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On the Generalization of Interval Valued $\left(\in, \in \vee q_{\widetilde{k}}\right)$-Fuzzy bi-Ideals in Ordered Semigroups

Muhammad S. Ali Khan[a], Saleem Abdullah[b], Kostaq Hila[c*]

[a]Department of Mathematics, Hazara University, Mansehra, KPK, Pakistan
[b]Department of Mathematics, Abdul Wali Khan University, Mardan, KPK, Pakistan
[c]Department of Mathematical Engineering, Polytechnic University of Tirana, Albania

E-mail: sajjadalimath@yahoo.com
E-mail: saleemabdullah81@yahoo.com
E-mail: kostaq_hila@yahoo.com

Abstract. In this paper, we introduce a new sort of interval valued $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-fuzzy bi-ideal in ordered semigroups which is the generalization of interval valued $\left(\in, \in \vee q_{\widetilde{k}}\right)$-fuzzy bi-ideal and interval valued $(\in, \in \vee q)$-fuzzy bi-ideal of ordered semigroups. We give examples in which we show that these structures are more general than previous one. Finally, we characterize ordered semigroup by the property of interval valued $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-implication based fuzzy bi-ideals.

Keywords: Interval valued fuzzy bi-ideal, Interval valued $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-fuzzy bi-ideal, Implication based interval valued $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-fuzzy bi-ideal.

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*Corresponding Author

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1. Introduction

Zadeh in 1965 proposed fundamental concept of a fuzzy set [25]. Kuroki [15, 16, 17] studied the notion of fuzzy set in semigroups. A systematic exposition of fuzzy semigroups was given by Mordeson et al. [18], where one can find theoretical results on fuzzy semigroups and their use in fuzzy coding, fuzzy finite state machines and fuzzy languages. In [19], the monograph by Mordeson and Malik deals with the application of fuzzy approach to the concepts of automata and formal languages. In [20], Murali proposed the concept of a fuzzy point belonging to a fuzzy subset under a natural equivalence on fuzzy subset. The notion of a fuzzy set in topological structure has been introduced in [22]. The idea of quasi-coincidence of a fuzzy point with a fuzzy set, played a vital role to generate some different types of fuzzy subgroups. Bhakat and Das [1, 2], gave the concepts of (α, β)-fuzzy subgroups by using the belongs to relation $(\in)$ and quasi-coincident with relation (q) between a fuzzy point and a fuzzy subgroup and introduced the concept of an $(\in, \in \vee q)$-fuzzy subgroup. Jun and Song in [9], introduced the notion of (α, β)-fuzzy interior ideals of a semigroup. Kazanci and Yamak studied $(\in, \in \vee q)$-fuzzy bi-ideals of a semigroup in [10]. In [21], Shabir et al., characterized regular semigroups by (α, β)-fuzzy ideals. The generalization of $(\in, \in \vee q)$-fuzzy ideals of a BCK/BCI-algebra was discussed by Jun et al. in [6]. In mathematics, an ordered semigroup is a semigroup together with a partial order that is compatible with the semigroup operation. Ordered semigroups have many applications in the theory of sequential machines, formal languages, computer arithmetic and error-correcting codes. The concept of a fuzzy bi-ideal in ordered semigroups was first introduced by Kehayopulu and Tsingelis in [12], where some basic properties of fuzzy bi-ideals were discussed. A theory of fuzzy generalized sets on ordered semigroups can be developed. For further study on generalized fuzzy sets in ordered semigroups, we refer the reader to [3, 8, 13]. The notion of general form of the quasi-coincidence of a fuzzy point with a fuzzy set is initiated by Jun in [7]. In [11], Kang generalized the concept of $(\in, \in \vee q_k^\delta)$-fuzzy subsemigroup and initiated the notion of $(\in, \in \vee q_k)$-fuzzy subsemigroup of semigroups. Interval valued fuzzy sets were introduced independently by Zadeh [26], Grattan-Guiness [4], John [5], in the same year, where the value of the membership functions are intervals of numbers in place of the numbers. Interval valued fuzzy sets were initiated as a natural extension of fuzzy sets. Thillaigovindan and Chinnadurai in [23] introduced the concept of interval valued fuzzy ideals (bi-ideals, interior ideals and quasi-ideals) in semigroups. In [14], the authors initiated a new generalization of interval valued fuzzy bi-ideals in ordered semigroup called an interval valued $\left(\in, \in \vee q_{\widetilde{k}}\right)$-fuzzy bi-ideal of an ordered semigroup.

In this article, we initiate a new more general form of an interval valued $(\in, \in \vee q)$-fuzzy bi-ideal and interval valued $\left(\in, \in \vee q_{\widetilde{k}}\right)$-fuzzy bi-ideal of an

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ordered semigroup. We introduce the notion of an interval valued $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-fuzzy bi-ideal of an ordered semigroup and give examples which are interval valued $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-fuzzy bi-ideals but not interval valued $(\in, \in \vee q)$-fuzzy bi-ideals and interval valued $\left(\in, \in \vee q_{\widetilde{k}}\right)$-fuzzy bi-ideal. We discuss characterizations of interval valued $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-fuzzy bi-ideals in ordered semigroups. We finally consider characterizations of an interval valued fuzzy bi-ideal and an interval valued $(\in, \in \vee q_{k}^{\delta})$-fuzzy bi-ideals by using implication operators and the notion of implication-based interval valued fuzzy bi-ideals. The important achievement of the study with an interval valued $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-fuzzy bi-ideal is that the notion of interval valued $\left(\in, \in \vee q_{\widetilde{k}}\right)$-fuzzy bi-ideal and interval valued $(\in, \in \vee q)$-fuzzy bi-ideal are special cases of an interval valued $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-fuzzy bi-ideals.

2. Preliminaries

By an ordered semigroup (or po-semigroup) we mean a structure $(S, \cdot, \leq)$ in which the following are satisfied:

$(OS1)$ $(S, \cdot)$ is a semigroup,

$(OS2)$ $(S, \leq)$ is a poset,

$(OS3)$ $(\forall x, y, a \in S)\,(x \leq y \Rightarrow a \cdot x \leq a \cdot y,\ x \cdot a \leq a \cdot y)$.

A nonempty subset A of an ordered semigroup S is said to be a subsemigroup of S if $A^2 \subseteq A$. A non-empty subset F of an ordered semigroup S is called a left (resp. right) ideal of S if it satisfies

(1) $(\forall x \in S)$ $(y \in F)$ $(x \leq y \Rightarrow x \in A)$,

(2) $SA \subseteq A(\text{resp.} AS \subseteq A)$.

A non-empty subset A of S is called an ideal if it is both a left and a right ideal of S.

A non-empty subset A of an ordered semigroup S is called a bi-ideal of S if

(1) $(\forall x \in S)$ $(y \in F)$ $(x \leq y \Rightarrow x \in A)$,

(2) $A^2 \subseteq A$;

(3) $ASA \subseteq A$.

A fuzzy subset λ of a universe X is a function from X into the unit closed interval $[0, 1]$, that is $\lambda : X \to [0, 1]$.

Let λ be a fuzzy set of a semigroup S and $t \in [0, 1]$, the set $U(\lambda; t) = \{x \in S | \lambda(x) \geq t\}$ is called a level subset of the fuzzy set λ.

Let λ be a fuzzy subset of S and $t \in [0, 1]$. Then the set $U(\lambda; t) = \{x \in S : \lambda(x) \geq t\}$ is called the level subset of S.

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Let A be a non-empty subset of S. We denote by λ_A, the characteristic function of A, that is the mapping of S into $[0,1]$ defined by

$$\lambda_A(x) = \begin{cases} 1, \text{ if } x \in A \\ 0, \text{ if } x \notin A \end{cases}$$

Obviously λ_A is a fuzzy subset of S.

Let $(S, \cdot, \leq)$ be an ordered semigroup and λ a fuzzy subset of S. Then

λ of S is said to be a fuzzy subsemigroup of S if $(\forall u, v \in S)(\lambda(xy) \geq \lambda(x) \wedge \lambda(y))$.

A fuzzy subsemigroup λ is said to be a fuzzy bi-ideal of S if:

1) $(\forall x, y \in S)(x \leq y \Rightarrow \lambda(x) \geq \lambda(y))$, 2) $(\forall x, y, z \in S)(\lambda(xyz) \geq \lambda(x) \wedge \lambda(z))$

A fuzzy set λ in a set S of the form

$$\lambda(y) := \begin{cases} t \in (0,1] & \text{if } y = x, \\ 0 & \text{if } y \neq x, \end{cases} \tag{2.1}$$

is said to be a *fuzzy point* with support x and value t and is denoted by (x, t).

For a fuzzy subset λ in S, a fuzzy point (x, t) is said to

- be contained in λ denoted $(x, t) \in \lambda$, if $\lambda(x) \geq t$.
- be quasi-coincident with λ, denoted by $(x, t)\, q\lambda$, if $\lambda(x) + t > 1$.

For a fuzzy subset λ and fuzzy point (x, t) in a set S, we say that

- $(x, t) \in \wedge q\lambda$ if $(x, t) \in \lambda$ or $(x, t)\, q\lambda$.

For a fuzzy subset λ and fuzzy point (x, t) in a set S, and $k \in [0, 1)$ we say that $(x, t)\, q_k\lambda$ if $\lambda(x) + t + k > 1$ and $(x, t) \in \vee q_k\lambda$ if $(x, t) \in \lambda$ or$(x, t)\, q_k\lambda$. Jun et. al. in [7], considered the general form of the symbol $(x, t)\, q_k\lambda$ and $(x, t) \in \vee q_k\lambda$ as follows: For a fuzzy point (x, t) and fuzzy subset λ in a set X, we say that

i) $(x, t)\, q^{\delta}\lambda$ if $\lambda(x) + t > \delta$,

ii) $(x, t)\, q_k^{\delta}\lambda$ if $\lambda(x) + t + k > \delta$,

iii) $(x, t) \in \vee q_k^{\delta}\lambda$ if $(x, t) \in \lambda$ or $(x, t)\, q_k^{\delta}\lambda$;

iv) $(x, t)\, \overline{\alpha}\lambda$ if $(x, t)\, \alpha\lambda$ does not hold. For $\alpha \in \{\in, q, \in \vee q, \in \vee q_k, q_k^{\delta}, \in \vee q_k^{\delta}\}$.

where $k \in [0, 1)$ and $k < \delta$ in $[0, 1]$. Obviously, $(x, t)\, q^{\delta}\lambda$ implies $(x, t)\, q_k^{\delta}\lambda$.

By an interval number $\widetilde{x}$ we mean an interval $[x^-, x^+]$ where $0 \leq x^- < x^+ \leq 1$. The set of all interval numbers is denoted by $D[0, 1]$. The interval $[x, x]$ can be simply identified by the number $x \in [0, 1]$. For the interval numbers $\widetilde{x}_i = [x_i^-, x_i^+]$, $\widetilde{y}_i = [y_i^-, y_i^+]$, for all $i \in I$, we define the following:

i) $r\max\{\widetilde{x}_i, \widetilde{y}_i\} = \left[\max\left(x_i^-, y_i^-\right), \max\left(x_i^+, y_i^+\right)\right]$,

ii) $r\min\{\widetilde{x}_i, \widetilde{y}_i\} = \left[\min\left(x_i^-, y_i^-\right), \min\left(x_i^+, y_i^+\right)\right]$,

iii) $r\inf \widetilde{x}_i = \left[\bigwedge_{i \in I} x_i^-, \bigwedge_{i \in I} x_i^+\right]$, $r\sup \widetilde{x}_i = \left[\bigvee_{i \in I} x_i^-, \bigvee_{i \in I} x_i^+\right]$

iv) $\widetilde{x}_1 \leq \widetilde{x}_2 \Leftrightarrow x_1^- \leq x_2^-$ and $x_1^+ \leq x_2^+$

v) $\widetilde{x}_2 = \widetilde{x}_2 \Leftrightarrow x_1^- = x_2^-$ and $x_1^+ = x_2^+$

vi) $\widetilde{x}_2 < \widetilde{x}_2 \Leftrightarrow x_1^- < x_2^-$ and $x_1^+ < x_2^+$

vii) $k\widetilde{x}_2 = [kx_i^-, kx_i^+]$

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Clearly $(D[0,1], \leq, \wedge, \vee)$ forms a complete lattice where $\widetilde{0} = [0,0]$ is its least element and $\widetilde{1} = [1,1]$ is its greatest element. The interval valued fuzzy subsets deliver a more suitable explanation of uncertainty than the traditional fuzzy subsets. Therefore it is significant to use interval valued fuzzy subsets in applications. One of the key applications of fuzzy subsets is fuzzy control, and one of the greatest computationally concentrated parts of fuzzy control is the defuzzification. As a conversion to interval valued fuzzy subsets typically increase the extent of computations, it is vitally important to design faster algorithms for the corresponding defuzzification. An interval valued fuzzy subset $\widetilde{\lambda} : X \to D[0,1]$ is the set $\widetilde{\lambda} = \{x \in X | [\lambda^-(x), \lambda^+(x)] \in D[0,1]\}$, where $\lambda^- : X \to [0,1]$ and $\lambda^+ : X \to [0,1]$ are fuzzy subsets such that $0 \leq \lambda^-(x) < \lambda^+(x) \leq 1$ for all $x \in X$ and $[\lambda^-(x), \lambda^+(x)]$ is the interval degree of membership function of an element x to the set $\widetilde{\lambda}$.

Let $\widetilde{\lambda}$ be an interval valued fuzzy subset of X. Then for every $\widetilde{0} \leq \widetilde{t} \leq \widetilde{1}$, the crisp set $U\left(\widetilde{\lambda}; \widetilde{t}\right) = \left\{x \in X | \widetilde{\lambda}(x) \geq \widetilde{t}\right\}$ is said to be the level set of $\widetilde{\lambda}$.

Note that, since every $x \in [0,1]$ is in correspondence with the interval $[x,x] \in D[0,1]$, therefore a fuzzy set is a particular case of the interval valued fuzzy sets.

For any $\widetilde{\lambda} = [\lambda^-, \lambda^+]$ and $\widetilde{t} = [t^-, t^+]$, we define

$$\widetilde{\lambda}(x) + \widetilde{t} = \left[\lambda^-(x) + t^-, \lambda^+(x) + t^+\right],$$

for all $x \in X$. In particular, if $\lambda^-(x) + t^- > 1$ and $\lambda^+(x) + t^+ > 1$, we write $\widetilde{\lambda}(x) + \widetilde{t} > [1,1]$.

An interval valued fuzzy subset $\widetilde{\lambda}$ of a set S of the form

$$\widetilde{\lambda}(y) := \begin{cases} \widetilde{t} \in D[0,1] & \text{if } y = x \\ [0,0] & \text{if } y \neq x \end{cases}$$

is said to be an interval valued fuzzy point with support x and value $\widetilde{t}$ and is denoted by $\left(x, \widetilde{t}\right)$.

