

## A Note on Absolute Central Automorphisms of Finite $p$ -Groups

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**ABSTRACT.** Let  $G$  be a finite group. The automorphism  $\sigma$  of a group  $G$  is said to be an absolute central automorphism, if for all  $x \in G$ ,  $x^{-1}x^\sigma \in L(G)$ , where  $L(G)$  be the absolute centre of  $G$ . In this paper, we study some properties of absolute central automorphisms of a given finite  $p$ -group.

**Keywords:** Absolute centre, Absolute central automorphisms, Finite  $p$ -groups.

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### 1. INTRODUCTION

Let  $G$  be a finite group and  $N$  a characteristic subgroup of  $G$ . Suppose  $\sigma$  is an automorphism of  $G$ . If  $(Ng)^\sigma = Ng$  for all  $g$  in  $G$  or equivalently  $\sigma$  induces the identity automorphism on  $G/N$ , we shall say  $\sigma$  centralizes  $G/N$ . We let  $\text{Aut}^N(G)$  denote the group of all automorphisms of  $G$  centralizing  $G/N$ . Clearly  $\sigma \in \text{Aut}^N(G)$  if and only if  $x^{-1}x^\sigma \in N$  for all  $x \in G$ . Now let  $M$  be a normal subgroup of  $G$ . Let us denote by  $C_{\text{Aut}^N(G)}(M)$  the group of all automorphisms of  $\text{Aut}^N(G)$  centralizing  $M$ . Various authors have studied the groups  $\text{Aut}^Z(G)$ , the central automorphisms of  $G$ , where  $Z = Z(G)$ ,  $\text{Aut}^{G'}(G)$ , the IA-automorphisms of  $G$ , where  $G'$  stands for the commutator subgroup of  $G$ , and  $\text{Aut}^\Phi(G)$ , where  $\Phi$  denote the Frattini subgroup of  $G$ , the intersection of all maximal subgroups of  $G$ , see for example [14, 17, 19, 20]. For any

element  $g \in G$  and  $\sigma \in \text{Aut}(G)$ , the element  $[g, \sigma] = g^{-1}g^\sigma$  is called the autocommutator of  $g$  and  $\sigma$ . Also inductively, for all  $\sigma_1, \sigma_2, \dots, \sigma_n \in \text{Aut}(G)$ , define  $[g, \sigma_1, \sigma_2, \dots, \sigma_n] = [[g, \sigma_1, \sigma_2, \dots, \sigma_{n-1}], \sigma_n]$ . Hegarty [7], generalized the concept of centre into absolute centre  $L(G)$  of a group  $G$  as

$$L(G) = \{g \in G \mid [g, \sigma] = 1, \forall \sigma \in \text{Aut}(G)\}.$$

One can easily check that the absolute centre is a characteristic subgroup contained in the centre of  $G$ . Also he introduced the concept of the absolute central automorphism. An automorphism  $\sigma$  of  $G$  is called an absolute central automorphism if  $\sigma$  centralizes  $G/L(G)$ . We denote the set of all absolute central automorphisms of  $G$  by  $\text{Aut}^L(G)$ . Singh and Gumber [18], Kaboutari Farimani [9], also Shabani-Attar [17] have given some necessary and sufficient conditions for a finite non-abelian  $p$ -group such that all absolute central automorphisms are inner. In this paper, we will characterize the finite non-abelian  $p$ -groups  $G$  such that  $\text{Aut}^L(G) = \text{Aut}^{G'}(G)$ . Then, we determine the finite non-abelian  $p$ -groups  $G$  with cyclic Frattini subgroup for which  $\text{Aut}^L(G) = \text{Aut}^\Phi(G)$ . Finally, we classify all finite  $p$ -groups  $G$  of order  $p^n$  ( $3 \leq n \leq 5$ ), such that  $\text{Aut}^L(G) = \text{Inn}(G)$ .

Throughout this paper all groups are assumed to be finite and  $p$  always denotes a prime number. Most of our notation is standard, and can be found in [5], for example. In particular, a  $p$ -group  $G$  is said to be extraspecial if  $G' = Z(G) = \Phi(G)$  is of order  $p$ . Let  $L_1(G) = L(G)$  and for  $n \geq 2$ , define  $L_n(G)$  inductively as

$$L_n(G) = \{g \in G \mid [g, \sigma_1, \sigma_2, \dots, \sigma_n] = 1, \forall \sigma_1, \sigma_2, \dots, \sigma_n \in \text{Aut}(G)\}.$$

