n}{p+1} \rfloor}(n) = C_p(n); \text{ for all } n \geq 1.$$

**Proposition 3.3.** The recurrence relation of the incomplete balancing  $p$ -number is defined as:

$$B_p^{k+1}(n) = 6B_p^k(n-1) - B_p^k(n-p-1); \quad 0 \leq k \leq \frac{n-p-3}{p+1}. \quad (3.3)$$

*Proof.* By using Definition 3.1, the right hand side of (3.3) can be written as

$$\begin{aligned}
 & 6 \sum_{j=0}^{k+1} (-1)^j \binom{n-pj-2}{j} 6^{n-(p+1)j-2} - \sum_{j=0}^k (-1)^j \binom{n-p-pj-2}{j} 6^{n-p-(p+1)j-2} \\
 &= \sum_{j=0}^{k+1} (-1)^j \binom{n-pj-2}{j} 6^{n-(p+1)j-1} - \sum_{j=1}^{k+1} (-1)^{j-1} \binom{n-pj-2}{j-1} 6^{n-(p+1)j-1} \\
 &= \sum_{j=0}^{k+1} (-1)^j \binom{n-pj-2}{j} 6^{n-(p+1)j-1} + \sum_{j=0}^{k+1} (-1)^j \binom{n-pj-2}{j-1} 6^{n-(p+1)j-1} \\
 &\quad - \binom{n-2}{-1} 6^{n-1} \\
 &= \sum_{j=0}^{k+1} \left[ \binom{n-pj-2}{j} + \binom{n-pj-2}{j-1} \right] (-1)^j 6^{n-(p+1)j-1} \\
 &= \sum_{j=0}^{k+1} \binom{n-pj-1}{j} (-1)^j 6^{n-(p+1)j-1} \\
 &= B_p^{k+1}(n),
 \end{aligned}$$

and the result follows.  $\square$

By virtue of Proposition 3.3 and equation (3.1), we get the following identity.

$$B_p^k(n) = 6B_p^k(n-1) - B_p^k(n-p-1) + (-1)^k \binom{n-p(k+1)-2}{k} 6^{n-(p+1)(k+1)-1}. \quad (3.4)$$

**Proposition 3.4.**

$$\sum_{j=0}^h \binom{h}{j} (-1)^{j+h} 6^j B_p^{k+j}(n+p(j-1)) = B_p^{k+h}(n+(p+1)h-p); \quad (3.5)$$

$$\left( 0 \leq k \leq \frac{n-p-h-1}{p+1} \right).$$

*Proof.* We shall prove this property by using principle of mathematical induction on  $h$ . The above sum (3.5) clearly holds for  $h = 0$  and  $h = 1$ . Let us assume it holds for certain  $h > 1$ . We will show that it holds for  $h \rightarrow h+1$ , now we

have

$$\begin{aligned}
& \sum_{j=0}^{h+1} \binom{h+1}{j} (-1)^{j+h+1} 6^j B_p^{k+j} (n + p(j-1)) \\
&= \sum_{j=0}^{h+1} (-1)^{j+h+1} \left( \binom{h}{j} + \binom{h}{j-1} \right) B_p^{k+j} (n + p(j-1)) 6^j \\
&= \sum_{j=0}^{h+1} (-1)^{j+h+1} \binom{h}{j} B_p^{k+j} (n + p(j-1)) 6^j + \sum_{j=0}^{h+1} (-1)^{j+h+1} \binom{h}{j-1} \\
&\quad \times B_p^{k+j} (n + p(j-1)) 6^j \\
&= -B_p^{k+h} (n + (p+1)h - p) + \sum_{j=-1}^h (-1)^{j+h+2} \binom{h}{j} B_p^{k+j+1} (n + pj) 6^{j+1} \\
&= -B_p^{k+h} (n + (p+1)h - p) + \sum_{j=0}^h (-1)^{j+h+2} \binom{h}{j} B_p^{k+j+1} (n + pj) 6^j \cdot 6 \\
&\quad + (-1)^{h+1} \binom{h}{-1} B_p^k (n - p) \\
&= -B_p^{k+h} (n + (p+1)h - p) + 6 \sum_{j=0}^h (-1)^{j+h} \binom{h}{j} B_p^{k+j+1} (n + pj) 6^j \\
&= -B_p^{k+h} (n + (p+1)h - p) + 6B_p^{k+h+1} (n + (p+1)h) \\
&= B_p^{k+h+1} (n + (p+1)h + 1),
\end{aligned}$$

