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On the Properties of Balancing and Lucas-Balancing p-Numbers

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ABSTRACT. The main goal of this paper is to develop a new generalization of balancing and Lucas-balancing sequences namely balancing and Lucas-balancing p-numbers and derive several identities related to them. Some combinatorial forms of these numbers are also presented.

Keywords: Balancing p-numbers, Lucas-balancing p-numbers, Incomplete balancing p-numbers, Incomplete Lucas-balancing p-numbers.

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1. Introduction

A number sequence closely associated to the famous Fibonacci sequence is the balancing sequence. Behera and Panda [1] in 1999 defined a natural number n as a balancing number if it is the solution of a simple Diophantine equation $1+2+\cdots+(n-1)=(n+1)+(n+2)+\cdots+(n+r)$, calling r as the balancer corresponding to n. In general if B_n denotes the n-th balancing number, then the balancing sequence is defined recursively as $B_n=6B_{n-1}-B_{n-2}$, for $n\geq 2$ with seeds $B_0=0$ and $B_1=1$. The sequence companion to balancing sequence is the Lucas-balancing sequence whose recurrence relation is given by $C_n=6C_{n-1}-C_{n-2}$, for $n\geq 2$ with seeds $C_0=1$ and $C_1=3$, where C_n denotes the n-th Lucas-balancing number. It is known that the ratio of two adjacent

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balancing numbers B_n and Lucas-balancing numbers C_n tends to a definite proportion $3+\sqrt{8}$ as $n\to\infty$. This number $\lambda_1=3+\sqrt{8}$ and its conjugate $\lambda_2=3-\sqrt{8}$ are indeed the roots of the characteristic equation $x^2-6x+1=0$. Binet's formulas are well-known in the theory of the balancing numbers, these formulas allow all balancing numbers B_n and Lucas-balancing numbers C_n to be represented by the roots of the characteristic equation as

$$B_n = \frac{\lambda_1^n - \lambda_2^n}{2\sqrt{8}} \tag{1.1}$$

and

$$C_n = \frac{\lambda_1^n + \lambda_2^n}{2}. (1.2)$$

The theory of balancing numbers is broadly studied by many authors, the interested readers may see [1, 3, 4, 6, 11] for a detail review. The combinatorial forms for balancing and Lucas-balancing numbers were almost studied by Patel et al. [7]. They defined incomplete balancing and Lucas-balancing numbers as

$$B_n(k) = \sum_{j=0}^{k} (-1)^j \binom{n-1-j}{j} 6^{n-2j-1}; \ 0 \leqslant k \leqslant \tilde{n}, \ \tilde{n} = \lfloor \frac{n-1}{2} \rfloor$$

and

$$C_n(k) = 3\sum_{j=0}^k (-1)^j \frac{n}{n-j} \binom{n-j}{j} 6^{n-2j-1}; \ 0 \leqslant k \leqslant \hat{n}, \ \hat{n} = \lfloor \frac{n}{2} \rfloor.$$

Balancing and Lucas-balancing sequences are generalized in many ways. For details, one can see for example [2, 5, 6, 7, 9].

In this note, we generalize balancing and Lucas-balancing sequences by introducing balancing and Lucas-balancing p-numbers and deduce some of their properties. Further, we also present some of the combinatorial forms of these number sequences.

2. Balancing and Lucas-Balancing p-numbers

In this section we introduce balancing and Lucas-balancing p-numbers and establish some of their properties.

Definition 2.1. For any given non-negative integer p, the balancing p-sequence is recursively defined as

$$B_n(n) = 6B_n(n-1) - B_n(n-p-1), \tag{2.1}$$

with seeds

$$B_p(n) = 6^{n-1}$$
; for $n = 1, 2, ..., p+1$ and $B_p(0) = 0$. (2.2)

For different values of p the recurrence relation (2.1) generates some interesting known sequences. For example, for the case p=0, the recurrence relation (2.1) is reduced to the identity $B_0(n)=5B_0(n-1)$, which generates the sequence of power of five, that is $B_0(n)=\{5^0,5^1,5^2,5^3,\dots\}$ for $n=1,2,\dots$ with the given initials $B_0(0)=0$ and $B_0(1)=1$.

For the case p=1, the basic recurrence relation (2.1) takes the form $B_1(n)=6B_1(n-1)-B_1(n-2)$, with the initials $B_1(2)=6^1=6$ and $B_1(1)=6^0=1$ and which generates the classical balancing sequence $B_1(n)=B_n=\{1,6,35,204,1189,6930,\dots\}$ for all $n\in\mathbb{N}$.

Definition 2.2. For any given non-negative integer p, Lucas-balancing p-numbers are defined by the following recurrence relation:

$$C_p(n) = 6C_p(n-1) - C_p(n-p-1),$$
 (2.3)

with seeds

$$C_p(p+1) = 3\left(6^p - \frac{p+1}{6}\right)$$
 and $C_p(n) = 3 \cdot 6^{n-1}$, for $n = 1, 2, \dots, p$. (2.4)

Notice that $C_p(0) = \frac{p+1}{2}$. Furthermore, for the initials $C_1(1) = 3$ and $C_1(2) = 17$, the recurrence relation (2.3) generates the classical Lucas-balancing numbers $C_n = C_1(n) = \{3, 17, 99, 577, \dots\}$.