For an interval valued fuzzy subset $\widetilde{\lambda}$ of a set S, an interval valued fuzzy point $\left(x, \widetilde{t}\right)$ is said to

· be contained in $\widetilde{\lambda}$ denoted by $\left(x, \widetilde{t}\right) \in \widetilde{\lambda}$, if $\widetilde{\lambda}(x) \geq \widetilde{t}$.

· be quasi-coincident with $\widetilde{\lambda}$ denoted by $\left(x, \widetilde{t}\right) q \widetilde{\lambda}$ if $\widetilde{\lambda}(x) + \widetilde{t} > \widetilde{1}$. Where $\lambda^-(x) + t^- > 1$ and $\lambda^+(x) + t^+ > 1$

For an interval valued fuzzy point $\left(x, \widetilde{t}\right)$ and an interval valued fuzzy subset $\widetilde{\lambda}$ of a set S, we say that

· $\left(x, \widetilde{t}\right) \in \vee q \widetilde{\lambda}$ if $\left(x, \widetilde{t}\right) \in \widetilde{\lambda}$ or $\left(x, \widetilde{t}\right) q \widetilde{\lambda}$

· $\left(x, \widetilde{t}\right) \overline{\alpha} \widetilde{\lambda}$ if $\left(x, \widetilde{t}\right) \alpha \widetilde{\lambda}$ does not hold for $\alpha \in \{\in, q, \in \vee q\}$.

For an interval valued fuzzy subset $\widetilde{\lambda}$ of a set S, we say that an interval valued fuzzy point $\left(x, \widetilde{t}\right)$ is

· contained in $\widetilde{\lambda}$ denoted by $\left(x, \widetilde{t}\right) \in \widetilde{\lambda}$, if $\widetilde{\lambda}(x) \geq \widetilde{t}$.

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· quasi-coincident with $\widetilde{\lambda}$ denoted by $\left(x,\widetilde{t}\right) q_{\widetilde{k}}\widetilde{\lambda}$ if $\widetilde{\lambda}(x)+\widetilde{t}+\widetilde{k}>\widetilde{1}$. Where $\lambda^{-}(x)+t^{-}+k^{-}>1$ and $\lambda^{+}(x)+t^{+}+k^{+}>1$

For an interval valued fuzzy point $\left(x,\widetilde{t}\right)$ and an interval valued fuzzy subset $\widetilde{\lambda}$ of a set S, we say that

· $\left(x,\widetilde{t}\right)\in\vee q_{\widetilde{k}}\widetilde{\lambda}$ if $\left(x,\widetilde{t}\right)\in\widetilde{\lambda}$ or $\left(x,\widetilde{t}\right) q_{\widetilde{k}}\widetilde{\lambda}$

· $\left(x,\widetilde{t}\right)\overline{\alpha}\widetilde{\lambda}$ if $\left(x,\widetilde{t}\right)\alpha\widetilde{\lambda}$ does not hold for $\alpha\in\left\{q_{\widetilde{k}},\in\vee q_{\widetilde{k}}\right\}$.

3. Generalized interval valued $\left(\in,\in\vee q_{\widetilde{k}}\right)$-fuzzy bi-ideals

In what follows, let S be an ordered semigroup and let $\widetilde{k}=[k^{-},k^{+}]$ denote an arbitrary element of $D[0,1)$, $\widetilde{\delta}=[\delta^{-},\delta^{+}]$ denote an arbitrary element of $D(0,1]$, where $\widetilde{0}\leq\widetilde{k}<\widetilde{\delta}$, unless otherwise specified. For an interval valued fuzzy point $\left(x,\widetilde{t}\right)$ and an interval valued fuzzy subset $\widetilde{\lambda}$ of S , we say that

· $\left(x,\widetilde{t}\right) q_{\widetilde{k}}^{\widetilde{\delta}}\widetilde{\lambda}$ if $\widetilde{\lambda}(x)+\widetilde{t}+\widetilde{k}>\widetilde{1}$, where $\lambda^{-}(x)+t^{-}+k^{-}>\delta^{-}$ and $\lambda^{+}(x)+t^{+}+k^{+}>\delta^{+}$.

· $\left(x,\widetilde{t}\right)\in\vee q_{\widetilde{k}}^{\widetilde{\delta}}\widetilde{\lambda}$ if $\left(x,\widetilde{t}\right)\in\widetilde{\lambda}$ or $\left(x,\widetilde{t}\right) q_{\widetilde{k}}^{\widetilde{\delta}}\widetilde{\lambda}$

· $\left(x,\widetilde{t}\right)\overline{\alpha}\widetilde{\lambda}$ if $\left(x,\widetilde{t}\right)\alpha\widetilde{\lambda}$ does not hold for $\alpha\in\left\{q_{\widetilde{k}}^{\widetilde{\delta}},\in\vee q_{\widetilde{k}}^{\widetilde{\delta}}\right\}$.

Definition 3.1. An interval valued fuzzy subset $\widetilde{\lambda}$ of S is said to be an $\left(\in,\in\vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-fuzzy bi-ideal of S, if it satisfies the following conditions:

(i) $x\leq y$, $\left(y,\widetilde{t}\right)\in\widetilde{\lambda}\Rightarrow\left(x,\widetilde{t}\right)\in\vee q_{\widetilde{k}}^{\widetilde{\delta}}\widetilde{\lambda}$,

(ii) $\left(x,\widetilde{t_1}\right)\in\widetilde{\lambda}$, $\left(y,\widetilde{t_2}\right)\in\widetilde{\lambda}\Rightarrow\left(xy,r\min\left\{\widetilde{t_1},\widetilde{t_2}\right\}\right)\in\vee q_{\widetilde{k}}^{\widetilde{\delta}}\widetilde{\lambda}$,

(iii) $\left(x,\widetilde{t_1}\right)\in\widetilde{\lambda}$, $\left(z,\widetilde{t_2}\right)\in\widetilde{\lambda}\Rightarrow\left(xyz,r\min\left\{\widetilde{t_1},\widetilde{t_2}\right\}\right)\in\vee q_{\widetilde{k}}^{\widetilde{\delta}}\widetilde{\lambda}$, for all $x,y\in S$ and $\widetilde{t},\widetilde{t_1},\widetilde{t_2}\in D(0,1]$.

Theorem 3.2. *Let $\widetilde{\lambda}$ be an interval valued fuzzy subset of S. Then, the following are equivalent:*

(i) $\left(\forall\widetilde{t}\in D(\frac{\delta-k}{2},1]\right)\left(U\left(\widetilde{\lambda};\widetilde{t}\right)\neq\emptyset\text{ is a bi-ideal of }S\right)$.

(ii) $\widetilde{\lambda}$ *satisfy the following conditions:*

(iia) $(\forall x,y\in S)\left(x\leq y\Rightarrow\widetilde{\lambda}(x)\leq r\max\left\{\widetilde{\lambda}(y),\left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right]\right\}\right)$,

(iib) $(\forall x,y\in S)\left(r\min\left\{\widetilde{\lambda}(x),\widetilde{\lambda}(y)\right\}\leq r\max\left\{\widetilde{\lambda}(xy),\left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right]\right\}\right)$

(iic) $(\forall x,y,z\in S)\left(r\min\left\{\widetilde{\lambda}(x),\widetilde{\lambda}(z)\right\}\leq r\max\left\{\widetilde{\lambda}(xyz),\left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right]\right\}\right)$

Proof. Suppose that $U\left(\widetilde{\lambda};\widetilde{t}\right)\neq\emptyset$ is a bi-ideal of S for all $\widetilde{t}\in(\frac{\delta-k}{2},1]$. If there exist $x,y\in S$ such that the condition (iia) is not satisfied, that is, there exist $x,y\in S$ with $x\leq y$ such that $\widetilde{\lambda}(y)>r\max\left\{\widetilde{\lambda}(x),\left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right]\right\}$. Then $\widetilde{\lambda}(y)\in D(\frac{\delta-k}{2},1]$ and $y\in U\left(\widetilde{\lambda};\widetilde{\lambda}(x)\right)$. But $\widetilde{\lambda}(x)<\widetilde{\lambda}(y)$. It implies that $y\notin U\left(\widetilde{\lambda};\widetilde{\lambda}(x)\right)$, which is a contradiction. Hence, condition (iia) is satisfied. Now assume that (iib) is not valid, i.e., $\widetilde{t_1}=r\min\left\{\widetilde{\lambda}(x),\widetilde{\lambda}(y)\right\}>$

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$r\max\left\{\widetilde{\lambda}(xy), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\}$ for some $x, y \in S$. Then $\widetilde{t_1} \in D(\frac{\delta - k}{2}, 1]$ and $x, z \in U\left(\widetilde{\lambda}; \widetilde{t_1}\right)$. But $xy \notin U\left(\widetilde{\lambda}; \widetilde{t_1}\right)$, since $\widetilde{\lambda}(xy) < \widetilde{t_1}$. Which is a contradiction and hence (iib) satisfied for all $x, y \in S$. Assume that (iic) is not true, that is, $\widetilde{t_2} = r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z)\right\} > r\max\left\{\widetilde{\lambda}(xyz), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\}$ for some $x, y, z \in S$. Then $\widetilde{t_2} \in D(\frac{\delta - k}{2}, 1]$ and $x, z \in U\left(\widetilde{\lambda}; \widetilde{t_2}\right)$. But $xyz \notin U\left(\widetilde{\lambda}; \widetilde{t_2}\right)$, since $\widetilde{\lambda}(xyz) < \widetilde{t_2}$. This is a contradiction and hence (iib) is also satisfied.

Conversely, assume that $\widetilde{\lambda}$ satisfies the conditions (iia) (iib) and (iic). Assume that $U\left(\widetilde{\lambda}; \widetilde{t}\right) \neq \emptyset$ for all $\widetilde{t} \in D(\frac{\delta - k}{2}, 1]$. Let $x, y \in S$ be such that $x \leq y$ and $y \in U\left(\widetilde{\lambda}; \widetilde{t}\right)$. Then, $\widetilde{\lambda}(y) \geq \widetilde{t}$ and so $r\max\left\{\widetilde{\lambda}(x), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} \geq \widetilde{\lambda}(y) \geq \widetilde{t} > \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$. Hence, $\widetilde{\lambda}(x) \geq \widetilde{t}$, that is, $x \in U\left(\widetilde{\lambda}; \widetilde{t}\right)$. If $x, y \in U\left(\widetilde{\lambda}; \widetilde{t}\right)$, then from (iib) it implies that $r\max\left\{\widetilde{\lambda}(xy), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} \geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y)\right\} \geq \widetilde{t} > \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$. Hence, $\widetilde{\lambda}(xy) \geq \widetilde{t}$. That is, $xy \in U\left(\widetilde{\lambda}; \widetilde{t}\right)$.If $x, y, z \in U\left(\widetilde{\lambda}; \widetilde{t}\right)$, then from (iic), it implies that

$$\begin{aligned} r\max\left\{\widetilde{\lambda}(xyz), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} &\geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z)\right\} \\ &\geq \widetilde{t} > \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]. \end{aligned}$$

Hence, $\widetilde{\lambda}(xyz) \geq \widetilde{t}$. That is, $xyz \in U\left(\widetilde{\lambda}; \widetilde{t}\right)$. Therefore, $U\left(\widetilde{\lambda}; \widetilde{t}\right)$ is a bi-ideal of S for all $\widetilde{t} \in D(\frac{\delta - k}{2}, 1]$, with $U\left(\widetilde{\lambda}; \widetilde{t}\right) \neq \emptyset$. □

If we take $\widetilde{\delta} = [1, 1]$ in Theorem 3.2, then we have the following corollary.

Corollary 3.3. *Let $\widetilde{\lambda}$ be an interval valued fuzzy subset of S. Then, the following are equivalent:*

(i) $\left(\forall \widetilde{t} \in D(\frac{1-k}{2}, 1]\right)\left(U\left(\widetilde{\lambda}; \widetilde{t}\right) \neq \emptyset \text{ is a bi-ideal of } S\right)$.

(ii) $\widetilde{\lambda}$ satisfy the following conditions:

(iia) $(\forall x, y \in S)\left(x \leq y \Rightarrow \widetilde{\lambda}(x) \leq r\max\left\{\widetilde{\lambda}(y), \left[\frac{1-k^-}{2}, \frac{1-k^+}{2}\right]\right\}\right)$,

(iib) $(\forall x, y \in S)\left(r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y)\right\} \leq r\max\left\{\widetilde{\lambda}(xy), \left[\frac{1-k^-}{2}, \frac{1-k^+}{2}\right]\right\}\right)$

(iic) $(\forall x, y, z \in S)\left(r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z)\right\} \leq r\max\left\{\widetilde{\lambda}(xyz), \left[\frac{1-k^-}{2}, \frac{1-k^+}{2}\right]\right\}\right)$

If we take $\widetilde{\delta} = [1, 1]$ and $\widetilde{k} = [0, 0]$ in Theorem 3.2, then we have the following corollary.

Corollary 3.4. *Let $\widetilde{\lambda}$ be an interval valued fuzzy subset of S. Then, the following are equivalent:*

(i) $\left(\forall \widetilde{t} \in D(\frac{1}{2}, 1]\right)\left(U\left(\widetilde{\lambda}; \widetilde{t}\right) \neq \emptyset \text{ is a bi-ideal of } S\right)$.

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(ii) $\widetilde{\lambda}$ *satisfy the following conditions:*
(iia) $(\forall x, y \in S)\left(x \leq y \Rightarrow \widetilde{\lambda}(x) \leq r\max\left\{\widetilde{\lambda}(y), \left[\frac{1}{2}, \frac{1}{2}\right]\right\}\right)$,
(iib) $(\forall x, y \in S)\left(r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y)\right\} \leq r\max\left\{\widetilde{\lambda}(xy), \left[\frac{1-k^-}{2}, \frac{1-k^+}{2}\right]\right\}\right)$
(ii) $(\forall x, y, z \in S)\left(r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z)\right\} \leq r\max\left\{\widetilde{\lambda}(xyz), \left[\frac{1}{2}, \frac{1}{2}\right]\right\}\right)$

Theorem 3.5. *An interval valued fuzzy subset* $\widetilde{\lambda}$ *of* S *is an interval valued* $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$*-fuzzy bi-ideal of* S *if and only if:*
(i) $\left(\widetilde{\lambda}(y) \geq r\min\left\{\widetilde{\lambda}(x), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} \text{ with } x \leq y\right)$
(ii) $\widetilde{\lambda}(xy) \geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\}$
(iii) $\widetilde{\lambda}(xyz) \geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\}$
for all $x, y, z \in S$.