which follows the result.  $\square$

**Proposition 3.5.** *Let  $k$  be a non-negative integer. For  $n \geq (p+1)k + p + 2$ , we have*

$$\sum_{j=0}^{h-1} 6^{h-1-j} B_p^k (n - p + j) = 6^h B_p^{k+1} (n) - B_p^{k+1} (n + h). \quad (3.6)$$

*Proof.* We shall prove this by using mathematical induction on  $h$ . The result is obvious for  $h = 1$  and  $h = 2$  by using (3.3).

Let us assume the given statement (3.6) is true for  $h = t$  that is

$$\sum_{j=0}^{t-1} 6^{t-1-j} B_p^k (n - p + j) = 6^t B_p^{k+1} (n) - B_p^{k+1} (n + t).$$

Now it is enough to show that the sum (3.6) is true for  $h = t + 1$  :

$$\sum_{j=0}^t 6^{t-j} B_p^k (n - p + j) = 6^{t+1} B_p^{k+1} (n) - B_p^{k+1} (n + t + 1).$$

This implies

$$6 \sum_{j=0}^{t-1} 6^{t-1-j} B_p^k(n-p+j) + B_p^k(n-p+t) = 6^{t+1} B_p^{k+1}(n) - B_p^{k+1}(n+t+1).$$

The above equality gives

$$6^{t+1} B_p^{k+1}(n) - 6 B_p^{k+1}(n+t) + B_p^k(n-p+t) = 6^{t+1} B_p^{k+1}(n) - B_p^{k+1}(n+t+1).$$

Further simplification results

$$B_p^{k+1}(n+t+1) = 6 B_p^{k+1}(n+t) - B_p^k(n-p+t).$$

This completes the result in view of (3.3).  $\square$

**Proposition 3.6.**

$$2C_p^k(n) = 6B_p^k(n) - (p+1)B_p^{k-1}(n-p); \quad 0 \leq k \leq \lfloor \frac{n-1}{p+1} \rfloor. \quad (3.7)$$

*Proof.* The right hand side of (3.7) can be written as

$$\begin{aligned} & 6 \sum_{j=0}^k (-1)^j \binom{n-pj-1}{j} 6^{n-(p+1)j-1} - (p+1) \sum_{j=0}^{k-1} (-1)^j \binom{n-p-pj-1}{j} \\ & \quad \times 6^{n-p-(p+1)j-1} \\ &= 6 \sum_{j=0}^k (-1)^j \binom{n-pj-1}{j} 6^{n-(p+1)j-1} - (p+1) \sum_{j=1}^k (-1)^{j-1} \binom{n-pj-1}{j-1} \\ & \quad \times 6^{n-(p+1)j} \\ &= \sum_{j=0}^k (-1)^j \binom{n-pj-1}{j} 6^{n-(p+1)j} + (p+1) \sum_{j=0}^k (-1)^j \binom{n-pj-1}{j-1} 6^{n-(p+1)j} \\ & \quad - (p+1) \binom{n-1}{-1} 6^n \\ &= \sum_{j=0}^k \left[ \binom{n-pj-1}{j} + (p+1) \binom{n-pj-1}{j-1} \right] (-1)^j 6^{n-(p+1)j} \\ &= 6 \sum_{j=0}^k (-1)^j \frac{n}{n-pj} \binom{n-pj}{j} 6^{n-(p+1)j-1} \\ &= 2C_p^k(n), \end{aligned}$$

and then the result follows.  $\square$

**Proposition 3.7.** *The recurrence relation of the incomplete Lucas-balancing  $p$ -numbers  $C_p^k(n)$  is*

$$C_p^{k+1}(n) = 6C_p^{k+1}(n-1) - C_p^k(n-p-1); \quad \left(0 \leq k \leq \frac{n-p-2}{p+1}\right). \quad (3.8)$$