Proposition 2.3. For any particular positive integer p the sum of the balancing p-numbers $B_p(n)$ for all non-negative integers n is

$$\sum_{i=0}^{n} B_p(i) = \frac{1}{4} \{ B_p(n+1) - \sum_{i=0}^{p-1} B_p(n-i) - B_p(p+1) + (6^p - 1) \}.$$

Proof. We will prove this by using the principle of mathematical induction on n. Clearly the result is true for n=0,1 and 2. Let us assume the statement is true for n=k, and is

$$\sum_{i=0}^{k} B_p(i) = \frac{1}{4} \{ B_p(k+1) - \sum_{i=0}^{p-1} B_p(k-i) - B_p(p+1) + 6^p - 1 \}.$$

Now $\sum_{i=0}^{k+1} B_p(i)$ can be written as

$$\begin{split} \sum_{i=0}^{k+1} B_p(i) &= \sum_{i=0}^k B_p(i) + B_p(k+1) \\ &= \frac{1}{4} \{ B_p(k+1) - \sum_{i=0}^{p-1} B_p(k-i) - B_p(p+1) + 6^p - 1 \} + B_p(k+1) \\ &= \frac{1}{4} \{ 5B_p(k+1) - \sum_{i=0}^{p-1} B_p(k-i) - B_p(p+1) + 6^p - 1 \} \\ &= \frac{1}{4} \{ 6B_p(k+1) - B_p(k-p+1) - B_p(k+1) - \sum_{i=0}^{p-2} B_p(k-i) \\ &- B_p(p+1) + 6^p - 1 \} \\ &= \frac{1}{4} \{ 6B_p(k+1) - B_p(k-p+1) - \sum_{i=0}^{p-1} B_p(k+1-i) \\ &- B_p(p+1) + 6^p - 1 \} \\ &= \frac{1}{4} \{ B_p(k+2) - \sum_{i=0}^{p-1} B_p(k+1-i) - B_p(p+1) + 6^p - 1 \}, \end{split}$$

which proves the result.

Proposition 2.4. For any particular positive integer p the sum of the Lucasbalancing p-numbers $C_p(n)$ for all positive integer n is

$$\sum_{i=1}^{n} C_p(i) = \frac{1}{4} \{ C_p(n+1) - \sum_{i=0}^{p-1} C_p(n-i) - C_p(p+1) + 3(6^p - 1) \}.$$

Proof. The proof has similar approach to the above.

As the limit of the ratio of two adjacent balancing and Lucas-balancing pnumbers $B_p(n)$ and $C_p(n)$ respectively tends to a definite proportion, we have

$$\lim_{n \to \infty} \frac{B_p(n)}{B_p(n-1)} = x.$$

Which imply by recurrence formula that

$$\frac{B_p(n)}{B_p(n-1)} = \frac{6B_p(n-1) - B_p(n-p-1)}{B_p(n-1)}$$
$$= 6 - \frac{1}{\frac{B_p(n-1)}{B_p(n-p-1)}}.$$

It follows that

$$\frac{B_p(n)}{B_p(n-1)} = 6 - \frac{1}{\frac{B_p(n-1)B_p(n-2)\cdots B_p(n-p)}{B_p(n-2)B_p(n-3)\cdots B_P(n-p-1)}}.$$

Taking $\lim_{n\to\infty}$ on both sides, we get the result

$$x^{p+1} - 6x^p + 1 = 0. (2.5)$$

The result (2.5) is the algebraic equation of (p+1)-th degree and has (p+1) roots namely be $x_1, x_2, x_3, ..., x_{p+1}$. Now we examine the equation (2.5) for different values of p. By taking p=0, (2.5) is the trivial equation x=5, and for p=1, (2.5) becomes $x^2-6x+1=0$. After solving this equation, we get two defined roots λ_1 and λ_2 , and has Binet's formulas (1.1) and (1.2).

Now we derive the Binet's formula for $B_p(n)$ and $C_p(n)$. Let x_1, x_2, \ldots, x_p , x_{p+1} be roots of the polynomial equation $x^{p+1} - 6x^p + 1 = 0$, then the Binet's formulas for balancing and Lucas-balancing p-numbers with p > 0, are of the forms

$$B_p(n) = k_1 x_1^n + k_2 x_2^n + \dots + k_{p+1} x_{p+1}^n$$
(2.6)

and

$$C_p(n) = a_1 x_1^n + a_2 x_2^n + \dots + a_{p+1} x_{p+1}^n,$$
 (2.7)

respectively, where $k_1, k_2, \ldots, k_{p+1}$ and $a_1, a_2, \ldots, a_{p+1}$ are coefficient constants. By considering the balancing p-numbers given by the recurrence relation (2.1) and by using (2.2) and (2.6), we will get a set of following results.