Proof. Assume that $\widetilde{\lambda}$ is an interval valued $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-fuzzy bi-ideal of S. Let $x, y \in S$ be such that $x \leq y$. If $\widetilde{\lambda}(y) < \widetilde{\lambda}(x)$, then $\widetilde{\lambda}(y) < \widetilde{t} \leq \widetilde{\lambda}(x)$ for some $\widetilde{t} \in D\left(0, \frac{\delta - k}{2}\right)$. It follows that $\left(x, \widetilde{t}\right) \in \widetilde{\lambda}$, but $\left(y, \widetilde{t}\right) \overline{\in} \widetilde{\lambda}$. Since $\widetilde{\lambda}(y) + \widetilde{t} < 2\widetilde{t} < \widetilde{\delta} - \widetilde{k}$, we have $\left(y, \widetilde{t}\right) \overline{q_{\widetilde{k}}^{\widetilde{\delta}}} \widetilde{\lambda}$. Therefore, $\left(y, \widetilde{t}\right) \overline{\in \vee q_{\widetilde{k}}^{\widetilde{\delta}}} \widetilde{\lambda}$, which is a contradiction. Hence, $\widetilde{\lambda}(y) \geq \widetilde{\lambda}(x)$. Now if $\widetilde{\lambda}(x) \geq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$, then $\left(x, \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right) \in \widetilde{\lambda}$ and so $\left(y, \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right) \in \vee q_{\widetilde{k}}^{\widetilde{\delta}} \widetilde{\lambda}$, it follows that $\widetilde{\lambda}(y) \geq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$ or $\widetilde{\lambda}(y) + \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right] > \widetilde{\delta} - \widetilde{k}$. Hence, $\widetilde{\lambda}(y) \geq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$. Otherwise, $\widetilde{\lambda}(y) + \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right] < \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right] + \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right] = \widetilde{\delta} - \widetilde{k}$, which is a contradiction. Therefore, $\widetilde{\lambda}(y) \geq r\min\left\{\widetilde{\lambda}(x), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\}$ for all $x, y \in S$ with $x \leq y$. Let $x, y \in S$ be such that $r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y)\right\} < \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$. We suppose that $\widetilde{\lambda}(xy) \geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y)\right\}$. If not, then we choose $\widetilde{t} \in D\left(0, \frac{\delta - k}{2}\right)$ such that $\widetilde{\lambda}(xy) < \widetilde{t} \leq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y)\right\}$. It implies that $\left(x, \widetilde{t}\right) \in \widetilde{\lambda}$ and $\left(y, \widetilde{t}\right) \in \widetilde{\lambda}$, but $\left(xy, \widetilde{t}\right) \overline{\in} \widetilde{\lambda}$ and $\widetilde{\lambda}(xy) + \widetilde{t} < 2\widetilde{t} < \widetilde{\delta} - \widetilde{k}$ i.e, $\left(xy, \widetilde{t}\right) \overline{\in \vee q_{\widetilde{k}}^{\widetilde{\delta}}} \widetilde{\lambda}$. This is a contradiction. Hence, $\widetilde{\lambda}(xy) \geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y)\right\}$ for all $x, y \in S$. If $r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y)\right\} \geq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$, then $\left(x, \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right) \in \widetilde{\lambda}$ and so $\left(y, \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right) \in \widetilde{\lambda}$. Thus,

$$\left(xy, \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right) = \left(xy, r\min\left\{\begin{array}{c}\left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right], \\ \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\end{array}\right\}\right) \in \vee q_{\widetilde{k}}^{\widetilde{\delta}} \widetilde{\lambda}$$

and so $\widetilde{\lambda}(xy) \geq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$ or $\widetilde{\lambda}(xy) + \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right] > \widetilde{\delta} - \widetilde{k}$. If $\widetilde{\lambda}(xy) < \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$, then $\widetilde{\lambda}(xy) + \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right] < \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right] +$

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$\left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right]=\widetilde{\delta}-\widetilde{k}$, which is a contradiction. Hence, $\widetilde{\lambda}\left(xy\right)\geq\left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right]$. Therefore, $\widetilde{\lambda}\left(xy\right)\geq r\min\left\{\widetilde{\lambda}\left(x\right),\widetilde{\lambda}\left(y\right),\left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right]\right\}$.

Let $x,y,z\in S$ be such that $r\min\left\{\widetilde{\lambda}\left(x\right),\widetilde{\lambda}\left(z\right)\right\}<\left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right]$. We suppose that $\widetilde{\lambda}\left(xyz\right)\geq r\min\left\{\widetilde{\lambda}\left(x\right),\widetilde{\lambda}\left(z\right)\right\}$. If not, then we choose $\widetilde{t}\in D\left(0,\frac{\delta-k}{2}\right)$ such that $\widetilde{\lambda}\left(xyz\right)<\widetilde{t}\leq r\min\left\{\widetilde{\lambda}\left(x\right),\widetilde{\lambda}\left(z\right)\right\}$. It implies that $\left(x,\widetilde{t}\right)\in\widetilde{\lambda}$ and $\left(z,\widetilde{t}\right)\in\widetilde{\lambda}$, but $\left(xyz,\widetilde{t}\right)\overline{\in}\widetilde{\lambda}$ and $\widetilde{\lambda}\left(xyz\right)+\widetilde{t}<2\widetilde{t}<\widetilde{\delta}-\widetilde{k}$ i.e, $\left(xyz,\widetilde{t}\right)\overline{\in\vee q_{\widetilde{k}}^{\widetilde{\delta}}}\widetilde{\lambda}$. This is a contradiction. Hence, $\widetilde{\lambda}\left(xyz\right)\geq r\min\left\{\widetilde{\lambda}\left(x\right),\widetilde{\lambda}\left(z\right)\right\}$ for all $x,z\in S$. If $r\min\left\{\widetilde{\lambda}\left(x\right),\widetilde{\lambda}\left(z\right)\right\}\geq\left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right]$, then $\left(x,\left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right]\right)\in\widetilde{\lambda}$ and so $\left(z,\left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right]\right)\in\widetilde{\lambda}$. Thus,

$$\left(xyz,\left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right]\right)=\left(xyz,r\min\left\{\begin{array}{c}\left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right],\\ \left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right]\end{array}\right\}\right)\in\vee q_{\widetilde{k}}^{\widetilde{\delta}}\widetilde{\lambda}$$

and so $\widetilde{\lambda}\left(xyz\right)\geq\left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right]$ or $\widetilde{\lambda}\left(xyz\right)+\left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right]>\widetilde{\delta}-\widetilde{k}$. If $\widetilde{\lambda}\left(xyz\right)<\left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right]$, then $\widetilde{\lambda}\left(xyz\right)+\left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right]<\left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right]+\left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right]=\widetilde{\delta}-\widetilde{k}$, which is a contradiction. Hence, $\widetilde{\lambda}\left(xyz\right)\geq\left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right]$. Therefore, $\widetilde{\lambda}\left(xyz\right)\geq r\min\left\{\widetilde{\lambda}\left(x\right),\widetilde{\lambda}\left(z\right),\left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right]\right\}$.

Conversely, let $\widetilde{\lambda}$ be an interval valued fuzzy subset of S that satisfies the three conditions $(i),(ii)$ and (iii). Let $x,y\in S$ and $\widetilde{t}\in D(0,1]$ be such that $x\leq y$ and $\left(x,\widetilde{t}\right)\in\widetilde{\lambda}$. Then, $\widetilde{\lambda}\left(x\right)\geq\widetilde{t}$, and so

$$\begin{aligned}\widetilde{\lambda}\left(x\right) &\geq r\min\left\{\widetilde{\lambda}\left(y\right),\left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right]\right\}\\ &\geq r\min\left\{\widetilde{t},\left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right]\right\}\\ &=\begin{cases}\widetilde{t} & \text{if } \widetilde{t}\leq\left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right]\\ \left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right] & \text{if } \widetilde{t}>\left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right]\end{cases}\end{aligned}$$

It implies that $\left(x,\widetilde{t}\right)\in\widetilde{\lambda}$ or $\widetilde{\lambda}\left(x\right)+\widetilde{t}\geq\left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right]+\widetilde{t}>\widetilde{\delta}-\widetilde{k}$, that is $\left(x,\widetilde{t}\right)q_{\widetilde{k}}^{\widetilde{\delta}}\widetilde{\lambda}$. Hence, $\left(x,\widetilde{t}\right)\in\vee q_{\widetilde{k}}^{\widetilde{\delta}}\widetilde{\lambda}$. Let $x,y\in S$ and $\widetilde{t_x},\widetilde{t_y}\in D(0,1]$ be such that $\left(x,\widetilde{t_x}\right)\in\widetilde{\lambda}$ and $\left(y,\widetilde{t_y}\right)\in\widetilde{\lambda}$. Then, $\widetilde{\lambda}\left(x\right)\geq\widetilde{t_x}$ and $\widetilde{\lambda}\left(y\right)\geq\widetilde{t_y}$. It implies from

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(ii) that

$$\begin{aligned}\widetilde{\lambda}(xy) &\geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} \\ &\geq r\min\left\{\widetilde{t_x}, \widetilde{t_y}, \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} \\ &= \begin{cases} r\min\left\{\widetilde{t_x}, \widetilde{t_y}\right\} & \text{if } r\min\left\{\widetilde{t_x}, \widetilde{t_y}\right\} \leq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right] \\ \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right] & \text{if } r\min\left\{\widetilde{t_x}, \widetilde{t_y}\right\} > \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right] \end{cases}.\end{aligned}$$

So that $\left(xy, r\min\left\{\widetilde{t_1}, \widetilde{t_2}\right\}\right) \in \widetilde{\lambda}$ or $\widetilde{\lambda}(xy) + r\min\left\{\widetilde{t_x}, \widetilde{t_y}\right\} \geq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right] +$ $r\min\left\{\widetilde{t_x}, \widetilde{t_y}\right\} > \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right] + \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right] = \widetilde{\delta} - \widetilde{k}$, that is, $\left(xy, r\min\left\{\widetilde{t_x}, \widetilde{t_z}\right\}\right) q_{\widetilde{k}}^{\widetilde{\delta}} \widetilde{\lambda}$. Thus, $\left(xy, r\min\left\{\widetilde{t_x}, \widetilde{t_y}\right\}\right) \in \vee q_{\widetilde{k}}^{\widetilde{\delta}} \widetilde{\lambda}$. Therefore, $\widetilde{\lambda}$ is an $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-fuzzy bi-ideal of S. Let $x, y, z \in S$ and $\widetilde{t_x}, \widetilde{t_z} \in D(0,1]$ be such that $\left(x, \widetilde{t_x}\right) \in \widetilde{\lambda}$ and $\left(z, \widetilde{t_z}\right) \in \widetilde{\lambda}$. Then, $\widetilde{\lambda}(x) \geq \widetilde{t_x}$ and $\widetilde{\lambda}(z) \geq \widetilde{t_z}$. It implies from (iii) that

$$\begin{aligned}\widetilde{\lambda}(xyz) &\geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} \\ &\geq r\min\left\{\widetilde{t_x}, \widetilde{t_z}, \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} \\ &= \begin{cases} r\min\left\{\widetilde{t_x}, \widetilde{t_z}\right\} & \text{if } r\min\left\{\widetilde{t_x}, \widetilde{t_z}\right\} \leq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right] \\ \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right] & \text{if } r\min\left\{\widetilde{t_x}, \widetilde{t_z}\right\} > \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right] \end{cases}.\end{aligned}$$

So that $\left(xyz, r\min\left\{\widetilde{t_1}, \widetilde{t_2}\right\}\right) \in \widetilde{\lambda}$ or $\widetilde{\lambda}(xyz) + r\min\left\{\widetilde{t_x}, \widetilde{t_z}\right\} \geq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right] +$ $r\min\left\{\widetilde{t_x}, \widetilde{t_z}\right\} > \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right] + \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right] = \widetilde{\delta} - \widetilde{k}$, that is,

$$\left(xyz, r\min\left\{\widetilde{t_x}, \widetilde{t_z}\right\}\right) q_{\widetilde{k}}^{\widetilde{\delta}} \widetilde{\lambda}.$$

Thus, $\left(xyz, r\min\left\{\widetilde{t_x}, \widetilde{t_z}\right\}\right) \in \vee q_{\widetilde{k}}^{\widetilde{\delta}} \widetilde{\lambda}$. Therefore, $\widetilde{\lambda}$ is an $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-fuzzy bi-ideal of S. □

If we take $\widetilde{\delta} = [1,1]$ in Theorem 3.5, then we have the following corollary.

Corollary 3.6. *An interval valued fuzzy subset* $\widetilde{\lambda}$ *of* S *is an interval valued* $\left(\in, \in \vee q_{\widetilde{k}}\right)$*-fuzzy bi-ideal of* S *if and only if:*

(i) $\left(\widetilde{\lambda}(y) \geq r\min\left\{\widetilde{\lambda}(x), \left[\frac{1-k^-}{2}, \frac{1-k^+}{2}\right]\right\}\right.$ *with* $\left.x \leq y\right)$

(ii) $\widetilde{\lambda}(xy) \geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y), \left[\frac{1-k^-}{2}, \frac{1-k^+}{2}\right]\right\}$

(iii) $\widetilde{\lambda}(xyz) \geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z), \left[\frac{1-k^-}{2}, \frac{1-k^+}{2}\right]\right\}$

for all $x, y, z \in S$.

If we take $\widetilde{\delta} = [1,1]$ and $\widetilde{k} = [0,0]$ in Theorem 3.5, then we have the following corollary.

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Corollary 3.7. *An interval valued fuzzy subset $\widetilde{\lambda}$ of S is an interval valued $(\in, \in \vee q)$-fuzzy bi-ideal of S if and only if:*

(i) $\left(\widetilde{\lambda}(y) \geq r\min\left\{\widetilde{\lambda}(x), \left[\frac{1}{2}, \frac{1}{2}\right]\right\}\right.$ *with* $\left. x \leq y\right)$

(ii) $\widetilde{\lambda}(xy) \geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y), \left[\frac{1-k^-}{2}, \frac{1-k^+}{2}\right]\right\}$

(iii) $\widetilde{\lambda}(xyz) \geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z), \left[\frac{1}{2}, \frac{1}{2}\right]\right\}$

for all $x, y, z \in S$.

Example 3.8. Consider the ordered semigroup $S = \{1, 2, 3, 4\}$ with the following multiplication table and ordered relation " $\leq$ ".