*Proof.* Applying (3.3) and (3.7), we have

$$\begin{aligned} & 2C_p^{k+1}(n) \\ &= 6\left(6B_p^{k+1}(n-1) - (p+1)B_p^k(n-p-1)\right) - \left(6B_p^k(n-p-1) - (p+1)\right. \\ & \quad \left.\times B_p^{k-1}(n-2p-1)\right) \\ &= 6\left(2C_p^{k+1}(n-1)\right) - 2C_p^k(n-p-1). \end{aligned}$$

Hence,  $C_p^{k+1}(n) = 6C_p^{k+1}(n-1) - C_p^k(n-p-1)$ .  $\square$

Here we observe that by applying (3.2), the above relation (3.8) can be transformed into the non-homogeneous relation

$$\begin{aligned} C_p^k(n) &= 6C_p^k(n-1) - C_p^k(n-p-1) + 3(-1)^k \frac{n-p-1}{n-(k+1)p-1} \\ & \quad \times \binom{n-(k+1)p-1}{k} 6^{n-(k+1)p-k-2}. \end{aligned} \quad (3.9)$$

**Proposition 3.8.** For  $0 \leq k \leq \frac{n-p-h}{p+1}$ , we have

$$\sum_{j=0}^h \binom{h}{j} (-1)^{j+h} 6^j C_p^{k+j}(n+p(j-1)) = C_p^{k+h}(n+(p+1)h-p).$$

*Proof.* The proof is similar to Proposition 3.4.  $\square$

**Proposition 3.9.** Let  $k$  be a non-negative integer. For  $n \geq (p+1)(k+1)$ , the identity

$$\sum_{j=0}^{h-1} 6^{h-1-j} C_p^k(n-p+j) = 6^h C_p^{k+1}(n) - C_p^{k+1}(n+h)$$

holds.

*Proof.* The proof is analogous to Proposition 3.5.  $\square$

The following result which has already proved in [8] is useful while finding the generating functions of  $B_p^k(n)$  and  $C_p^k(n)$ .

**Lemma 3.10.** Let  $\{s_n\}_{n=0}^\infty$  be a complex sequence satisfying the non-homogeneous recurrence relation

$$s_n = 6s_{n-1} - s_{n-p-1} + r_n, \quad n > p,$$

where  $r_n$  is a given complex sequence. Then the generating function  $S_p^k(t)$  of the sequence  $\{s_n\}_{n=0}^\infty$  is

$$S_p^k(t) = \frac{s_0 - r_0 + \sum_{i=1}^p t^i (s_i - 6s_{i-1} - r_i) + G(t)}{1 - 6t + t^{p+1}},$$

where  $G(t)$  denotes the generating function of  $\{r_n\}$ .

**Theorem 3.11.** *The generating function of the incomplete balancing  $p$ -numbers  $B_p^k(n)$  ( $k = 0, 1, 2, 3, \dots$ ) is given by*

$$\begin{aligned} R_p^k(t) &= \sum_{j=0}^{\infty} B_p^j(k) t^j \\ &= t^{k(p+1)+1} \left[ \left\{ B_p(k(p+1)+1) + \sum_{i=1}^p t^i (B_p(k(p+1)+i+1) - 6B_p(k(p+1)+i)) \right\} (1-6t)^{k+1} + (-1)^k t^{p+1} \right] \\ &\quad \left[ (1-6t+t^{p+1})(1-6t)^{k+1} \right]^{-1}. \end{aligned}$$