Similarly by considering (2.3) and by using (2.4) and (2.7), we get

Solving the above sets of equations, we get the approximate values of all constants $k_1, k_2, \ldots, k_{p+1}$ and $a_1, a_2, \ldots, a_{p+1}$.

For the case p=1, the characteristic equation $x^{p+1}-6x^p+1=0$ is $x^2-6x+1=0$, which implies the roots $x_1=\lambda_1=3+\sqrt{8}$ and $x_2=\lambda_2=\frac{1}{\lambda_1}=3-\sqrt{8}$. Hence for p=1, equation (2.6) becomes the Binet's formula for balancing 1-number and is

$$B_1(n) = k_1 x_1^n + k_2 x_2^n = k_1 (3 + \sqrt{8})^n + k_2 (3 - \sqrt{8})^n$$
 (2.10)

To find out the values of k_1 and k_2 , use equation (2.8) and get $k_1 = \frac{1}{2\sqrt{8}}$ and $k_2 = \frac{-1}{2\sqrt{8}}$. Hence by manipulating k_1 and k_2 in (2.10), we get the desired Binet's formula (1.1).

In a similar way we find the Binet's formula for Lucas-balancing 1-numbers $C_1(n)$, equation (2.7) implies

$$C_1(n) = a_1 x_1^n + a_2 x_2^n = a_1 (3 + \sqrt{8})^n + a_2 (3 - \sqrt{8})^n.$$
 (2.11)

To find out the values of a_1 and a_2 , use equation (2.9) and get $a_1 = \frac{1}{2}$ and $a_2 = \frac{1}{2}$. Hence by manipulating a_1 and a_2 in (2.11), we get the desired Binet's formula (1.2).

For p = 2, from the algebraic equation $x^{p+1} - 6x^p + 1 = 0$ we get $x^3 - 6x^2 + 1 = 0$, which gives $x_1 = -0.39543$, $x_2 = 0.42347$ and $x_3 = 5.9720$. Again for p = 2, the Binet's formula (2.6) and equation (2.8) become

$$B_2(n) = k_1 x_1^n + k_2 x_2^n + k_3 x_3^n (2.12)$$

and

$$k_1 + k_2 + k_3 = 0;$$

 $k_1x_1 + k_2x_2 + k_3x_3 = 1;$
 $k_1x_1^2 + k_2x_2^2 + k_3x_3^2 = 6,$

respectively, and solving this system of equations, we get $k_1 = -0.0758435$, $k_2 = -0.0931908$ and $k_3 = 0.169034$.

Finally, (2.12) can be written as

$$B_2(n) = (-0.0758435)(-0.39543)^n + (-0.0931908)(0.42347)^n + (0.169034)(5.9720)^n,$$

which is the Binet's formula for balancing 2-numbers for any integers $n = 0, \pm 1, \pm 2, \pm 3, \dots$

Similarly we can calculate the Binet's formula for the Lucas-balancing 2-numbers. Put p=2 in the algebraic equation $x^{p+1}-6x^p+1=0$, we get the desired equation $x^3-6x^2+1=0$, which acquire same roots $x_1=-0.39543$, $x_2=0.42347$ and $x_3=5.9720$. Again by using p=2, the Binet's formula (2.7) and equation (2.9) become

$$C_2(n) = a_1 x_1^n + a_2 x_2^n + a_3 x_3^n (2.13)$$

and

$$a_1 + a_2 + a_3 = \frac{3}{2};$$

$$a_1x_1 + a_2x_2 + a_3x_3 = 3;$$

$$a_1x_1^2 + a_2x_2^2 + a_3x_3^2 = 18,$$

respectively and solving this system of equations, we get $a_1 = 0.499979$, $a_2 = 0.500028$ and $a_3 = 0.499993$.

Finally, (2.13) can be written as

 $C_2(n) = (0.499979)(-0.39543)^n + (0.500028)(0.42347)^n + (0.499993)(5.9720)^n$

which is the Binet's formula for the Lucas-balancing 2-numbers for any integers $n=0,\pm 1,\pm 2,\pm 3,\cdots$.

In this way we can find out the Binet's formulas for all remaining balancing and Lucas-balancing p-numbers for occurrence of $p=3,4,\cdots$. In general the Binet's formulas for balancing and Lucas-balancing p-numbers are of the form given by (2.6) and (2.7) in which the coefficients $k_1, k_2, \cdots, k_{p+1}$ and $a_1, a_2, \cdots, a_{p+1}$ can be calculated by using the equations (2.8) and (2.9).

Before going to prove the following theorem it is better to discuss one more thing, that is, if $x_1, x_2, x_3, \dots, x_{p+1}$ are roots of the characteristic equation $x^{p+1} - 6x^p + 1 = 0$, then these roots can be written in balancing and Lucasbalancing p-numbers as in form:

$$x_k^n = 6.x_k^{n-1} - x_k^{n-p-1} = x_k(6.x_k^{n-2} - x_k^{n-p-2}) = x_k \cdot x_k^{n-1}, \tag{2.14}$$

for all integer values n and $k = 1, 2, 3, \dots, p + 1$.