A group  $G$  is called autonilpotent of class at most  $n$  if  $L_n(G) = G$ , for some  $n \in \mathbb{N}$ . If  $\sigma$  is an automorphism of  $G$  and  $x$  is an element of  $G$ , we write  $x^\sigma$  for the image of  $x$  under  $\sigma$  and  $o(x)$  for the order of  $x$ . For a finite group  $G$ ,  $\exp(G)$ ,  $d(G)$  and  $\text{cl}(G)$ , denote the exponent of  $G$ , minimal number of generators of  $G$  and the nilpotency class of  $G$ , respectively. Recall that a group  $G$  is called a central product of its subgroups  $G_1, \dots, G_n$  if  $G = G_1 \cdots G_n$  and  $[G_i, G_j] = 1$ , for all  $1 \leq i < j \leq n$ . In this situation, we shall write  $G = G_1 * \cdots * G_n$ . For  $s \geq 1$ , we use the notation  $G^{*s}$  for the iterated central product defined by  $G^{*s} = G * G^{*(s-1)}$  with  $G^{*1} = G$ , where  $G$  is a finite  $p$ -group. We also make the convention  $G^{*0} = 1$ . Finally, we use  $X^n$  for the direct product of  $n$ -copies of a group  $X$ ,  $C_n$  for the cyclic group of order  $n$  where  $n \geq 1$ , as usual,  $D_8$  for the dihedral group,  $Q_8$  for the quaternion group, of order 8, respectively and  $M_p(n, m)$  and  $M_p(n, m, 1)$  for the minimal non-abelian  $p$ -groups of order  $p^{n+m}$  and  $p^{n+m+1}$  defined respectively by

$$\langle a, b \mid a^{p^n} = b^{p^m} = 1, a^b = a^{1+p^{n-1}} \rangle,$$

where  $n \geq 2$ ,  $m \geq 1$  and

$$\langle a, b, c \mid a^{p^n} = b^{p^m} = c^p = 1, [a, b] = c, [a, c] = [b, c] = 1 \rangle,$$

where  $n \geq m \geq 1$  and if  $p = 2$ , then  $m + n > 2$ .

## 2. PRELIMINARY RESULTS

In this section we give some results which will be used in the rest of the paper.

Let  $G$  and  $H$  be any two groups. We denote by  $\text{Hom}(G, H)$  the set of all homomorphisms from  $G$  into  $H$ . Clearly, if  $H$  is an abelian group, then  $\text{Hom}(G, H)$  forms an abelian group under the following operation  $(fg)(x) = f(x)g(x)$ , for all  $f, g \in \text{Hom}(G, H)$  and  $x \in G$ .

The following lemma is a well-known.

**Lemma 2.1.** *Let  $A, B$  and  $C$  be finite abelian groups. Then*

- (i)  $\text{Hom}(A \times B, C) \cong \text{Hom}(A, C) \times \text{Hom}(B, C)$ ;
- (ii)  $\text{Hom}(A, B \times C) \cong \text{Hom}(A, B) \times \text{Hom}(A, C)$ ;
- (iii)  $\text{Hom}(C_m, C_n) \cong C_e$ , where  $e$  is the greatest common divisor of  $m$  and  $n$ .

We have the following theorem due to Müller [14].

**Theorem 2.2.** [14, Theorem] *If  $G$  is a finite  $p$ -group which is neither elementary abelian nor extraspecial, then  $\text{Aut}^{\Phi}(G)/\text{Inn}(G)$  is a non-trivial normal  $p$ -subgroup of the group of outer automorphisms of  $G$ .*

The following preliminary lemma is well-known result [19, Lemma 2.2].

**Lemma 2.3.** *Let  $G$  be a group and  $M, N$  be normal subgroups of  $G$  with  $N \leq M$  and  $C_N(M) \leq Z(G)$ . Then  $C_{\text{Aut}^N(G)}(M) \cong \text{Hom}(G/M, C_N(M))$ .*

**Corollary 2.4.** *If  $G$  is a finite group, then*

$$C_{\text{Aut}^L(G)}(Z(G)) \cong \text{Hom}(G/Z(G), L(G)),$$

where  $L = L(G)$ .

Moghaddam and Safa [12], proved that for a finite group  $G$ ,

$$\text{Aut}^L(G) \cong \text{Hom}(G/L(G), L(G)).$$

The following theorem states a useful result for finite  $p$ -groups.