*Proof.* We prove this theorem by using Lemma 3.10. Let  $k$  be a fixed positive integer, from (3.1) and (3.4), we have

$$B_p^k(n) = 0; \quad \text{if } 0 \leq n < k(p+1)+1$$

and

$$B_p^k(k(p+1)+1) = B_p(k(p+1)+1),$$

$$B_p^k(k(p+1)+2) = B_p(k(p+1)+2),$$

$$\dots\dots\dots,$$

$$B_p^k(k(p+1)+p+1) = B_p(k(p+1)+p+1),$$

and that

$$\begin{aligned} B_p^k(n) &= 6B_p^k(n-1) - B_p^k(n-p-1) \\ &\quad + (-1)^k \binom{n-p(k+1)-2}{n-k(p+1)-p-2} 6^{n-k(p+1)-p-2}, \end{aligned}$$

if  $n \geq k(p+1)+p+2$ .

We let

$$s_0 = B_p^k(k(p+1)+1), \quad s_1 = B_p^k(k(p+1)+2), \dots, s_p = B_p^k(k(p+1)+p+1)$$

and  $s_n = B_p^k(n+k(p+1)+1)$ . Suppose that  $r_0 = r_1 = r_2 = \dots = r_p = 0$  and

$$r_n = \binom{n-(p+1)+k}{n-(p+1)} (-1)^k 6^{n-(p+1)}.$$

Thus we can easily derive that the generating function of the sequence  $r_n$  is (see p.355 of [10] )

$$G(t) = \frac{(-1)^k t^{p+1}}{(1-6t)^{k+1}}.$$

Then in view of Lemma 3.10, the generating function

$$\begin{aligned} S_p^k(t)(1-6t+t^{p+1}) - \frac{(-1)^k t^{p+1}}{(1-6t)^{k+1}} \\ = B_p^k(k(p+1)+1) + \sum_{i=1}^p t^i (B_p(k(p+1)+i+1) \\ - 6B_p(k(p+1)+i)), \end{aligned}$$

which implies

$$\begin{aligned} S_p^k(t) = & \left[ \{ B_p^k(k(p+1)+1) + \sum_{i=1}^p t^i (B_p(k(p+1)+i+1) - 6B_p(k(p+1)+ \right. \\ & \left. i)) \} (1-6t)^{k+1} + (-1)^k t^{p+1} \right] \left[ (1-6t+t^{p+1})(1-6t)^{k+1} \right]^{-1}. \end{aligned}$$

Finally, we conclude that

$$R_p^k(t) = t^{k(p+1)+1} S_p^k(t).$$

This completes the proof.  $\square$

**Theorem 3.12.** *The generating function of the incomplete Lucas-balancing  $p$ -numbers  $C_p^k(n)$  ( $k = 0, 1, 2, 3, \dots$ ) is given by*

$$\begin{aligned} W_p^k(t) &= \sum_{j=0}^{\infty} C_p^j(k) t^j \\ &= t^{k(p+1)} \left[ \left\{ C_p(k(p+1)) + \sum_{i=1}^p t^i (C_p(k(p+1)+i) - 6C_p(k(p+1)+ \right. \right. \\ &\quad \left. \left. i-1)) \right\} (1-6t)^{k+1} + (-1)^k 3t^{p+1} (p(1-t)+1) \right] \left[ (1-6t+t^{p+1}) \cdot \right. \\ &\quad \left. (1-6t)^{k+1} \right]^{-1}. \end{aligned}$$

*Proof.* We prove this theorem by using Lemma 3.10. Let  $k$  be a fixed positive integer, from (3.2) and (3.9), we have

$$C_p^k(n) = 0; \quad \text{if } 0 \leq n < k(p+1)$$

and

$$\begin{aligned} C_p^k(k(p+1)) &= C_p(k(p+1)), \\ C_p^k(k(p+1)+1) &= C_p(k(p+1)+1), \\ &\dots\dots\dots, \\ C_p^k(k(p+1)+p) &= C_p(k(p+1)+p), \end{aligned}$$

and that

$$C_p^k(n) = 6C_p^k(n-1) - C_p^k(n-p-1) + 3(-1)^k \frac{n-p-1}{n-(k+1)p-1} \cdot \binom{n-p(k+1)-1}{n-k(p+1)-p-1} 6^{n-k(p+1)-p-2},$$

if  $n \geq k(p+1) + p + 1$ . We let

$s_0 = C_p^k(k(p+1))$ ,  $s_1 = C_p^k(k(p+1)+1)$ ,  $\dots$ ,  $s_p = C_p^k(k(p+1)+p)$  and  $s_n = C_p^k(n+k(p+1))$ .