$\cdot$	a	b	c	d
a	a	b	b	d
b	b	b	b	b
c	c	b	b	b
d	d	d	d	b

$\leq := \{(a,a), (b,b), (c,c), (d,d), (a,d), (a,c), (c,d), (b,c), (b,d), (c,d), (d,c)\}$

Let $\widetilde{\lambda}$ be an interval valued fuzzy subset defined by

$$\widetilde{\lambda}(x) \begin{cases} [0.7, 0.8] & \text{it } x \in \{a, c\} \\ [0.3, 0.4] & \text{if } x = b \\ [0.4, 0.5] & \text{if } x = d \end{cases}$$

Let, $\widetilde{t_1}, \widetilde{t_2} \in D(0,1]$ such that $\widetilde{t_1} = [0.1, 0.2], \widetilde{t_1} = [0.2, 0.3]$. Then, $\widetilde{\lambda}$ is an interval valued $\left(\in, \in \vee q^{[0.3,0.4]}_{[0.1,0.2]}\right)$-fuzzy bi-ideal of S.

From Theorem 3.5, it is clear that every interval valued fuzzy bi-ideal, an interval valued $(\in, \in \vee q)$-fuzzy bi-ideal and interval valued $(\in, \in \vee q_{\widetilde{k}})$-fuzzy bi-ideal of S is an interval valued $\left(\in, \in \vee q^{\widetilde{\delta}}_{\widetilde{k}}\right)$-fuzzy bi-ideal of S. But the converse is not true. The following example shows this.

Example 3.9. The interval valued $\left(\in, \in \vee q^{\widetilde{\delta}}_{\widetilde{k}}\right)$-fuzzy bi-ideal of S is not an interval valued fuzzy bi-ideal, not an interval valued $(\in, \in \vee q)$-fuzzy bi-ideal and not interval valued $(\in, \in \vee q_{\widetilde{k}})$-fuzzy bi-ideal of S. Consider an ordered semigroup in Example 3.8. Since $b < d$, $\widetilde{\lambda}(b) = [0.3, 0.4] \ngeq [0.4, 0.5] = \widetilde{\lambda}(d)$, which shows that $\widetilde{\lambda}$ is not an interval valued fuzzy bi-ideal of S.

Also, $\widetilde{\lambda}(d \cdot d \cdot d) = \widetilde{\lambda}(b) = [0.3, 0.4] \ngeq [0.4, 0.5] = r\min\left\{\widetilde{\lambda}(d), \widetilde{\lambda}(d), \left[\frac{1}{2}, \frac{1}{2}\right]\right\}$, which shows that $\widetilde{\lambda}$ is not an interval valued $(\in, \in \vee q)$-fuzzy bi-ideal of S.

Moreover,

$$\widetilde{\lambda}(d \cdot d \cdot d) = \widetilde{\lambda}(b) = [0.3, 0.4] \ngeq [0.25, 0.35] = r\min\left\{\widetilde{\lambda}(d), \widetilde{\lambda}(d), \left[\frac{1-k^-}{2}, \frac{1-k^+}{2}\right]\right\}.$$

And so $\widetilde{\lambda}$ is not an interval valued $(\in, \in \vee q_{\widetilde{k}})$-fuzzy bi-ideal of S.

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In the following theorem we give a sufficient condition for interval valued $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-fuzzy bi-ideal of S to be an interval valued fuzzy bi-ideal of S.

Theorem 3.10. *Let $\widetilde{\lambda}$ be an interval valued $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-fuzzy bi-ideal of S. If $\widetilde{\lambda}(x) < \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$ for all $x \in S$, then $\widetilde{\lambda}$ is an interval valued fuzzy bi-ideal of S.*

Proof. The proof follows from Theorem 3.5. □

If we take $\widetilde{\delta} = [1, 1]$ in Theorem 3.10, then we have the following corollary.

Corollary 3.11. *Let $\widetilde{\lambda}$ be an interval valued $\left(\in, \in \vee q_{\widetilde{k}}\right)$-fuzzy bi-ideal of S. If $\widetilde{\lambda}(x) < \left[\frac{1-k^-}{2}, \frac{1-k^+}{2}\right]$ for all $x \in S$, then $\widetilde{\lambda}$ is an interval valued fuzzy bi-ideal of S.*

If we take $\widetilde{\delta} = [1, 1]$ and $\widetilde{k} = [0, 0]$ in Theorem 3.10, then we have the following corollary.

Corollary 3.12. *Let $\widetilde{\lambda}$ be an interval valued $(\in, \in \vee q)$-fuzzy bi-ideal of S. If $\widetilde{\lambda}(x) < \left[\frac{1}{2}, \frac{1}{2}\right]$ for all $x \in S$, then $\widetilde{\lambda}$ is an interval valued fuzzy bi-ideal of S.*

Theorem 3.13. *Let $[0, 0] \leq \widetilde{k} < \widetilde{r} < [1, 1]$ and $[0, 0] < \widetilde{\delta} < \widetilde{s} \leq [1, 1]$. Then, every interval valued $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-fuzzy bi-ideal of S is an interval valued $\left(\in, \in \vee q_{\widetilde{r}}^{\widetilde{s}}\right)$-fuzzy bi-ideal of S.*

Proof. Straightforward. □

The converse of Theorem 3.13 is not true. The following example shows this.

Example 3.14. Consider the ordered semigroup of Example 3.8. Let $\widetilde{\lambda}$ be an interval valued fuzzy subset of S defined by

$$\widetilde{\lambda}(x) = \begin{cases} [0.4, 0.5] & \text{if } x = a \\ [0.2, 0.3] & \text{if } x \in \{b, c\} \\ [0.1, 0.2] & \text{if } x = d \end{cases}$$

Then, $\widetilde{\lambda}$ is an interval valued $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-fuzzy bi-ideal of S with $\widetilde{k} \geq [0.3, 0.3]$ and $\widetilde{\delta} \geq [0.5, 0.6]$. Note that $3 < 4$ and

$$\begin{aligned} \widetilde{\lambda}(4) &= [0.1, 0.2] < [0.26, 0.28] = \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right] \\ &= r\min\left\{\widetilde{\lambda}(x), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} \end{aligned}$$

where $\widetilde{k} = [0.13, -0.4]$ and $\widetilde{\delta} = [0.4, 0.5]$. Thus, $\widetilde{\lambda}$ doesn't satisfy the first condition of Theorem 3.5, and hence $\widetilde{\lambda}$ is not an interval valued $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-fuzzy bi-ideal of S with $\widetilde{k} = [0.13, -0.4]$ and $\widetilde{\delta} = [0.4, 0.5]$. This shows that the converse of Theorem 3.13 is not true.

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Theorem 3.15. *For an interval valued fuzzy subset $\widetilde{\lambda}$ in S, the following are equivalent:*

(*i*) $\widetilde{\lambda}$ *is an interval valued* $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$*-fuzzy bi-ideal of* S.

(*ii*) $\left(\forall \widetilde{t} \in D(0, \frac{\delta-k}{2}]\right)\left(U\left(\widetilde{\lambda}; \widetilde{t}\right) \neq \emptyset \Rightarrow U\left(\widetilde{\lambda}; \widetilde{t}\right) \text{ is a bi-ideal of } S\right)$.

Proof. Suppose that $\widetilde{\lambda}$ is an interval valued $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-fuzzy bi-ideal of S and $\widetilde{t} \in D(0, \frac{\delta-k}{2})$ is such that $U\left(\widetilde{\lambda}; \widetilde{t}\right) \neq \emptyset$.
By Theorem 3.5(i), we have $\widetilde{\lambda}(x) \geq r\min\left\{\widetilde{\lambda}(y), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\}$ for any $x, y \in S$ with $x \leq y$ and $y \in U\left(\widetilde{\lambda}; \widetilde{t}\right)$. It implies that $\widetilde{\lambda}(x) \geq r\min\left\{\widetilde{t}, \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} = \widetilde{t}$ so that $x \in U\left(\widetilde{\lambda}; \widetilde{t}\right)$. Let $x, y \in U\left(\widetilde{\lambda}; \widetilde{t}\right)$. Then, $\widetilde{\lambda}(x) \geq \widetilde{t}$ and $\widetilde{\lambda}(y) \geq \widetilde{t}$. By Theorem 3.5($ii$) we have that

$$\begin{aligned} \widetilde{\lambda}(xy) &\geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} \\ &\geq r\min\left\{\widetilde{t}, \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} = \widetilde{t}. \end{aligned}$$

Thus, $xy \in U\left(\widetilde{\lambda}; \widetilde{t}\right)$. Therefore, $U\left(\widetilde{\lambda}; \widetilde{t}\right)$ is a bi-ideal of S, where $\widetilde{t} \in D(0, \frac{\delta-k}{2}]$.
Now let $x, z \in U\left(\widetilde{\lambda}; \widetilde{t}\right)$. Then, $\widetilde{\lambda}(x) \geq \widetilde{t}$ and $\widetilde{\lambda}(z) \geq \widetilde{t}$. Theorem 3.5($ii$) implies that

$$\begin{aligned} \widetilde{\lambda}(xyz) &\geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} \\ &\geq r\min\left\{\widetilde{t}, \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} = \widetilde{t}. \end{aligned}$$

Thus, $xyz \in U\left(\widetilde{\lambda}; \widetilde{t}\right)$. Therefore, $U\left(\widetilde{\lambda}; \widetilde{t}\right)$ is a bi-ideal of S where $\widetilde{t} \in D(0, \frac{\delta-k}{2}]$.

Conversely, let $\widetilde{\lambda}$ be an interval valued fuzzy subset of S such that $U\left(\widetilde{\lambda}; \widetilde{t}\right) \neq \emptyset$ is a bi-ideal of S for all $\widetilde{t} \in D(0, \frac{\delta-k}{2}]$. If there exist $x, y \in S$ with $x \leq y$ and $y \in \left(\widetilde{\lambda}; \widetilde{t}\right)$ such that $\widetilde{\lambda}(x) < r\min\left\{\widetilde{\lambda}(y), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\}$, then $\widetilde{\lambda}(y) < \widetilde{t}_x \leq r\min\left\{\widetilde{\lambda}(x), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\}$ for some $\widetilde{t}_x \in D(0, \frac{\delta-k}{2})$ and so $x \in U\left(\widetilde{\lambda}; \widetilde{t}\right)$ which is a contradiction. Therefore, $\widetilde{\lambda}(x) \geq r\min\left\{\widetilde{\lambda}(y), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\}$ for all $x, y \in S$ with $x \leq y$. Assume that for all $x, y, z \in S$, let $\widetilde{\lambda}(xy) < r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\}$. Then,

$$\widetilde{\lambda}(xy) < \widetilde{t}_{xy} \leq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\}$$

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for some $\widetilde{t}_{xy} \in D(0, \frac{\delta-k}{2})$. It implies that $x \in U\left(\widetilde{\lambda}; \widetilde{t}_{xy}\right)$ and $y \in U\left(\widetilde{\lambda}; \widetilde{t}_{xy}\right)$, but $xy \notin U\left(\widetilde{\lambda}; \widetilde{t}\right)$ which is a contradiction. Hence,

$$\widetilde{\lambda}(xy) \geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y), \left[\frac{\delta^{-}-k^{-}}{2}, \frac{\delta^{+}-k^{+}}{2}\right]\right\}.$$

Suppose that for all $x, y, z \in S$, let $\widetilde{\lambda}(xyz) < r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z), \left[\frac{\delta^{-}-k^{-}}{2}, \frac{\delta^{+}-k^{+}}{2}\right]\right\}$. Then, $\widetilde{\lambda}(xyz) < \widetilde{t}_{xyz} \leq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z), \left[\frac{\delta^{-}-k^{-}}{2}, \frac{\delta^{+}-k^{+}}{2}\right]\right\}$ for some $\widetilde{t}_{xyz} \in D(0, \frac{\delta-k}{2})$. It implies that $x \in U\left(\widetilde{\lambda}; \widetilde{t}_{xyz}\right)$ and $z \in U\left(\widetilde{\lambda}; \widetilde{t}_{xyz}\right)$, but $xyz \notin U\left(\widetilde{\lambda}; \widetilde{t}\right)$ which is a contradiction. Hence,

$$\widetilde{\lambda}(xyz) \geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z), \left[\frac{\delta^{-}-k^{-}}{2}, \frac{\delta^{+}-k^{+}}{2}\right]\right\}.$$

Therefore, by Theorem 3.5, we conclude that $\widetilde{\lambda}$ is an interval valued $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-fuzzy bi-ideal of S. □

If we take $\widetilde{\delta} = [1,1]$, in Theorem 3.15, then we have the following corollary.

Corollary 3.16. *For an interval valued fuzzy subset $\widetilde{\lambda}$ in S, the following are equivalent:*

(i) $\widetilde{\lambda}$ *is an interval valued $\left(\in, \in \vee q_{\widetilde{k}}\right)$-fuzzy bi-ideal of S.*

(ii) $\left(\forall \widetilde{t} \in D(0, \frac{1-k}{2}]\right)\left(U\left(\widetilde{\lambda}; \widetilde{t}\right) \neq \emptyset \Rightarrow U\left(\widetilde{\lambda}; \widetilde{t}\right) \text{ is a bi-ideal of } S\right)$.

If we take $\widetilde{\delta} = [1,1]$, and $\widetilde{k} = [0,0]$ in Theorem 3.15, then we have the following corollary.

Corollary 3.17. *For an interval valued fuzzy subset $\widetilde{\lambda}$ in S, the following are equivalent:*

(i) $\widetilde{\lambda}$ *is an interval valued $(\in, \in \vee q)$-fuzzy bi-ideal of S.*

(ii) $\left(\forall \widetilde{t} \in D(0, \frac{1}{2}]\right)\left(U\left(\widetilde{\lambda}; \widetilde{t}\right) \neq \emptyset \Rightarrow U\left(\widetilde{\lambda}; \widetilde{t}\right) \text{ is a bi-ideal of } S\right)$.

For any interval valued fuzzy subset of S and $\widetilde{t} \in D(0, \frac{\delta-k}{2}]$, we consider the following subsets:

$$Q_{\widetilde{k}}^{\widetilde{\delta}}\left(\widetilde{\lambda}; \widetilde{t}\right) := \left\{x \in S \mid \left(x, \widetilde{t}\right) q_{\widetilde{k}}^{\widetilde{\delta}} \widetilde{\lambda}\right\} \text{ and } \left[\widetilde{\lambda}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}} := \left\{x \in S \mid \left(x, \widetilde{t}\right) \in \vee q_{\widetilde{k}}^{\widetilde{\delta}} \widetilde{\lambda}\right\}.$$

It is clear that $\left[\widetilde{\lambda}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}} = U\left(\widetilde{\lambda}; \widetilde{t}\right) \cup Q_{\widetilde{k}}^{\widetilde{\delta}}\left(\widetilde{\lambda}; \widetilde{t}\right)$.

Theorem 3.18. *Let, $\widetilde{\lambda}$ be an interval valued $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-fuzzy bi-ideal of S, then*

$\left(\forall \widetilde{t} \in D(\frac{\delta-k}{2}, 1]\right)\left(Q_{\widetilde{k}}^{\widetilde{\delta}}\left(\widetilde{\lambda}; \widetilde{t}\right) \neq \emptyset \Rightarrow Q_{\widetilde{k}}^{\widetilde{\delta}}\left(\widetilde{\lambda}; \widetilde{t}\right) \text{ is a bi-ideal of } S\right)$.