Theorem 2.5. For any given positive integers p(p > 0), balancing p-numbers can be written for $(n = 0, \pm 1, \pm 2, \pm 3, \cdots)$ in the form:

$$B_p(n) = k_1 x_1^n + k_2 x_2^n + \dots + k_{p+1} x_{p+1}^n, \tag{2.15}$$

where k_1, k_2, \dots, k_{p+1} are coefficient constants and x_1, x_2, \dots, x_{p+1} are roots of the polynomial equation $x^{p+1} - 6x^p + 1 = 0$.

Proof. We can easily find out the first p-terms for $n=0,1,2,\cdots,p$ of the balancing p-numbers by using (2.6), (2.8) and algebraic equation $x^{p+1}-6x^p+1=0$. Now our seek is to prove $B_p(n)=k_1x_1^n+k_2x_2^n+\cdots+k_{p+1}x_{p+1}^n$ for remaining positive integers. For the case n=p+1, we have

$$B_{p}(p+1) = k_{1}x_{1}^{p+1} + k_{2}x_{2}^{p+1} + \dots + k_{p+1}x_{p+1}^{p+1}$$

$$= 6[k_{1}x_{1}^{p} + k_{2}x_{2}^{p} + \dots + k_{p+1}x_{p+1}^{p}] - [k_{1}x_{1}^{0} + k_{2}x_{2}^{0} + \dots + k_{p+1}x_{p+1}^{0}].$$

Therefore according to (2.8), we have

$$B_p(p+1) = 6B_p(p) - B_p(0),$$

which is the basic recurrence relation (2.1) for n = p + 1.

Similarly it is easy to prove that equation (2.15) is true for all remaining positive values from n = p + 2.

Finally, we have to prove equation (2.15) is true for all negative values of n. For the case n = -1:

$$B_p(-1) = k_1 x_1^{-1} + k_2 x_2^{-1} + \dots + k_{p+1} x_{p+1}^{-1}.$$
 (2.16)

Let write (2.14) in the form:

$$x_k^{n-p-1} = 6 \cdot x_k^{n-1} - x_k^n. (2.17)$$

By puting n = p in (2.17), we get

$$x_k^{-1} = 6 \cdot x_k^{p-1} - x_k^p. (2.18)$$

Apply (2.18) in (2.16), we get

$$B_p(-1) = 6[k_1x_1^{p-1} + k_2x_2^{p-1} + \dots + k_{p+1}x_{p+1}^{p-1}] - [k_1x_1^p + k_2x_2^p + \dots + k_{p+1}x_{p+1}^p].$$
(2.19)

Using (2.8), expression (2.19) will become

$$B_p(-1) = 6B_p(p-1) - B_p(p) = 0,$$

which is the balancing p-number $B_p(-1) = 0$.

Similarly, for negative values of $n=-2,-3,-4,\cdots$, we will get all balancing p-numbers. Hence the equation (2.15) is true for all $n=0,\pm 1,\pm 2,\pm 3,\cdots$. This completes the proof.

Using a similar approach to Theorem 2.5, we can also prove the following theorem for Lucas-balancing p-numbers

Theorem 2.6. For any given positive integers p(p > 0), Lucas-balancing p-numbers can be written for $(n = 0, \pm 1, \pm 2, \pm 3, \cdots)$ in the form:

$$C_n(n) = a_1 x_1^n + a_2 x_2^n + \dots + a_{n+1} x_{n+1}^n$$

where a_1, a_2, \dots, a_{p+1} are coefficient constants and x_1, x_2, \dots, x_{p+1} are roots of the polynomial equation $x^{p+1} - 6x^p + 1 = 0$.

3. Incomplete balancing and Lucas-balancing p-numbers

In this section we introduce incomplete balancing and Lucas-balancing pnumbers and present some of their properties.

Definition 3.1. The incomplete balancing *p*-numbers denoted by $B_p^k(n)$ are defined by

$$B_p^k(n) = \sum_{j=0}^k (-1)^j \binom{n-1-pj}{j} 6^{n-(p+1)j-1}, \left(n=1,2,3,\cdots;0 \leqslant k \leqslant \lfloor \frac{n-1}{p+1} \rfloor\right). \tag{3.1}$$

In a similar manner incomplete Lucas-balancing p-numbers can also be defined as follows:

Definition 3.2. The incomplete Lucas-balancing *p*-numbers denoted by $C_p^k(n)$ are defined by

$$C_p^k(n) = 3\sum_{j=0}^k (-1)^j \frac{n}{n-pj} \binom{n-pj}{j} 6^{n-(p+1)j-1}, \qquad (3.2)$$
$$\left(n = 1, 2, 3, \dots; 0 \leqslant k \leqslant \lfloor \frac{n}{p+1} \rfloor\right).$$

Notice that $B_1^{\lfloor \frac{n-1}{2} \rfloor}(n) = B_n$, $C_1^{\lfloor \frac{n}{2} \rfloor}(n) = C_n$ and $B_1^k(n) = B_n(k)$, $C_1^k(n) = C_n(k)$.