**Theorem 2.5.** *Let  $G$  be a finite  $p$ -group different from  $C_2$ . Then  $\text{Aut}^L(G) \cong \text{Hom}(G, L(G))$ .*

*Proof.* Let  $\theta \in \text{Aut}^L(G)$ . We define the map  $f_\theta : G \rightarrow L(G)$  by  $f_\theta(g) = g^{-1}g^\theta$ . It is easy to see that  $f_\theta$  is a homomorphism, and  $\theta \mapsto f_\theta$  is an injective map from  $\text{Aut}^L(G)$  to  $\text{Hom}(G, L(G))$ . Conversely, assume that  $f \in \text{Hom}(G, L(G))$ . Then we define  $\theta = \theta_f : G \rightarrow G$  by  $g^\theta = gf(g)$ . Since by [11, Corollary 3.7],  $g^{-1}g^\theta \in L(G) \leq \Phi(G)$ , for every element  $g \in G$ , we may write  $G$  as the product of the image of  $\theta$  and the Frattini subgroup of  $G$  and so the image of  $\theta$  must be  $G$  itself. Hence  $\theta$  is an automorphism of  $G$ . Now  $\theta = \theta_f \in \text{Aut}^L(G)$  and  $f_{\theta_f} = f$ . Finally, suppose that  $\alpha, \beta \in \text{Aut}^L(G)$ . Then for any  $x \in G$ ,

$$f_{\alpha\beta}(x) = x^{-1}x^{\alpha\beta} = x^{-1}(xx^{-1}x^\alpha)^\beta = x^{-1}x^\beta x^{-1}x^\alpha = x^{-1}x^\alpha x^{-1}x^\beta,$$

since  $x^{-1}x^\alpha \in L(G)$ . Thus  $f_{\alpha\beta}(x) = f_\alpha(x)f_\beta(x)$  and so  $\theta \mapsto f_\theta$  is a homomorphism, which completes the proof.  $\square$

We next give a necessary and sufficient condition on a finite  $p$ -group  $G$  for the group  $\text{Aut}^L(G)$  to be elementary abelian.

**Corollary 2.6.** *Let  $G$  be a finite  $p$ -group. Then  $\text{Aut}^L(G)$  is elementary abelian if and only if  $\exp(G/G') = p$  or  $\exp(L(G)) = p$ .*

*Proof.* It is straightforward by Lemma 2.1 and Theorem 2.5.  $\square$

### 3. MAIN RESULTS

For a finite abelian  $p$ -group  $G$ ,  $|L(G)| = 1, 2$  by [11, Lemma 4.4] and so  $|\text{Aut}^L(G)| = 1$  or  $\text{Aut}^L(G) \cong C_2^d$ , with  $d = d(G)$ . Thus we may assume that  $G$  is a non-abelian  $p$ -group. In this section, first we characterize the finite non-abelian  $p$ -groups  $G$  such that  $\text{Aut}^L(G) = \text{Aut}^{G'}(G)$ . Then, we determine the finite non-abelian  $p$ -groups  $G$  with cyclic Frattini subgroup for which  $\text{Aut}^L(G) = \text{Aut}^\Phi(G)$ .

In [9], Kaboutari Farimani proved the following two results giving some information of absolute central automorphisms of a finite  $p$ -group.

**Lemma 3.1.** *Let  $G$  be a finite non-abelian  $p$ -group. Then  $C_{\text{Aut}^L(G)}(Z(G)) = \text{Inn}(G)$  if and only if  $G/L(G)$  is abelian and  $L(G)$  is cyclic.*

**Theorem 3.2.** *Let  $G$  be a finite non-abelian  $p$ -group. Then  $\text{Aut}^L(G) = \text{Inn}(G)$  if and only if  $G/L(G)$  is abelian,  $L(G)$  is cyclic and  $Z(G) = L(G)G^{p^n}$  where  $\exp(L(G)) = p^n$ .*

Note that the Theorem 3.2 yields the following corollary that is the Corollary 1 of Singh and Gumber [18].

Let  $G$  be a finite non-abelian  $p$ -group such that  $G' \leq L(G)$ . Let  $G/Z(G) = C_{p^{\alpha_1}} \times C_{p^{\alpha_2}} \times \cdots \times C_{p^{\alpha_r}}$ , where  $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_r \geq 1$ . Also let  $G/L(G) = C_{p^{\beta_1}} \times C_{p^{\beta_2}} \times \cdots \times C_{p^{\beta_s}}$ , where  $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_s \geq 1$  and  $L(G) = C_{p^{\gamma_1}} \times$

$C_{p^{\gamma_2}} \times \cdots \times C_{p^{\gamma_t}}$ , where  $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_t \geq 1$ . Since  $G/Z(G)$  is a quotient group of  $G/L(G)$  by [2, Section 25],  $r \leq s$  and  $\alpha_i \leq \beta_i$  for all  $1 \leq i \leq r$ .