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Proof. Suppose that $\widetilde{\lambda}$ is an interval valued $\left(\in,\in\vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-fuzzy bi-ideal of S. Let, $\widetilde{t}\in D(\frac{\delta-k}{2},1]$ be such that $Q_{\widetilde{k}}^{\widetilde{\delta}}\left(\widetilde{\lambda};\widetilde{t}\right)\neq\emptyset$. Let $y\in Q_{\widetilde{k}}^{\widetilde{\delta}}\left(\widetilde{\lambda};\widetilde{t}\right)$ and $x\in S$ be such that $x\leq y$. Then, $\widetilde{\lambda}(y)+\widetilde{t}>\widetilde{\delta}-\widetilde{k}$. Thus by Theorem 3.5$(i)$, we have

$$\begin{aligned}\widetilde{\lambda}(x) &\geq r\min\left\{\widetilde{\lambda}(y),\left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right]\right\}\\ &=\begin{cases}\left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right] \text{ if } \widetilde{\lambda}(y)\geq\left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right]\\ \widetilde{\lambda}(y) \text{ if } \widetilde{\lambda}(y)<\left[\frac{\delta^{-}-k^{-}}{2},\frac{\delta^{+}-k^{+}}{2}\right]\end{cases}\\ &>\widetilde{\delta}-\widetilde{t}-\widetilde{k}.\end{aligned}$$

Thus, $x\in Q_{\widetilde{k}}^{\widetilde{\delta}}\left(\widetilde{\lambda};\widetilde{t}\right)$. Let $x,y\in Q_{\widetilde{k}}^{\widetilde{\delta}}\left(\widetilde{\lambda};\widetilde{t}\right)$. Then, $\widetilde{\lambda}(x)+\widetilde{t}>\widetilde{\delta}-\widetilde{k}$ and $\widetilde{\lambda}(y)+\widetilde{t}>\widetilde{\delta}-\widetilde{k}$. Thus by Theorem 3.5$(ii)$, it follows that

$$\begin{aligned}\widetilde{\lambda}(xy) &\geq r\min\left\{\widetilde{\lambda}(x),\widetilde{\lambda}(y),\left[\frac{1-k^{-}}{2},\frac{1-k^{+}}{2}\right]\right\}\\ &=\begin{cases}r\min\left\{\widetilde{\lambda}(x),\widetilde{\lambda}(y)\right\} \text{ if } r\min\left\{\widetilde{\lambda}(x),\widetilde{\lambda}(y)\right\}<\left[\frac{1-k^{-}}{2},\frac{1-k^{+}}{2}\right]\\ \left[\frac{1-k^{-}}{2},\frac{1-k^{+}}{2}\right] \text{ if } r\min\left\{\widetilde{\lambda}(x),\widetilde{\lambda}(y)\right\}\geq\left[\frac{1-k^{-}}{2},\frac{1-k^{+}}{2}\right]\end{cases}\\ &>\widetilde{\delta}-\widetilde{t}-\widetilde{k}.\end{aligned}$$

Thus, $xyz\in Q_{\widetilde{k}}^{\widetilde{\delta}}\left(\widetilde{\lambda};\widetilde{t}\right)$. Now let $x,z\in Q_{\widetilde{k}}^{\widetilde{\delta}}\left(\widetilde{\lambda};\widetilde{t}\right)$. Then, $\widetilde{\lambda}(x)+\widetilde{t}>\widetilde{\delta}-\widetilde{k}$ and $\widetilde{\lambda}(z)+\widetilde{t}>\widetilde{\delta}-\widetilde{k}$. Thus by Theorem 3.5$(iii)$, it follows that

$$\begin{aligned}\widetilde{\lambda}(xyz) &\geq r\min\left\{\widetilde{\lambda}(x),\widetilde{\lambda}(z),\left[\frac{1-k^{-}}{2},\frac{1-k^{+}}{2}\right]\right\}\\ &=\begin{cases}r\min\left\{\widetilde{\lambda}(x),\widetilde{\lambda}(z)\right\} \text{ if } r\min\left\{\widetilde{\lambda}(x),\widetilde{\lambda}(z)\right\}<\left[\frac{1-k^{-}}{2},\frac{1-k^{+}}{2}\right]\\ \left[\frac{1-k^{-}}{2},\frac{1-k^{+}}{2}\right] \text{ if } r\min\left\{\widetilde{\lambda}(x),\widetilde{\lambda}(z)\right\}\geq\left[\frac{1-k^{-}}{2},\frac{1-k^{+}}{2}\right]\end{cases}\\ &>\widetilde{\delta}-\widetilde{t}-\widetilde{k}.\end{aligned}$$

Thus, $xyz\in Q_{\widetilde{k}}^{\widetilde{\delta}}\left(\widetilde{\lambda};\widetilde{t}\right)$. Therefore, $Q_{\widetilde{k}}^{\widetilde{\delta}}\left(\widetilde{\lambda};\widetilde{t}\right)$ is a bi-ideal of S. □

If we take $\widetilde{\delta}=[1,1]$, in Theorem 3.18, then we have the following corollary.

Corollary 3.19. *Let, $\widetilde{\lambda}$ be an interval valued $\left(\in,\in\vee q_{\widetilde{k}}\right)$-fuzzy bi-ideal of S, then*

$$\left(\forall\widetilde{t}\in D(\tfrac{1-k}{2},1]\right)\left(Q_{\widetilde{k}}\left(\widetilde{\lambda};\widetilde{t}\right)\neq\emptyset\Rightarrow Q_{\widetilde{k}}\left(\widetilde{\lambda};\widetilde{t}\right)\text{ is a bi-ideal of } S.\right)$$

If we take $\widetilde{\delta}=[1,1]$, and $\widetilde{k}=[0,0]$ in Theorem 3.18, then we have the following corollary.

Corollary 3.20. *Let, $\widetilde{\lambda}$ be an interval valued $(\in,\in\vee q)$-fuzzy bi-ideal of S, then*

$$\left(\forall\widetilde{t}\in D(\tfrac{1}{2},1]\right)\left(Q\left(\widetilde{\lambda};\widetilde{t}\right)\neq\emptyset\Rightarrow Q\left(\widetilde{\lambda};\widetilde{t}\right)\text{ is a bi-ideal of } S.\right)$$

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Theorem 3.21. *For an interval valued fuzzy subset* $\widetilde{\lambda}$ *of* S*, the following are equivalent:*

(i) $\widetilde{\lambda}$ *is an interval valued* $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$*-fuzzy bi-ideal of* S*.*

(ii) $\left(\forall \widetilde{t} \in D(0,\delta]\right)\left(\left[\widetilde{\lambda}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}} \neq \emptyset \Rightarrow \left[\widetilde{\lambda}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}}\right.$ *is a bi-ideal of* S*.*$\Big)$

Proof. Suppose that $\widetilde{\lambda}$ is an interval valued $\left(\in, \in \vee q_{\widetilde{k}}\right)$-fuzzy bi-ideal of S and $\widetilde{t} \in D(0,\delta]$ is such that $\left[\widetilde{\lambda}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}} \neq \emptyset$. Let $x \in \left[\widetilde{\lambda}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}}$ and $y \in S$ be such that $x \leq y$. Then, $y \in U\left(\widetilde{\lambda};\widetilde{t}\right)$ or $y \in Q_{\widetilde{k}}^{\widetilde{\delta}}\left(\widetilde{\lambda};\widetilde{t}\right)$. That is $\widetilde{\lambda}(y) \geq \widetilde{t}$ or $\widetilde{\lambda}(y) + \widetilde{t} > \widetilde{\delta} - \widetilde{k}$. Thus, by Theorem 3.5$(i)$, we get

$$\widetilde{\lambda}(x) \geq r\min\left\{\widetilde{\lambda}(y), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\}. \qquad (a)$$

So we consider two cases: If $\widetilde{\lambda}(y) \leq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$ and $\widetilde{\lambda}(y) > \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$. For the first case, if $\widetilde{\lambda}(y) \leq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$, Then $\widetilde{\lambda}(x) \geq \widetilde{\lambda}(y)$. Thus if $\widetilde{\lambda}(y) \geq \widetilde{t}$ for some $\widetilde{t} \in D(0,\delta]$. Then, $\widetilde{\lambda}(x) \geq \widetilde{t}$ and so $y \in U\left(\widetilde{\lambda};\widetilde{t}\right) \subseteq \left[\widetilde{\lambda}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}}$. If $\widetilde{\lambda}(y) + \widetilde{t} > \widetilde{\delta} - \widetilde{k}$, then $\widetilde{\lambda}(x) + \widetilde{t} \geq \widetilde{\lambda}(y) + \widetilde{t} > \widetilde{\delta} - \widetilde{k}$. This shows that $\left(x, \widetilde{t}\right) q_{\widetilde{k}}^{\widetilde{\delta}} \widetilde{\lambda}$, that is, $x \in Q_{\widetilde{k}}^{\widetilde{\delta}}\left(\widetilde{\lambda};\widetilde{t}\right) \subseteq \left[\widetilde{\lambda}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}}$.

From the second case and (a) we have, $\widetilde{\lambda}(x) > \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$. If $\widetilde{t} \leq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$, then $\widetilde{\lambda}(x) \geq \widetilde{t}$ and hence $x \in U\left(\widetilde{\lambda};\widetilde{t}\right) \subseteq \left[\widetilde{\lambda}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}}$. If $\widetilde{t} > \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$, then $\widetilde{\lambda}(x) + \widetilde{t} > \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right] + \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right] = \widetilde{\delta} - \widetilde{k}$. It follows that $x \in Q_{\widetilde{k}}^{\widetilde{\delta}}\left(\widetilde{\lambda};\widetilde{t}\right) \subseteq \left[\widetilde{\lambda}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}}$. Therefore, $\left[\widetilde{\lambda}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}}$ satisfies the first condition of being a bi-ideal in order semigroup S.

Let, $x, y \in \left[\widetilde{\lambda}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}}$. Then, $x \in U\left(\widetilde{\lambda};\widetilde{t}\right)$ or $x \in Q_{\widetilde{k}}^{\widetilde{\delta}}\left(\widetilde{\lambda};\widetilde{t}\right)$ and $y \in U\left(\widetilde{\lambda};\widetilde{t}\right)$ or $y \in Q_{\widetilde{k}}^{\widetilde{\delta}}\left(\widetilde{\lambda};\widetilde{t}\right)$. That is $\widetilde{\lambda}(x) \geq \widetilde{t}$ or $\widetilde{\lambda}(x) + \widetilde{t} > \widetilde{\delta} - \widetilde{k}$ and $\widetilde{\lambda}(y) \geq \widetilde{t}$ or $\widetilde{\lambda}(y) + \widetilde{t} > \widetilde{\delta} - \widetilde{k}$. Thus we consider the following four cases:

(i) If $\widetilde{\lambda}(x) \geq \widetilde{t}$ and $\widetilde{\lambda}(y) \geq \widetilde{t}$.

(ii) If $\widetilde{\lambda}(x) \geq \widetilde{t}$ and $\widetilde{\lambda}(y) + \widetilde{t} > \widetilde{\delta} - \widetilde{k}$.

(iii) If $\widetilde{\lambda}(x) + \widetilde{t} > \widetilde{\delta} - \widetilde{k}$ and $\widetilde{\lambda}(y) \geq \widetilde{t}$.

(iv) If $\widetilde{\lambda}(x) + \widetilde{t} > \widetilde{\delta} - \widetilde{k}$ and $\widetilde{\lambda}(y) + \widetilde{t} > \widetilde{\delta} - \widetilde{k}$.

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Case (i) : From Theorem 3.5(ii), it follows that

$$\begin{aligned}
\widetilde{\lambda}(xy) &\geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} \\
&\geq r\min\left\{\widetilde{t}, \widetilde{t}, \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} \\
&= r\min\left\{\widetilde{t}, \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} \\
&= \begin{cases} \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right] \text{ if } \widetilde{t} > \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right] \\ \widetilde{t} \text{ if } \widetilde{t} \leq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right] \end{cases}.
\end{aligned}$$

This implies that $xy \in U\left(\widetilde{\lambda}; \widetilde{t}\right)$ or $\widetilde{\lambda}(xy) + \widetilde{t} > \widetilde{\delta} - \widetilde{k} > \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right] + \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right] = \widetilde{\delta} - \widetilde{k}$. That is, $xy \in Q_{\widetilde{k}}^{\widetilde{\delta}}\left(\widetilde{\lambda}; \widetilde{t}\right)$. Hence, $xy \in \left[\widetilde{\lambda}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}}$.

Case (ii) : Assume that $\widetilde{t} > \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$. Then $\widetilde{\delta} - \widetilde{t} - \widetilde{k} \leq \widetilde{\delta} - \widetilde{t} < \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$ and hence,

$$\begin{aligned}
\widetilde{\lambda}(xy) &\geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} \\
&= \begin{cases} r\min\left\{\widetilde{\lambda}(y), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} > \widetilde{\delta} - \widetilde{k} \\ \text{if } r\min\left\{\widetilde{\lambda}(y), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} \leq \widetilde{\lambda}(x) \\ \widetilde{\lambda}(x) \geq \widetilde{t} \quad \text{if } r\min\left\{\widetilde{\lambda}(y), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} > \widetilde{\lambda}(x) \end{cases}
\end{aligned}$$

Thus, $xy \in U\left(\widetilde{\lambda}; \widetilde{t}\right) \cup Q_{\widetilde{k}}^{\widetilde{\delta}}\left(\widetilde{\lambda}; \widetilde{t}\right) = \left[\widetilde{\lambda}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}}$.

Now suppose that $\widetilde{t} \leq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$. Then,

$$\begin{aligned}
\widetilde{\lambda}(xy) &\geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} \\
&= \begin{cases} r\min\left\{\widetilde{\lambda}(x), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} \\ > \widetilde{\delta} - \widetilde{k} \text{ if } r\min\left\{\widetilde{\lambda}(x), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} \leq \widetilde{\lambda}(y) \\ \widetilde{\lambda}(y) \geq \widetilde{\delta} - \widetilde{t} - \widetilde{k} \quad \text{if } r\min\left\{\widetilde{\lambda}(x), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} > \widetilde{\lambda}(y) \end{cases}
\end{aligned}$$

Thus, $xy \in U\left(\widetilde{\lambda}; \widetilde{t}\right) \cup Q_{\widetilde{k}}^{\widetilde{\delta}}\left(\widetilde{\lambda}; \widetilde{t}\right) = \left[\widetilde{\lambda}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}}$.

Case (iii) : The proof of case (iii) is similar to that of case (ii).