Some cases based on definitions 3.1 and 3.2 are

$$B_p^0 = 6^{n-1}$$
; for all $n \ge 1$,

$$B^1_p(n) = 6^{n-1} - 6^{n-p-2}(n-p-1); \text{ for all } n \geqslant p+2,$$

$$B_p^2(n) = 6^{n-1} - (n-1-p)6^{n-p-2} + 3(n-2p-1)(n-2p-2)6^{n-2p-4};$$
 for all $n \ge 2p+1$,

$$B_p^{\lfloor \frac{n-1}{p+1} \rfloor}(n) = B_p(n); \text{ for all } n \geqslant 1,$$

$$C_p^0(n) = 3.6^{n-1}$$
; for all $n \ge 1$,

$$C_p^1(n) = 3[6^{n-1} - n6^{n-p-2}]; \text{ for all } n \geqslant p+1,$$

$$C_p^2(n) = 3[6^{n-1} - n6^{n-p-2} + 3n(n-2p-1)6^{n-2p-4}]; \text{ for all } n \geqslant 2p+2$$

and

$$C_p^{\lfloor \frac{n}{p+1} \rfloor}(n) = C_p(n); \text{ for all } n \geqslant 1.$$

Proposition 3.3. The recurrence relation of the incomplete balancing p-number is defined as:

$$B_p^{k+1}(n) = 6B_p^{k+1}(n-1) - B_p^k(n-p-1); \ 0 \le k \le \frac{n-p-3}{p+1}.$$
 (3.3)

Proof. By using Definition 3.1, the right hand side of (3.3) can be written as

$$\begin{split} &6\sum_{j=0}^{k+1}(-1)^{j}\binom{n-pj-2}{j}6^{n-(p+1)j-2} - \sum_{j=0}^{k}(-1)^{j}\binom{n-p-pj-2}{j}6^{n-p-(p+1)j-2} \\ &= \sum_{j=0}^{k+1}(-1)^{j}\binom{n-pj-2}{j}6^{n-(p+1)j-1} - \sum_{j=1}^{k+1}(-1)^{j-1}\binom{n-pj-2}{j-1}6^{n-(p+1)j-1} \\ &= \sum_{j=0}^{k+1}(-1)^{j}\binom{n-pj-2}{j}6^{n-(p+1)j-1} + \sum_{j=0}^{k+1}(-1)^{j}\binom{n-pj-2}{j-1}6^{n-(p+1)j-1} \\ &- \binom{n-2}{-1}6^{n-1} \\ &= \sum_{j=0}^{k+1}\left[\binom{n-pj-2}{j} + \binom{n-pj-2}{j-1}\right](-1)^{j}6^{n-(p+1)j-1} \\ &= \sum_{j=0}^{k+1}\binom{n-pj-1}{j}(-1)^{j}6^{n-(p+1)j-1} \\ &= B_p^{k+1}(n), \end{split}$$

and the result follows.

By virtue of Proposition 3.3 and equation (3.1), we get the following identity.

$$B_p^k(n) = 6B_p^k(n-1) - B_p^k(n-p-1) + (-1)^k \binom{n-p(k+1)-2}{k} 6^{n-(p+1)(k+1)-1}. \tag{3.4}$$

Proposition 3.4.

$$\sum_{j=0}^{h} \binom{h}{j} (-1)^{j+h} 6^{j} B_{p}^{k+j} (n+p(j-1)) = B_{p}^{k+h} (n+(p+1)h-p); \qquad (3.5)$$

$$\left(0 \leqslant k \leqslant \frac{n-p-h-1}{p+1}\right).$$

Proof. We shall prove this property by using principle of mathematical induction on h. The above sum (3.5) clearly holds for h = 0 and h = 1. Let us assume it holds for certain h > 1. We will show that it holds for $h \to h + 1$, now we

have

which follows the result.