Suppose that  $r_0 = r_1 = r_2 = \dots = r_p = 0$  and

$$r_n = \frac{n+k(p+1)-p-1}{n+k-p-1} \binom{n-(p+1)+k}{n-(p+1)} 3(-1)^k 6^{n-(p+2)}.$$

Then the generating function of the sequence  $r_n$  is (p.355, [10] )

$$G(t) = \frac{(-1)^k 3t^{p+1}(p(1-t)+1)}{(1-6t)^{k+1}}.$$

By virtue of Lemma 3.10, the generating function

$$\begin{aligned} S_p^k(t)(1 - 6t + t^{p+1}) - \frac{(-1)^k 3t^{p+1}(p(1-t)+1)}{(1-6t)^{k+1}} \\ = C_p^k(k(p+1)) + \sum_{i=1}^p t^i (C_p(k(p+1)+i) - 6C_p(k(p+1)+i-1)). \end{aligned}$$

Further simplification gives

$$\begin{aligned} S_p^k(t) &= \left[ \{ C_p^k(k(p+1)) + \sum_{i=1}^p t^i (C_p(k(p+1)+i) - 6C_p(k(p+1)+i-1)) \} \cdot \right. \\ &\quad \left. (1-6t)^{k+1} + (-1)^k 3t^{p+1}(p(1-t)+1) \right] \left[ (1-6t+t^{p+1}) \cdot \right. \\ &\quad \left. (1-6t)^{k+1} \right]^{-1}. \end{aligned}$$

Finally, we conclude that

$$W_p^k(t) = t^{k(p+1)} S_p^k(t),$$

and hence the proof.  $\square$

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## REFERENCES

1. A. Behera, G. K. Panda, On the Square Roots of Triangular Numbers, *Fibonacci Quarterly*, **37**(2), (1999), 98-105.
2. A. Bérczes, K. Liptai, I. Pink, On Generalized Balancing Sequences, *Fibonacci Quarterly*, **48**(2), (2010), 121-128.
3. P. Catarino, H. Campos, P. Vasco, On Some Identities for Balancing and Cobalancing Numbers, *Annales Mathematicae et Informaticae*, **45**, (2015), 11-24.
4. T. Kovacs, K. Liptai, P. Olajos, On  $(a, b)$ -Balancing Numbers, *Publicationes Mathematicae Debrecen*, **77**(3-4), (2010), 485-498.
5. K. Liptai, F. Luca, A. Pintér, L. Szalay, Generalized Balancing Numbers, *Indagationes Mathematicae*, **20**(1), (2009), 87-100.
6. G. K. Panda, P. K. Ray, Cobalancing Numbers and Cobalancers, *International Journal of Mathematics and Mathematical Sciences*, **2005**(8), (2005), 1189-1200.
7. B. K. Patel, N. Irmak, P. K. Ray, Incomplete Balancing and Lucas-Balancing Numbers, *Mathematical Reports*, **20**(1), (2018), 59-72.
8. A. Pintér, H. M. Srivastava, Generating Functions of the Incomplete Fibonacci and Lucas Numbers, *Rendiconti del Circolo Matematico di Palermo*, **48**(3), (1999), 591-596.
9. P. K. Ray, Balancing Polynomials and Their Derivatives, *Ukrainian Mathematical Journal*, **69**(4), (2017), 646-663.
10. H. M. Srivastava, H. L. Manocha, *A Treatise on Generating Functions*, Halsted pres (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1984.
11. T. Szakács, Multiplying Balancing Numbers, *Acta Universitatis Sapientiae, Mathematica*, **3**(1) (2011), 90-96.