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Case (iv) : Suppose that $\widetilde{t} > \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$. Then, $\widetilde{\delta} - \widetilde{t} - \widetilde{k} < \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$. Hence,

$$\begin{aligned} \widetilde{\lambda}(xy) &\geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} \\ &= \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right] \\ &> \widetilde{\delta} - \widetilde{t} - \widetilde{k}. \end{aligned}$$

whenever $r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y)\right\} < \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$. Thus, $xy \in Q_{\widetilde{k}}^{\widetilde{\delta}}\left(\widetilde{\lambda}; \widetilde{t}\right) \subseteq \left[\widetilde{\lambda}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}}$. If $\widetilde{t} \leq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$, then

$$\begin{aligned} \widetilde{\lambda}(xy) &\geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} \\ &= \begin{cases} \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right] \text{ if } r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y)\right\} \geq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]. \\ r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y)\right\} > \widetilde{\delta} - \widetilde{t} - \widetilde{k} \\ \text{if } r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y)\right\} < \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]. \end{cases} \end{aligned}$$

This implies that, $xy \in U\left(\widetilde{\lambda}; \widetilde{t}\right) \cup Q_{\widetilde{k}}^{\widetilde{\delta}}\left(\widetilde{\lambda}; \widetilde{t}\right) = \left[\widetilde{\lambda}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}}$.

Now let, $x, z \in \left[\widetilde{\lambda}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}}$. Then, $x \in U\left(\widetilde{\lambda}; \widetilde{t}\right)$ or $x \in Q_{\widetilde{k}}^{\widetilde{\delta}}\left(\widetilde{\lambda}; \widetilde{t}\right)$ and $z \in U\left(\widetilde{\lambda}; \widetilde{t}\right)$ or $z \in Q_{\widetilde{k}}^{\widetilde{\delta}}\left(\widetilde{\lambda}; \widetilde{t}\right)$. That is, $\widetilde{\lambda}(x) \geq \widetilde{t}$ or $\widetilde{\lambda}(z) + \widetilde{t} > \widetilde{\delta} - \widetilde{k}$ and $\widetilde{\lambda}(z) \geq \widetilde{t}$ or $\widetilde{\lambda}(z) + \widetilde{t} > \widetilde{\delta} - \widetilde{k}$. Thus we consider the following four cases:

(i) If $\widetilde{\lambda}(x) \geq \widetilde{t}$ and $\widetilde{\lambda}(z) \geq \widetilde{t}$.
(ii) If $\widetilde{\lambda}(x) \geq \widetilde{t}$ and $\widetilde{\lambda}(z) + \widetilde{t} > \widetilde{\delta} - \widetilde{k}$.
(iii) If $\widetilde{\lambda}(x) + \widetilde{t} > \widetilde{\delta} - \widetilde{k}$ and $\widetilde{\lambda}(z) \geq \widetilde{t}$.
(iv) If $\widetilde{\lambda}(x) + \widetilde{t} > \widetilde{\delta} - \widetilde{k}$ and $\widetilde{\lambda}(z) + \widetilde{t} > \widetilde{\delta} - \widetilde{k}$.

Case (i) : From Theorem 3.5(ii), it follows that

$$\begin{aligned} \widetilde{\lambda}(xyz) &\geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} \\ &\geq r\min\left\{\widetilde{t}, \widetilde{t}, \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} \\ &= r\min\left\{\widetilde{t}, \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} \\ &= \begin{cases} \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right] \text{ if } \widetilde{t} > \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right] \\ \widetilde{t} \text{ if } \widetilde{t} \leq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right] \end{cases}. \end{aligned}$$

This implies that $xyz \in U\left(\widetilde{\lambda}; \widetilde{t}\right)$ or $\widetilde{\lambda}(xyz) + \widetilde{t} > \widetilde{\delta} - \widetilde{k} > \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right] + \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right] = \widetilde{\delta} - \widetilde{k}$. That is, $xyz \in Q_{\widetilde{k}}^{\widetilde{\delta}}\left(\widetilde{\lambda}; \widetilde{t}\right)$. Hence, $xyz \in \left[\widetilde{\lambda}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}}$.

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Case (ii) : Assume that $\widetilde{t} > \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$. Then $\widetilde{\delta} - \widetilde{t} - \widetilde{k} \leq \widetilde{\delta} - \widetilde{t} < \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$ and hence,

$$\begin{aligned} \widetilde{\lambda}(xyz) &\geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} \\ &= \begin{cases} r\min\left\{\widetilde{\lambda}(z), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} > \widetilde{\delta} - \widetilde{k} \\ \quad \text{if } r\min\left\{\widetilde{\lambda}(z), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} \leq \widetilde{\lambda}(x) \\ \widetilde{\lambda}(x) \geq \widetilde{t} \quad \text{if } r\min\left\{\widetilde{\lambda}(z), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} > \widetilde{\lambda}(x) \end{cases} \end{aligned}$$

Thus, $xy \in U\left(\widetilde{\lambda}; \widetilde{t}\right) \cup Q^{\widetilde{\delta}}_{\widetilde{k}}\left(\widetilde{\lambda}; \widetilde{t}\right) = \left[\widetilde{\lambda}^{\widetilde{\delta}}_{\widetilde{k}}\right]_{\widetilde{t}}$.

Now suppose that $\widetilde{t} \leq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$. Then,

$$\begin{aligned} \widetilde{\lambda}(xyz) &\geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} \\ &= \begin{cases} r\min\left\{\widetilde{\lambda}(x), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} > \widetilde{\delta} - \widetilde{k} \\ \quad \text{if } r\min\left\{\widetilde{\lambda}(x), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} \leq \widetilde{\lambda}(z) \\ \widetilde{\lambda}(z) \geq \widetilde{\delta} - \widetilde{t} - \widetilde{k} \quad \text{if } r\min\left\{\widetilde{\lambda}(x), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} > \widetilde{\lambda}(z) \end{cases} \end{aligned}$$

Thus, $xyz \in U\left(\widetilde{\lambda}; \widetilde{t}\right) \cup Q^{\widetilde{\delta}}_{\widetilde{k}}\left(\widetilde{\lambda}; \widetilde{t}\right) = \left[\widetilde{\lambda}^{\widetilde{\delta}}_{\widetilde{k}}\right]_{\widetilde{t}}$.

Case (iii) : The proof of case (iii) is similar to case (ii).

Case (iv) : Suppose that $\widetilde{t} > \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$. Then, $\widetilde{\delta} - \widetilde{t} - \widetilde{k} < \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$. Hence,

$$\begin{aligned} \widetilde{\lambda}(xyz) &\geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} \\ &= \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right] \\ &> \widetilde{\delta} - \widetilde{t} - \widetilde{k}. \end{aligned}$$

whenever $r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z)\right\} < \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$. Thus, $xyz \in Q^{\widetilde{\delta}}_{\widetilde{k}}\left(\widetilde{\lambda}; \widetilde{t}\right) \subseteq \left[\widetilde{\lambda}^{\widetilde{\delta}}_{\widetilde{k}}\right]_{\widetilde{t}}$. If $\widetilde{t} \leq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$, then

$$\begin{aligned} \widetilde{\lambda}(xyz) &\geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} \\ &= \begin{cases} \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right] \text{ if } r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z)\right\} \geq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]. \\ r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z)\right\} > \widetilde{\delta} - \widetilde{t} - \widetilde{k} \\ \quad \text{if } r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z)\right\} < \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]. \end{cases} \end{aligned}$$

This implies that, $xyz \in U\left(\widetilde{\lambda}; \widetilde{t}\right) \cup Q^{\widetilde{\delta}}_{\widetilde{k}}\left(\widetilde{\lambda}; \widetilde{t}\right) = \left[\widetilde{\lambda}^{\widetilde{\delta}}_{\widetilde{k}}\right]_{\widetilde{t}}$.

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Conversely, assume that (ii) is valid. If there exist $x, y \in S$ such that $x \leq y$ and

$$\widetilde{\lambda}(x) < r \min\left\{\widetilde{\lambda}(y), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\},$$

then, $\widetilde{\lambda}(x) < \widetilde{t}_x \leq r \min\left\{\widetilde{\lambda}(y), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\}$ for some $\widetilde{t}_x \in D(0, \frac{\delta - k}{2}]$. It implies that $x \in U\left(\widetilde{\lambda}; \widetilde{t}_x\right) \subseteq \left[\widetilde{\lambda}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}}$ but $x \notin Q_{\widetilde{k}}^{\widetilde{\delta}}\left(\widetilde{\lambda}; \widetilde{t}\right)$. Also, we have $\widetilde{\lambda}(x) + \widetilde{t}_x < 2\widetilde{t}_x \leq \widetilde{\delta} - \widetilde{k}$, and hence $\left(x, \widetilde{t}_x\right) \overline{q_{\widetilde{k}}^{\widetilde{\delta}}} \widetilde{\lambda}$, that is $x \notin Q_{\widetilde{k}}^{\widetilde{\delta}}\left(\widetilde{\lambda}; \widetilde{t}_x\right)$. Therefore, $x \notin \left[\widetilde{\lambda}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}_x}$, which is a contradiction. Hence, $\widetilde{\lambda}(x) \geq r \min\left\{\widetilde{\lambda}(y), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\}$ for all $x, y \in S$ with $x \leq y$. Assume that there exist $x, y \in S$ such that $\widetilde{\lambda}(xy) < r \min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\}$, then $\widetilde{\lambda}(xy) < \widetilde{t} \leq r \min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\}$ for some $\widetilde{t} \in D(0, \frac{\delta - k}{2}]$. It implies that $x \in U\left(\widetilde{\lambda}; \widetilde{t}\right) \subseteq \left[\widetilde{\lambda}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}}$ and $y \in U\left(\widetilde{\lambda}; \widetilde{t}\right) \subseteq \left[\widetilde{\lambda}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}}$, thus by second condition of being bi-ideal in ordered semigroup S, we have $xy \subseteq \left[\widetilde{\lambda}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}}$. Thus, $\widetilde{\lambda}(xy) + \widetilde{t}$ or $\widetilde{\lambda}(xyz) + \widetilde{t} \geq \widetilde{\delta} - \widetilde{k}$, which is a contradiction. Hence, $\widetilde{\lambda}(xy) \geq r \min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\}$. Now suppose that there exist $x, y, z \in S$ such that $\widetilde{\lambda}(xyz) < r \min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\}$, then $\widetilde{\lambda}(xyz) < \widetilde{t} \leq r \min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\}$ for some $\widetilde{t} \in D(0, \frac{\delta - k}{2}]$. It implies that $x \in U\left(\widetilde{\lambda}; \widetilde{t}\right) \subseteq \left[\widetilde{\lambda}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}}$ and $z \in U\left(\widetilde{\lambda}; \widetilde{t}\right) \subseteq \left[\widetilde{\lambda}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}}$, thus by second condition of bi-ideal in order semigroup S we have, $xyz \subseteq \left[\widetilde{\lambda}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}}$. Thus, $\widetilde{\lambda}(xyz) + \widetilde{t}$ or $\widetilde{\lambda}(xyz) + \widetilde{t} \geq \widetilde{\delta} - \widetilde{k}$, which is a contradiction. Hence, $\widetilde{\lambda}(xyz) \geq r \min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\}$. Therefore, by Theorem 3.5, we conclude that $\widetilde{\lambda}$ is an interval valued $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-fuzzy bi-ideal of S. □

If we take $\widetilde{\delta} = [1, 1]$, in Theorem 3.21, then we have the following corollary.

Corollary 3.22. *For interval valued fuzzy subset* $\widetilde{\lambda}$ *of* S*, the following are equivalent:*

(i) $\widetilde{\lambda}$ *is an interval valued* $\left(\in, \in \vee q_{\widetilde{k}}\right)$*-fuzzy bi-ideal of* S.

(ii) $\left(\forall \widetilde{t} \in D(0, 1]\right)\left(\left[\widetilde{\lambda}_{\widetilde{k}}\right]_{\widetilde{t}} \neq \emptyset \Rightarrow \left[\widetilde{\lambda}_{\widetilde{k}}\right]_{\widetilde{t}}\right.$ *is a bi-ideal of* $S.\Big)$

If we take $\widetilde{\delta} = [1, 1]$, and $\widetilde{k} = [0, 0]$ in Theorem 3.21, then we have the following corollary.

Corollary 3.23. *For interval valued fuzzy subset* $\widetilde{\lambda}$ *of* S*, the following are equivalent:*

(i) $\widetilde{\lambda}$ *is an interval valued* $(\in, \in \vee q)$*-fuzzy bi-ideal of* S.

(ii) $\left(\forall \widetilde{t} \in D(0, 1]\right)\left(\left[\widetilde{\lambda}\right]_{\widetilde{t}} \neq \emptyset \Rightarrow \left[\widetilde{\lambda}\right]_{\widetilde{t}}\right.$ *is a bi-ideal of* $S.\Big)$

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An interval valued fuzzy subset $\widetilde{\lambda}$of S is called proper if $Im(\widetilde{\lambda})$ has at least two elements. Two interval valued fuzzy subsets are called equivalent if they have same family of level subsets. Otherwise, they are called non-equivalent.

Theorem 3.24. *Let $\widetilde{\lambda}$ be an interval valued $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-fuzzy bi-ideal of S such that*

$$\#\left\{\widetilde{\lambda}(x)\,|\widetilde{\lambda}(x) < \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} \geq 2.$$

Then, there exist two proper non-equivalent interval valued $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-fuzzy bi-ideal of S such that $\widetilde{\lambda}$ can be expressed as the union of them.