$$\begin{split} &\sum_{j=0}^{h+1} \binom{h+1}{j} (-1)^{j+h+1} 6^j B_p^{k+j} (n+p(j-1)) \\ &= \sum_{j=0}^{h+1} (-1)^{j+h+1} \binom{h}{j} + \binom{h}{j-1} B_p^{k+j} (n+p(j-1)) 6^j \\ &= \sum_{j=0}^{h+1} (-1)^{j+h+1} \binom{h}{j} B_p^{k+j} \binom{n+p(j-1)}{6^j} + \sum_{j=0}^{h+1} (-1)^{j+h+1} \binom{h}{j-1} \\ &\times B_p^{k+j} \binom{n+p(j-1)}{6^j} \\ &= -B_p^{k+h} (n+(p+1)h-p) + \sum_{j=-1}^{h} (-1)^{j+h+2} \binom{h}{j} B_p^{k+j+1} \binom{n+pj}{6^j} 6^{j+1} \\ &= -B_p^{k+h} (n+(p+1)h-p) + \sum_{j=0}^{h} (-1)^{j+h+2} \binom{h}{j} B_p^{k+j+1} \binom{n+pj}{6^j} 6^j .6 \\ &+ (-1)^{h+1} \binom{h}{-1} B_p^k (n-p) \\ &= -B_p^{k+h} (n+(p+1)h-p) + 6 \sum_{j=0}^{h} (-1)^{j+h} \binom{h}{j} B_p^{k+j+1} \binom{n+pj}{6^j} 6^j \\ &= -B_p^{k+h} (n+(p+1)h-p) + 6 B_p^{k+h+1} (n+(p+1)h) \\ &= B_p^{k+h+h+1} (n+(p+1)h+1), \end{split}$$

Proposition 3.5. Let k be a non-negative integer. For $n \ge (p+1)k + p + 2$

$$\sum_{j=0}^{h-1} 6^{h-1-j} B_p^k(n-p+j) = 6^h B_p^{k+1}(n) - B_p^{k+1}(n+h).$$
 (3.6)

Proof. We shall prove this by using mathematical induction on h. The result is obvious for h = 1 and h = 2 by using (3.3).

Let us assume the given statement (3.6) is true for h = t that is

$$\sum_{j=0}^{t-1} 6^{t-1-j} B_p^k(n-p+j) = 6^t B_p^{k+1}(n) - B_p^{k+1}(n+t).$$

Now it is enough to show that the sum (3.6) is true for h = t + 1:

$$\sum_{j=0}^{t} 6^{t-j} B_p^k(n-p+j) = 6^{t+1} B_p^{k+1}(n) - B_p^{k+1}(n+t+1).$$

This implies

$$6\sum_{j=0}^{t-1} 6^{t-1-j} B_p^k(n-p+j) + B_p^k(n-p+t) = 6^{t+1} B_p^{k+1}(n) - B_p^{k+1}(n+t+1).$$

The above equality gives

$$6^{t+1}B_p^{k+1}(n) - 6B_p^{k+1}(n+t) + B_p^k(n-p+t) = 6^{t+1}B_p^{k+1}(n) - B_p^{k+1}(n+t+1).$$

Further simplification results

$$B_p^{k+1}(n+t+1) = 6B_p^{k+1}(n+t) - B_p^k(n-p+t)$$

This completes the result in view of (3.3).

Proposition 3.6.

$$2C_p^k(n) = 6B_p^k(n) - (p+1)B_p^{k-1}(n-p); \ 0 \le k \le \lfloor \frac{n-1}{p+1} \rfloor. \tag{3.7}$$

Proof. The right hand side of (3.7) can be written as

$$\begin{split} &6\sum_{j=0}^{k}(-1)^{j}\binom{n-pj-1}{j}6^{n-(p+1)j-1}-(p+1)\sum_{j=0}^{k-1}(-1)^{j}\binom{n-p-pj-1}{j}\\ &\times 6^{n-p-(p+1)j-1}\\ &=6\sum_{j=0}^{k}(-1)^{j}\binom{n-pj-1}{j}6^{n-(p+1)j-1}-(p+1)\sum_{j=1}^{k}(-1)^{j-1}\binom{n-pj-1}{j-1}\\ &\times 6^{n-(p+1)j}\\ &=\sum_{j=0}^{k}(-1)^{j}\binom{n-pj-1}{j}6^{n-(p+1)j}+(p+1)\sum_{j=0}^{k}(-1)^{j}\binom{n-pj-1}{j-1}6^{n-(p+1)j}\\ &-(p+1)\binom{n-1}{-1}6^{n}\\ &=\sum_{j=0}^{k}\left[\binom{n-pj-1}{j}+(p+1)\binom{n-pj-1}{j-1}\right](-1)^{j}6^{n-(p+1)j}\\ &=6\sum_{j=0}^{k}(-1)^{j}\frac{n}{n-pj}\binom{n-pj}{j}6^{n-(p+1)j-1}\\ &=2C_{p}^{k}(n), \end{split}$$

and then the result follows.

Proposition 3.7. The recurrence relation of the incomplete Lucas-balancing p-numbers $C_p^k(n)$ is

$$C_p^{k+1}(n) = 6C_p^{k+1}(n-1) - C_p^k(n-p-1); \qquad \left(0 \le k \le \frac{n-p-2}{p+1}\right). \tag{3.8}$$

Proof. Applying (3.3) and (3.7), we have

$$\begin{split} &2C_p^{k+1}(n)\\ =&6\Big(6B_p^{k+1}(n-1)-(p+1)B_p^k(n-p-1)\Big)-\Big(6B_p^k(n-p-1)-(p+1)\\ &\times B_p^{k-1}(n-2p-1)\Big)\\ =&6\Big(2C_p^{k+1}(n-1)\Big)-2C_p^k(n-p-1). \end{split}$$
 Hence, $C_p^{k+1}(n)=6C_p^{k+1}(n-1)-C_p^k(n-p-1)$.