Proof. Let $\left\{\widetilde{\lambda}(x)\,|\widetilde{\lambda}(x) < \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} = \left\{\widetilde{t}_1, \widetilde{t}_2, ..., \widetilde{t}_n\right\}$, where, $\widetilde{t}_1 > \widetilde{t}_2 > ... > \widetilde{t}_n$ and $n \geq 2$. Then, the chain of $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-level bi-ideal of S is $\left[\widetilde{\lambda}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]} \subseteq \left[\widetilde{\lambda}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}_1} \subseteq \left[\widetilde{\lambda}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}_2} \subseteq ... \subseteq \left[\widetilde{\lambda}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}_n} = S$. Let $\widetilde{\Theta}$ and $\widetilde{\Xi}$ be interval valued fuzzy subset of S defined by

$$\widetilde{\Theta}(x) = \begin{cases} \widetilde{t}_1 & \text{if } x \in \left[\widetilde{\Theta}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}_1}, \\ \widetilde{t}_2 & \text{if } x \in \left[\widetilde{\Theta}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}_2} \setminus \left[\widetilde{\Theta}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}_1}, \\ & ... \\ \widetilde{t}_n & \text{if } x \in \left[\widetilde{\Theta}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}_n} \setminus \left[\widetilde{\Theta}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}_{n-1}} \end{cases}$$

and

$$\widetilde{\Xi}(x) = \begin{cases} \widetilde{\Xi}(x) & \text{if } x \in \left[\widetilde{\Xi}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]}, \\ \widetilde{k} & \text{if } x \in \left[\widetilde{\Xi}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}_2} \setminus \left[\widetilde{\Xi}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]}, \\ \widetilde{t}_3 & \text{if } x \in \left[\widetilde{\Xi}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}_2} \setminus \left[\widetilde{\Xi}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}_1}, \\ & ... \\ \widetilde{t}_n & \text{if } x \in \left[\widetilde{\Xi}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}_n} \setminus \left[\widetilde{\Xi}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}_{n-1}} \end{cases}$$

respectively, where $\widetilde{t}_3 < \widetilde{k} < \widetilde{t}_2$. Then, $\widetilde{\Theta}$ and $\widetilde{\Xi}$ an interval valued $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-fuzzy bi-ideal of S, and $\widetilde{\Theta}$,$\widetilde{\Xi} \in \widetilde{\lambda}$. The chain of $\left(\in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-level bi-ideal of $\widetilde{\Theta}$ and $\widetilde{\Xi}$ are given by $\left[\widetilde{\Theta}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}_1} \subseteq \left[\widetilde{\Theta}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}_2} \subseteq ... \subseteq \left[\widetilde{\Theta}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}_n}$ and $\left[\widetilde{\Xi}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}_1} \subseteq \left[\widetilde{\Xi}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}_2} \subseteq ... \subseteq \left[\widetilde{\Xi}_{\widetilde{k}}^{\widetilde{\delta}}\right]_{\widetilde{t}_n}$, respectively. Therefore, $\widetilde{\Theta}$ and $\widetilde{\Xi}$ are non equivalent and clearly, $\widetilde{\lambda} = \widetilde{\Theta} \cup \widetilde{\Xi}$. Hence proved. □

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4. Implication-based interval valued fuzzy bi-ideals

Fuzzy logic is an extension of set theoretic multi-valued logic in which the truth values are linguistic variables or terms of the linguistic variable truth. Some operators, for example $\wedge, \vee, \neg, \rightarrow$ in fuzzy logic are also defined by using truth tables and the extension principle can be applied to derive definitions of the operators. In fuzzy logic, the truth value of fuzzy proposition Φ is denoted by $[\Phi]$. For a universe X of discourse, we display the fuzzy logical and corresponding set-theoretical notations used in this paper

$$\begin{aligned} x &\in [\widetilde{\lambda}] = \widetilde{\lambda}(x), & (1)\\ [\Phi \vee \Psi] &= \min\{[\Phi], [\Psi]\}, & (2)\\ [\Phi &\rightarrow \Psi] = \min\{1, 1 - [\Phi] + [\Psi]\}, & (3)\\ [\forall \Phi(x)] &= \inf_{x \in X} [\Phi(x)], & (4)\\ \Phi \text{ if and only if } [\Phi] &= 1 \text{ for all valuation} & (5) \end{aligned}$$

The truth valuation rules given in (3) are those in the Ł ukasiewicz system of continuous-valued logic. Of course, various implication operators have been defined. We show only a selection of them in the following.

(a) Gaines-Rescher implication operator (I_{GR}) :

$$(I_{GR})(x, y) = \begin{cases} 1 & \text{if } x \leq y, \\ 0 & \text{otherwise.} \end{cases}$$

(b) Gödel implication operator (I_G) :

$$(I_G)(x, y) = \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{otherwise.} \end{cases}$$

(c) The contraposition of Gödel implication operator (I_{cG}) :

$$(I_{cG})(x, y) = \begin{cases} 1 & \text{if } x \leq y, \\ 1 - x & \text{otherwise.} \end{cases}$$

Ying [24] introduced the concept of fuzzifying topology. We can expand his/her idea to ordered semigroups, and we define an interval valued fuzzifying bi-ideal as follows.

Definition 4.1. An interval valued fuzzy subset $\widetilde{\lambda}$ of S is called an interval valued fuzzifying bi-ideal of S if it satisfies the following conditions:

(i) $(\forall x, y \in S)\left(x \leq y \Rightarrow \left(\models_{\widetilde{t}} \left[x \in \widetilde{\lambda}\right] \rightarrow \left[y \in \widetilde{\lambda}\right]\right)\right)$

(ii) $(\forall x, y \in S)\left(\models r\min\left\{\left[x \in \widetilde{\lambda}\right] \rightarrow \left[y \in \widetilde{\lambda}\right]\right\} \rightarrow \left[xy \in \widetilde{\lambda}\right]\right)$

(iii) $(\forall x, y, z \in S)\left(\models r\min\left\{\left[x \in \widetilde{\lambda}\right] \rightarrow \left[z \in \widetilde{\lambda}\right]\right\} \rightarrow \left[xyz \in \widetilde{\lambda}\right]\right)$

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In [11], the concept of $\widetilde{t}$-tautology is introduced, i.e., for all valuations:

$$\models \widetilde{\Phi} \text{ if and only if } \left[\widetilde{\Phi}\right] \geq \widetilde{t} \qquad (6)$$

Definition 4.2. An interval valued fuzzy subset $\widetilde{\lambda}$ of S and $\widetilde{t} \in D(0,1]$ is called a $\widetilde{t}$-implication-based interval valued fuzzy bi-ideal of S if it satisfies:

(iv) $(\forall x, y \in S)\left(x \leq y \Rightarrow \left(\models_{\widetilde{t}} \left[x \in \widetilde{\lambda}\right] \rightarrow \left[y \in \widetilde{\lambda}\right]\right)\right)$

(v) $(\forall x, y \in S)\left(\models_{\widetilde{t}} r\min\left\{\left[x \in \widetilde{\lambda}\right] \rightarrow \left[y \in \widetilde{\lambda}\right]\right\} \rightarrow \left[xy \in \widetilde{\lambda}\right]\right)$

(vi) $(\forall x, y, z \in S)\left(\models_{\widetilde{t}} r\min\left\{\left[x \in \widetilde{\lambda}\right] \rightarrow \left[z \in \widetilde{\lambda}\right]\right\} \rightarrow \left[xyz \in \widetilde{\lambda}\right]\right)$

Suppose I is an implication operator. Then $\widetilde{\lambda}$ is a $\widetilde{t}$-implication-based interval valued fuzzy bi-ideal of S if and only if the following conditions are satisfied for all $x, y, z \in S$.

(i) $x \leq y \Rightarrow I\left(\widetilde{\lambda}(x), \widetilde{\lambda}(y)\right) \geq \widetilde{t}$

(ii) $I\left(r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y), \widetilde{\lambda}(xy)\right\}\right) \geq \widetilde{t}$

(iii) $I\left(r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y), \widetilde{\lambda}(xyz)\right\}\right) \geq \widetilde{t}$

In the following, we characterize ordered semigroups by the properties of $\left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$-implication-based interval valued bi-ideals.

Theorem 4.3. *For any interval valued fuzzy subset $\widetilde{\lambda}$ of S, we have,*

(i) If $I = I_{GR}$, then $\widetilde{\lambda}$ is a $\frac{\delta-k}{2}$-implication-based interval fuzzy bi-ideal of S if and only if $\widetilde{\lambda}$ is an interval valued $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-fuzzy bi-ideal of S.

(ii) If $I = I_G$, then $\widetilde{\lambda}$ is a $\frac{\delta-k}{2}$-implication-based interval valued fuzzy bi-ideal of S if and only if $\widetilde{\lambda}$ is an interval valued $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-fuzzy bi-ideal of S.

(iii) If $I = I_{cG}$, then $\widetilde{\lambda}$ is a $\frac{\delta-k}{2}$-implication-based interval valued fuzzy bi-ideal of S if and only if $\widetilde{\lambda}$ satisfies the following conditions:

$(iiia)$ $(\forall x, y \in S)\left(x \leq y \Rightarrow r\max - \left\{\widetilde{\lambda}(y), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} \geq r\min\left\{\widetilde{\lambda}(y), \widetilde{\delta}\right\}\right)$

$(iiib)$ $r\max\left\{\widetilde{\lambda}(xy), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} \geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y), \widetilde{\delta}\right\}$

$(iiic)$ $r\max\left\{\widetilde{\lambda}(xyz), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} \geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z), \widetilde{\delta}\right\}$

for all, $x, y, z \in S$.

Proof. (i) : The proof of (i) is straightforward.

(ii) : Suppose that $\widetilde{\lambda}$ is a $\frac{\delta-k}{2}$- implication based interval valued fuzzy bi-ideal of S. Then,

(a) : $(\forall x, y \in S)\left(x \leq y \Rightarrow I_G\left(\widetilde{\lambda}(x), \widetilde{\lambda}(y)\right) \geq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right)$

(b) : $(\forall x, y \in S)\left(I_G\left(r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y)\right\}, \widetilde{\lambda}(xy)\right) \geq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right)$

(c) : $(\forall x, y, z \in S)\left(I_G\left(r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z)\right\}, \widetilde{\lambda}(xyz)\right) \geq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right)$

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Let $x, y \in S$ be such that $x \leq y$. Using (a) we have, $\widetilde{\lambda}(y) \geq \widetilde{\lambda}(x)$ or $\widetilde{\lambda}(x) > \widetilde{\lambda}(y) \geq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$, Hence, $\widetilde{\lambda}(x) \geq r\min\left\{\widetilde{\lambda}(y), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\}$. From (b) we have $\widetilde{\lambda}(xy) \geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y)\right\}$ or $r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y)\right\} > \widetilde{\lambda}(xy) \geq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$. Hence, $\widetilde{\lambda}(xy) \geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\}$. Therefore, $\widetilde{\lambda}(xy) > r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\}$. Now from (c), we have $\widetilde{\lambda}(xyz) \geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z)\right\}$ or $r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z)\right\} > \widetilde{\lambda}(xyz) \geq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$. Hence, $\widetilde{\lambda}(xyz) \geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\}$. Thus,

$$\widetilde{\lambda}(xyz) > r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\}.$$

Therefore, by Theorem 3.5, it follows that $\widetilde{\lambda}$ is an interval valued $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-fuzzy bi-ideal of S.

Conversely, assume that $\widetilde{\lambda}$ is an interval valued $\left(\in, \in \vee q_{\widetilde{k}}^{\widetilde{\delta}}\right)$-fuzzy bi-ideal of S. Let, $x, y \in S$ be such that $x \leq y$. Thus, by Theorem 3.5(i), it follows that

$$I_G\left(\widetilde{\lambda}(x), \widetilde{\lambda}(y)\right) = \begin{cases} \widetilde{\delta} \geq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right] \\ \text{if } r\min\left\{\widetilde{\lambda}(x), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} = \widetilde{\lambda}(x), \\ \widetilde{\lambda}(y) \geq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right] \\ \text{if } r\min\left\{\widetilde{\lambda}(x), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} = \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right] \end{cases}$$

From Theorem 3.5(ii), if
$r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} = r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y)\right\}$,
then $\widetilde{\lambda}(xy) \geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y)\right\}$.
Also, $I_G\left(r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y)\right\}, \widetilde{\lambda}(xy)\right) = \widetilde{\delta} \geq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$.
If $r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} = \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$, then $\widetilde{\lambda}(xy) \geq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$ and hence, $I_G\left(r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y)\right\}, \widetilde{\lambda}(xy)\right) \geq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$.

Now from Theorem 3.5(iii), if,

$$r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} = r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z)\right\},$$

then $\widetilde{\lambda}(xyz) \geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z)\right\}$. Also, $I_G\left(r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z)\right\}, \widetilde{\lambda}(xyz)\right) = \widetilde{\delta} \geq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$. If $r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} = \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$, then $\widetilde{\lambda}(xyz) \geq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$ and hence, $I_G\left(r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z)\right\}, \widetilde{\lambda}(xyz)\right) \geq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$. Therefore $\widetilde{\lambda}$ is a $\frac{\widetilde{\delta} - \widetilde{k}}{2}$-implication-based interval valued fuzzy bi-ideal of S

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(iii) : Assume that $\widetilde{\lambda}$ satisfies $(iiia)$ $(iiib)$ and $(iiic)$. Let $x, y \in S$ be such that $x \leq y$. In $(iiia)$, if $r\min\left\{\widetilde{\lambda}(x), \widetilde{\delta}\right\} = \widetilde{\delta}$, then $r\max\left\{\widetilde{\lambda}(y), \left[\frac{\delta^{-}-k^{-}}{2}, \frac{\delta^{+}-k^{+}}{2}\right]\right\} = \widetilde{\delta}$ and so $\widetilde{\lambda}(x) = \widetilde{\delta} \geq \widetilde{\lambda}(y)$. Consequently, $I_{cG}\left(\widetilde{\lambda}(x), \widetilde{\lambda}(y)\right) = \widetilde{\delta} \geq \left[\frac{\delta^{-}-k^{-}}{2}, \frac{\delta^{+}-k^{+}}{2}\right]$.
If $\widetilde{\lambda}(y) < \widetilde{\delta}$, then

$$r\max\left\{\widetilde{\lambda}(x), \left[\frac{\delta^{-}-k^{-}}{2}, \frac{\delta^{+}-k^{+}}{2}\right]\right\} \geq \widetilde{\lambda}(y) \qquad (a)$$

Now, if $\widetilde{\lambda}(x) > \left[\frac{\delta^{-}-k^{-}}{2}, \frac{\delta^{+}-k^{+}}{2}\right]$, in (a) then $\widetilde{\lambda}(x) \geq \widetilde{\lambda}(y)$ and so $I_{cG}\left(\widetilde{\lambda}(x), \widetilde{\lambda}(y)\right) = \widetilde{\delta} \geq \left[\frac{\delta^{-}-k^{-}}{2}, \frac{\delta^{+}-k^{+}}{2}\right]$. If $\widetilde{\lambda}(x) \leq \left[\frac{\delta^{-}-k^{-}}{2}, \frac{\delta^{+}-k^{+}}{2}\right]$, in (a) then $\widetilde{\lambda}(y) \leq \left[\frac{\delta^{-}-k^{-}}{2}, \frac{\delta^{+}-k^{+}}{2}\right]$.
Hence,

$$I_{cG}\left(\widetilde{\lambda}(x), \widetilde{\lambda}(y)\right) = \begin{cases} \widetilde{\delta} \geq \left[\frac{\delta^{-}-k^{-}}{2}, \frac{\delta^{+}-k^{+}}{2}\right] & \text{if } \widetilde{\lambda}(x) \geq \widetilde{\lambda}(y) \\ \widetilde{\delta} - \widetilde{\lambda}(y) \geq \left[\frac{\delta^{-}-k^{-}}{2}, \frac{\delta^{+}-k^{+}}{2}\right] & \text{otherwise.} \end{cases}$$