Here we observe that by applying (3.2), the above relation (3.8) can be transformed into the non-homogeneous relation

$$C_p^k(n) = 6C_p^k(n-1) - C_p^k(n-p-1) + 3(-1)^k \frac{n-p-1}{n-(k+1)p-1} \times \binom{n-(k+1)p-1}{k} 6^{n-(k+1)p-k-2}.$$
(3.9)

Proposition 3.8. For $0 \le k \le \frac{n-p-h}{p+1}$, we have

$$\sum_{j=0}^{h} \binom{h}{j} (-1)^{j+h} 6^{j} C_{p}^{k+j} \left(n + p(j-1) \right) = C_{p}^{k+h} \left(n + (p+1)h - p \right).$$

Proof. The proof is similar to Proposition 3.4.

Proposition 3.9. Let k be a non-negative integer. For $n \ge (p+1)(k+1)$, the identity

$$\sum_{j=0}^{h-1} 6^{h-1-j} C_p^k(n-p+j) = 6^h C_p^{k+1}(n) - C_p^{k+1}(n+h)$$

holds.

Proof. The proof is analogous to Proposition 3.5.

The following result which has already proved in [8] is useful while finding the generating functions of $B_p^k(n)$ and $C_p^k(n)$.

Lemma 3.10. Let $\{s_n\}_{n=0}^{\infty}$ be a complex sequence satisfying the non-homogeneous recurrence relation

$$s_n = 6s_{n-1} - s_{n-p-1} + r_n, \quad n > p,$$

where r_n is a given complex sequence. Then the generating function $S_p^k(t)$ of the sequence $\{s_n\}_{n=0}^{\infty}$ is

$$S_p^k(t) = \frac{s_0 - r_0 + \sum_{i=1}^p t^i (s_i - 6s_{i-1} - r_i) + G(t)}{1 - 6t + t^{p+1}}$$

where G(t) denotes the generating function of $\{r_n\}$.

Theorem 3.11. The generating function of the incomplete balancing p-numbers $B_n^k(n)$ (k = 0, 1, 2, 3, ...) is given by

$$\begin{split} R_p^k(t) &= \sum_{j=0}^\infty B_p^j(k)t^j \\ &= t^{k(p+1)+1} \Big[\Big\{ B_p(k(p+1)+1) + \sum_{i=1}^p t^i (B_p(k(p+1)+i+1) - 6B_p(k(p+1)+i)) \Big\} (1-6t)^{k+1} + (-1)^k t^{p+1} \Big] \cdot \\ &= \Big[(1-6t+t^{p+1})(1-6t)^{k+1} \Big]^{-1}. \end{split}$$

Proof. We prove this theorem by using Lemma 3.10. Let k be a fixed positive integer, from (3.1) and (3.4), we have

$$B_p^k(n) = 0; \quad if \ 0 \le n < k(p+1) + 1$$

and

$$B_p^k(k(p+1)+1) = B_p(k(p+1)+1),$$

 $B_p^k(k(p+1)+2) = B_p(k(p+1)+2),$
.....

$$B_p^k(k(p+1) + p + 1) = B_p(k(p+1) + p + 1),$$

and that

$$\begin{array}{lcl} B_p^k(n) & = & 6B_p^k(n-1) - B_p^k(n-p-1) \\ & & + (-1)^k \binom{n-p(k+1)-2}{n-k(p+1)-p-2} 6^{n-k(p+1)-p-2}, \end{array}$$

if $n \ge k(p+1) + p + 2$.

We let

$$s_0 = B_p^k(k(p+1)+1), \ s_1 = B_p^k(k(p+1)+2), ..., s_p = B_p^k(k(p+1)+p+1)$$

and $s_n = B_p^k(n + k(p+1) + 1)$. Suppose that $r_0 = r_1 = r_2 = \cdots = r_p = 0$ and

$$r_n = \binom{n - (p+1) + k}{n - (p+1)} (-1)^k 6^{n - (p+1)}.$$

Thus we can easily derive that the generating function of the sequence r_n is (see p.355 of [10])