In $(iiib)$, if $r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y), \widetilde{\delta}\right\} = \widetilde{\delta}$, then $r\max\left\{\widetilde{\lambda}(xy), \left[\frac{\delta^{-}-k^{-}}{2}, \frac{\delta^{+}-k^{+}}{2}\right]\right\} = \widetilde{\delta}$ and hence, $\widetilde{\lambda}(xy) = \widetilde{\delta} \geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y)\right\}$. Therefore,

$$I_{cG}\left(r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y)\right\}, \widetilde{\lambda}(xy)\right) = \widetilde{\delta} \geq \left[\frac{\delta^{-}-k^{-}}{2}, \frac{\delta^{+}-k^{+}}{2}\right].$$

If $r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y), \widetilde{\delta}\right\} = r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y)\right\}$, then

$$r\max\left\{\widetilde{\lambda}(xy), \left[\frac{\delta^{-}-k^{-}}{2}, \frac{\delta^{+}-k^{+}}{2}\right]\right\} \geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y)\right\}. \qquad (b)$$

Hence, if $r\max\left\{\widetilde{\lambda}(xy), \left[\frac{\delta^{-}-k^{-}}{2}, \frac{\delta^{+}-k^{+}}{2}\right]\right\} = \left[\frac{\delta^{-}-k^{-}}{2}, \frac{\delta^{+}-k^{+}}{2}\right]$, in (b) then $\widetilde{\lambda}(xy) \leq \left[\frac{\delta^{-}-k^{-}}{2}, \frac{\delta^{+}-k^{+}}{2}\right]$ and $r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y)\right\} \leq \left[\frac{\delta^{-}-k^{-}}{2}, \frac{\delta^{+}-k^{+}}{2}\right]$. Therefore,

$$I_{cG}\left(r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y)\right\}, \widetilde{\lambda}(xy)\right) = \widetilde{\delta} \geq \left[\frac{\delta^{-}-k^{-}}{2}, \frac{\delta^{+}-k^{+}}{2}\right],$$

whenever $\widetilde{\lambda}(xy) \geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y)\right\}$ and $I_{cG}\left(r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y)\right\}, \widetilde{\lambda}(xy)\right) = \widetilde{\delta} - r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y)\right\} \geq \left[\frac{\delta^{-}-k^{-}}{2}, \frac{\delta^{+}-k^{+}}{2}\right]$, whenever $\widetilde{\lambda}(xy) < r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y)\right\}$.
Now, if $r\max\left\{\widetilde{\lambda}(xy), \left[\frac{\delta^{-}-k^{-}}{2}, \frac{\delta^{+}-k^{+}}{2}\right]\right\} = \widetilde{\lambda}(xy)$, in (b) then $\widetilde{\lambda}(xy) \geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y)\right\}$ and so $I_{cG}\left(r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y)\right\}, \widetilde{\lambda}(xy)\right) = \widetilde{\delta} \geq \left[\frac{\delta^{-}-k^{-}}{2}, \frac{\delta^{+}-k^{+}}{2}\right]$.

Now in $(iiic)$, if $r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z), \widetilde{\delta}\right\} = \widetilde{\delta}$, then $r\max\left\{\widetilde{\lambda}(xyz), \left[\frac{\delta^{-}-k^{-}}{2}, \frac{\delta^{+}-k^{+}}{2}\right]\right\} = \widetilde{\delta}$ and hence, $\widetilde{\lambda}(xyz) = \widetilde{\delta} \geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y)\right\}$. Therefore,

$$I_{cG}\left(r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z)\right\}, \widetilde{\lambda}(xyz)\right) = \widetilde{\delta} \geq \left[\frac{\delta^{-}-k^{-}}{2}, \frac{\delta^{+}-k^{+}}{2}\right].$$

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If $r\min\left\{\widetilde{\lambda}(x),\widetilde{\lambda}(z),\widetilde{\delta}\right\} = r\min\left\{\widetilde{\lambda}(x),\widetilde{\lambda}(z)\right\}$, then

$$r\max\left\{\widetilde{\lambda}(xyz),\left[\frac{\delta^- - k^-}{2},\frac{\delta^+ - k^+}{2}\right]\right\} \geq r\min\left\{\widetilde{\lambda}(x),\widetilde{\lambda}(z)\right\}. \qquad (c)$$

Hence, if $r\max\left\{\widetilde{\lambda}(xyz),\left[\frac{\delta^- - k^-}{2},\frac{\delta^+ - k^+}{2}\right]\right\} = \left[\frac{\delta^- - k^-}{2},\frac{\delta^+ - k^+}{2}\right]$, in (c) then $\widetilde{\lambda}(xyz) \leq \left[\frac{\delta^- - k^-}{2},\frac{\delta^+ - k^+}{2}\right]$ and $r\min\left\{\widetilde{\lambda}(x),\widetilde{\lambda}(z)\right\} \leq \left[\frac{\delta^- - k^-}{2},\frac{\delta^+ - k^+}{2}\right]$. Therefore,

$$I_{cG}\left(r\min\left\{\widetilde{\lambda}(x),\widetilde{\lambda}(z)\right\},\widetilde{\lambda}(xyz)\right) = \widetilde{\delta} \geq \left[\frac{\delta^- - k^-}{2},\frac{\delta^+ - k^+}{2}\right],$$

whenever $\widetilde{\lambda}(xyz) \geq r\min\left\{\widetilde{\lambda}(x),\widetilde{\lambda}(z)\right\}$ and $I_{cG}\left(r\min\left\{\widetilde{\lambda}(x),\widetilde{\lambda}(z)\right\},\widetilde{\lambda}(xyz)\right) = \widetilde{\delta} - r\min\left\{\widetilde{\lambda}(x),\widetilde{\lambda}(z)\right\} \geq \left[\frac{\delta^- - k^-}{2},\frac{\delta^+ - k^+}{2}\right]$, whenever $\widetilde{\lambda}(xyz) < r\min\left\{\widetilde{\lambda}(x),\widetilde{\lambda}(z)\right\}$. Now, if $r\max\left\{\widetilde{\lambda}(xyz),\left[\frac{\delta^- - k^-}{2},\frac{\delta^+ - k^+}{2}\right]\right\} = \widetilde{\lambda}(xyz)$, in (c) then $\widetilde{\lambda}(xyz) \geq r\min\left\{\widetilde{\lambda}(x),\widetilde{\lambda}(z)\right\}$ and so $I_{cG}\left(r\min\left\{\widetilde{\lambda}(x),\widetilde{\lambda}(z)\right\},\widetilde{\lambda}(xyz)\right) = \widetilde{\delta} \geq \left[\frac{\delta^- - k^-}{2},\frac{\delta^+ - k^+}{2}\right]$. Therefore, $\widetilde{\lambda}$ is a $\frac{\widetilde{\delta}-\widetilde{k}}{2}$-implication-based interval valued fuzzy bi-ideal of S.

Conversely, suppose that $\widetilde{\lambda}$ is a $\frac{\widetilde{\delta}-\widetilde{k}}{2}$-implication-based interval valued fuzzy bi-ideal of S. Then

(a) $(\forall x,y\in S)\left(x\leq y \Rightarrow I_G\left(\widetilde{\lambda}(x),\widetilde{\lambda}(y)\right) \geq \left[\frac{\delta^- - k^-}{2},\frac{\delta^+ - k^+}{2}\right]\right)$,

(b) $(\forall x,y\in S)\left(I_G\left(r\min\left\{\widetilde{\lambda}(x),\widetilde{\lambda}(y)\right\},\widetilde{\lambda}(xy)\right) \geq \left[\frac{\delta^- - k^-}{2},\frac{\delta^+ - k^+}{2}\right]\right)$,

(c) $(\forall x,y,z\in S)\left(I_G\left(r\min\left\{\widetilde{\lambda}(x),\widetilde{\lambda}(z)\right\},\widetilde{\lambda}(xyz)\right) \geq \left[\frac{\delta^- - k^-}{2},\frac{\delta^+ - k^+}{2}\right]\right)$,

Let $x,y\in S$ be such that $x\leq y$. Then, from (a) it follows that $I_G\left(\widetilde{\lambda}(x),\widetilde{\lambda}(y)\right) = \widetilde{\delta}$ or $\widetilde{\delta} - \widetilde{\lambda}(y) \geq \left[\frac{\delta^- - k^-}{2},\frac{\delta^+ - k^+}{2}\right]$ so that $\widetilde{\lambda}(y) \leq \widetilde{\lambda}(x)$ or $\widetilde{\lambda}(y) \leq \left[\frac{\delta^- - k^-}{2},\frac{\delta^+ - k^+}{2}\right]$. Therefore,

$$r\max\left\{\widetilde{\lambda}(x),\left[\frac{\delta^- - k^-}{2},\frac{\delta^+ - k^+}{2}\right]\right\} \geq \widetilde{\lambda}(y) = r\min\left\{\widetilde{\lambda}(y),\widetilde{\delta}\right\}.$$

From (b), for all $x,y\in S$, it follows that $I_G\left(r\min\left\{\widetilde{\lambda}(x),\widetilde{\lambda}(y)\right\},\widetilde{\lambda}(xy)\right) = \widetilde{\delta}$, that is $r\min\left\{\widetilde{\lambda}(x),\widetilde{\lambda}(y)\right\} \leq \widetilde{\lambda}(xy)$, or $\widetilde{\delta} - r\min\left\{\widetilde{\lambda}(x),\widetilde{\lambda}(y)\right\} \geq \left[\frac{\delta^- - k^-}{2},\frac{\delta^+ - k^+}{2}\right]$. Hence,

$$\begin{aligned} r\max\left\{\widetilde{\lambda}(y),\left[\frac{\delta^- - k^-}{2},\frac{\delta^+ - k^+}{2}\right]\right\} &\geq r\min\left\{\widetilde{\lambda}(x),\widetilde{\lambda}(y)\right\} \\ &= r\min\left\{\widetilde{\lambda}(x),\widetilde{\lambda}(y),\widetilde{\delta}\right\}. \end{aligned}$$

Therefore, $r\max\left\{\widetilde{\lambda}(xy),\left[\frac{\delta^- - k^-}{2},\frac{\delta^+ - k^+}{2}\right]\right\} \geq r\min\left\{\widetilde{\lambda}(x),\widetilde{\lambda}(y)\right\} = r\min\left\{\widetilde{\lambda}(x),\widetilde{\lambda}(y),\widetilde{\delta}\right\}$ for all $x,y\in S$.

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Also from (c), for all $x, y, z \in S$, it follows that $I_G\left(r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z)\right\}, \widetilde{\lambda}(xyz)\right) = \widetilde{\delta}$, that is, $r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z)\right\} \leq \widetilde{\lambda}(xyz)$, or $\widetilde{\delta} - r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z)\right\} \geq \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]$. Hence,

$$\begin{aligned} r\max\left\{\widetilde{\lambda}(z), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} &\geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z)\right\} \\ &= r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z), \widetilde{\delta}\right\}. \end{aligned}$$

Therefore,

$$\begin{aligned} r\max\left\{\widetilde{\lambda}(xyz), \left[\frac{\delta^- - k^-}{2}, \frac{\delta^+ - k^+}{2}\right]\right\} &\geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z)\right\} \\ &= r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z), \widetilde{\delta}\right\} \end{aligned}$$

$x, y, z \in S$. □

If we take $\widetilde{\delta} = [1, 1]$, in Theorem 4.3, then we have the following corollary.

Corollary 4.4. *For any interval valued fuzzy subset $\widetilde{\lambda}$ of S, we have,*

(i) If $I = I_{GR}$, then $\widetilde{\lambda}$ is a $\frac{1-k}{2}$-implication-based interval fuzzy bi-ideal of S if and only if $\widetilde{\lambda}$ is an interval valued $(\in, \in \vee q_{\widetilde{k}})$-fuzzy bi-ideal of S.

(ii) If $I = I_G$, then $\widetilde{\lambda}$ is a $\frac{1-k}{2}$-implication-based interval valued fuzzy bi-ideal of S if and only if $\widetilde{\lambda}$ is an interval valued $(\in, \in \vee q_{\widetilde{k}})$-fuzzy bi-ideal of S.

(iii) If $I = I_{cG}$, then $\widetilde{\lambda}$ is a $\frac{1-k}{2}$-implication-based interval valued fuzzy bi-ideal of S if and only if $\widetilde{\lambda}$ satisfies the following conditions:

(iiia) $(\forall x, y \in S)\left(x \leq y \Rightarrow r\max - \left\{\widetilde{\lambda}(y), \left[\frac{1-k^-}{2}, \frac{1-k^+}{2}\right]\right\} \geq r\min\left\{\widetilde{\lambda}(y), [1, 1]\right\}\right)$

(iiib) $r\max\left\{\widetilde{\lambda}(xy), \left[\frac{1-k^-}{2}, \frac{1-k^+}{2}\right]\right\} \geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y), [1, 1]\right\}$

(iiic) $r\max\left\{\widetilde{\lambda}(xyz), \left[\frac{1-k^-}{2}, \frac{1-k^+}{2}\right]\right\} \geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z), [1, 1]\right\}$

for all, $x, y, z \in S$.

If we take $\widetilde{\delta} = [1, 1]$ and $\widetilde{k} = [0, 0]$ in Theorem 4.3, then we have the following corollary.

Corollary 4.5. *For any interval valued fuzzy subset $\widetilde{\lambda}$ of S, we have,*

(i) If $I = I_{GR}$, then $\widetilde{\lambda}$ is a $\frac{1}{2}$-implication-based interval fuzzy bi-ideal of S if and only if $\widetilde{\lambda}$ is an interval valued $(\in, \in \vee q)$-fuzzy bi-ideal of S.

(ii) If $I = I_G$, then $\widetilde{\lambda}$ is a $\frac{1}{2}$-implication-based interval valued fuzzy bi-ideal of S if and only if $\widetilde{\lambda}$ is an interval valued $(\in, \in \vee q)$-fuzzy bi-ideal of S.

(iii) If $I = I_{cG}$, then $\widetilde{\lambda}$ is a $\frac{1}{2}$-implication-based interval valued fuzzy bi-ideal of S if and only if $\widetilde{\lambda}$ satisfies the following conditions:

(iiia) $(\forall x, y \in S)\left(x \leq y \Rightarrow r\max - \left\{\widetilde{\lambda}(y), \left[\frac{1}{2}, \frac{1}{2}\right]\right\} \geq r\min\left\{\widetilde{\lambda}(y), [1, 1]\right\}\right)$

(iiib) $r\max\left\{\widetilde{\lambda}(xy), \left[\frac{1-k^-}{2}, \frac{1-k^+}{2}\right]\right\} \geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y), [1, 1]\right\}$

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$(iiic)$ $r\max\left\{\widetilde{\lambda}(xyz), \left[\frac{1}{2}, \frac{1}{2}\right]\right\} \geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y), [1,1]\right\}$
for all, $x, y, z \in S$.

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