$$G(t) = \frac{(-1)^k t^{p+1}}{(1-6t)^{k+1}}$$

Then in view of Lemma 3.10, the generating function

$$S_p^k(t)(1-6t+t^{p+1}) - \frac{(-1)^k t^{p+1}}{(1-6t)^{k+1}}$$

$$= B_p^k(k(p+1)+1) + \sum_{i=1}^p t^i (B_p(k(p+1)+i+1) - 6B_p(k(p+1)+i)),$$

which implies

$$S_p^k(t) = \left[\left\{ B_p^k(k(p+1)+1) + \sum_{i=1}^p t^i (B_p(k(p+1)+i+1) - 6B_p(k(p+1) + i+1) - 6B_p(k(p+1) + i+1) + \sum_{i=1}^p t^i (B_p(k(p+1)+i+1) - 6B_p(k(p+1) + i+1) - 6B_p(k(p+1) + i+1) + \sum_{i=1}^p t^i (B_p(k(p+1)+i+1) - 6B_p(k(p+1) + i+1) - 6B_p(k(p+1) + i+1) + \sum_{i=1}^p t^i (B_p(k(p+1)+i+1) - 6B_p(k(p+1) + i+1) - 6B_p(k(p+1) + i+1) + \sum_{i=1}^p t^i (B_p(k(p+1)+i+1) - 6B_p(k(p+1) + i+1) - 6B_p(k(p+1) + i+1) + \sum_{i=1}^p t^i (B_p(k(p+1)+i+1) - 6B_p(k(p+1) + i+1) - 6B_p(k(p+1) + i+1) + \sum_{i=1}^p t^i (B_p(k(p+1)+i+1) - 6B_p(k(p+1) + i+1) - 6B_p(k(p+1) + i+1) + \sum_{i=1}^p t^i (B_p(k(p+1)+i+1) - 6B_p(k(p+1) + i+1) - 6B_p(k(p+1) + i+1) + \sum_{i=1}^p t^i (B_p(k(p+1)+i+1) - 6B_p(k(p+1)+i+1) + \sum_{i=1}^p t^i (B_p(k(p+1)+i+1) + \sum_{i=1}^p t^i (B$$

Finally, we conclude that

$$R_p^k(t) = t^{k(p+1)+1} S_p^k(t).$$

This completes the proof.

Theorem 3.12. The generating function of the incomplete Lucas-balancing p-numbers $C_p^k(n)$ (k = 0, 1, 2, 3, ...) is given by

$$\begin{split} W_p^k(t) &= \sum_{j=0}^\infty C_p^j(k)t^j \\ &= t^{k(p+1)} \Big[\Big\{ C_p(k(p+1)) + \sum_{i=1}^p t^i (C_p(k(p+1)+i) - 6C_p(k(p+1) + i) - 6C_p(k(p+1) + i) \Big\} \\ &\qquad \qquad (1-6t)^{k+1} + (-1)^k 3t^{p+1} (p(1-t)+1) \Big] \Big[(1-6t+t^{p+1}) \cdot (1-6t)^{k+1} \Big]^{-1}. \end{split}$$

Proof. We prove this theorem by using Lemma 3.10. Let k be a fixed positive integer, from (3.2) and (3.9), we have

$$C_p^k(n) = 0; \quad if \ 0 \leqslant n < k(p+1)$$

and

$$C_p^k(k(p+1)) = C_p(k(p+1)),$$

$$C_p^k(k(p+1)+1) = C_p(k(p+1)+1),$$
....,
$$C_p^k(k(p+1)+p) = C_p(k(p+1)+p),$$

and that

$$C_p^k(n) = 6C_p^k(n-1) - C_p^k(n-p-1) + 3(-1)^k \frac{n-p-1}{n-(k+1)p-1} \cdot \binom{n-p(k+1)-1}{n-k(p+1)-p-1} 6^{n-k(p+1)-p-2};$$

 $s_0=C_p^k(k(p+1)),\ s_1=C_p^k(k(p+1)+1),\ \cdots,\ s_p=C_p^k(k(p+1)+p)$ and $s_n=C_p^k(n+k(p+1)).$ Suppose that $r_0=r_1=r_2=\cdots=r_p=0$ and if $n \ge k(p+1) + p + 1$. We let

$$r_n = \frac{n+k(p+1)-p-1}{n+k-p-1} \binom{n-(p+1)+k}{n-(p+1)} 3(-1)^k 6^{n-(p+2)}.$$

Then the generating function of the sequence r_n is (p.355, [10])

$$G(t) = \frac{(-1)^k 3t^{p+1}(p(1-t)+1)}{(1-6t)^{k+1}}.$$

By virtue of Lemma 3.10, the generating function

$$\begin{split} S_p^k(t)(1 & - 6t + t^{p+1}) - \frac{(-1)^k 3t^{p+1}(p(1-t)+1)}{(1-6t)^{k+1}} \\ & = C_p^k(k(p+1)) + \sum_{i=1}^p t^i (C_p(k(p+1)+i) - 6C_p(k(p+1)+i-1)). \end{split}$$

Further simplification gives

$$S_p^k(t) = \left[\left\{ C_p^k(k(p+1)) + \sum_{i=1}^p t^i (C_p(k(p+1)+i) - 6C_p(k(p+1)+i-1)) \right\} \cdot (1 - 6t)^{k+1} + (-1)^k 3t^{p+1} (p(1-t)+1) \right] \left[(1 - 6t + t^{p+1}) \cdot (1 - 6t)^{k+1} \right]^{-1}.$$

Finally, we conclude that

$$W_p^k(t) = t^{k(p+1)} S_p^k(t),$$

and hence the proof.

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