By the above notation, we prove the following corollary:

**Corollary 3.3.** [18, Corollary 1] *Let  $G$  be a finite non-abelian  $p$ -group. Then  $\text{Aut}^L(G) = \text{Inn}(G)$  if and only if  $G' \leq L(G)$ ,  $L(G)$  is cyclic and either  $L(G) = Z(G)$  or  $d(G/L(G)) = d(G/Z(G))$ ,  $\alpha_i = \gamma_1$  for  $1 \leq i \leq k$  and  $\alpha_i = \beta_i$  for  $k+1 \leq i \leq r$ , where  $k$  is the largest integer such that  $\beta_k > \gamma_1$ .*

*Proof.* First assume that  $\text{Aut}^L(G) = \text{Inn}(G)$ . Hence by Theorem 3.2,  $G' \leq L(G)$  and  $L(G)$  is cyclic. If  $\exp(G/L(G)) \leq \exp(L(G))$ , then

$$G/Z(G) \cong \text{Aut}^L(G) \cong \text{Hom}(G/L(G), L(G)) \cong G/L(G),$$

because  $L(G)$  is cyclic and by [12, Proposition 1]. Therefore  $L(G) = Z(G)$ . Next, let  $\exp(G/L(G)) > \exp(L(G))$  and  $k$  is the largest integer such that  $\beta_k > \gamma_1$ . Since  $L(G)$  and  $G/L(G)$  are abelian,

$$d(G/Z(G)) = d(\text{Hom}(G/L(G), L(G))) = d(G/L(G))d(L(G)) = d(G/L(G)).$$

Now we have  $\text{Hom}(G/L(G), L(G)) \cong C_{p^{\gamma_1}} \times C_{p^{\gamma_1}} \times \cdots \times C_{p^{\gamma_1}} \times C_{p^{\beta_{k+1}}} \times \cdots \times C_{p^{\beta_s}}$  and  $\text{Hom}(G/L(G), L(G)) \cong G/Z(G) = C_{p^{\alpha_1}} \times C_{p^{\alpha_2}} \times \cdots \times C_{p^{\alpha_r}}$ . Hence  $\alpha_1 = \alpha_2 = \cdots = \alpha_k = \gamma_1$  and  $\alpha_i = \beta_i$  for  $k+1 \leq i \leq r$ , as required.

Conversely if  $L(G) = Z(G)$ , then  $\exp(G/Z(G)) = \exp(G')|\exp(Z(G))$ , since  $G' \leq L(G)$  and by [13, Lemma 0.4]. Now

$$\text{Hom}(G/L(G), L(G)) = \text{Hom}(G/Z(G), Z(G)) \cong G/Z(G),$$

because  $Z(G)$  is cyclic and so  $\text{Aut}^L(G) = \text{Inn}(G)$ . Next assume that  $L(G) < Z(G)$ ,  $s = d(G/L(G)) = d(G/Z(G)) = r$ ,  $\alpha_i = \gamma_1$  for  $1 \leq i \leq k$  and  $\alpha_i = \beta_i$  for  $k+1 \leq i \leq r$ , where  $k$  is the largest integer such that  $\beta_k > \gamma_1$ . We claim that  $Z(G) = L(G)G^{p^{\gamma_1}}$ . Since  $\exp(G/Z(G)) = \exp(L(G))$ , we have  $L(G) \leq L(G)G^{p^{\gamma_1}} \leq Z(G)$ . It follows that  $G/Z(G)$  is a quotient group of  $G/L(G)G^{p^{\gamma_1}}$ . Now let  $G/L(G)G^{p^{\gamma_1}} = C_{p^{\gamma_1}} \times C_{p^{\delta_2}} \times \cdots \times C_{p^{\delta_r}}$ , where  $\delta_1 = \gamma_1 \geq \delta_2 \geq \cdots \geq \delta_r \geq 1$ , since  $d(G/L(G)) = d(G/L(G)G^{p^{\gamma_1}})$  and  $\exp(G/L(G)G^{p^{\gamma_1}}) = p^{\gamma_1}$ . Therefore  $\gamma_1 = \alpha_i \leq \delta_i \leq \gamma_1$  for  $1 \leq i \leq k$ , whence we have  $\delta_i = \gamma_1 = \alpha_i$  for  $1 \leq i \leq k$ . As  $\beta_i = \alpha_i \leq \delta_i \leq \beta_i$  for  $k+1 \leq i \leq r$ , it follows that  $\delta_i = \alpha_i = \beta_i$  for  $k+1 \leq i \leq r$ . Hence  $G/Z(G) = G/L(G)G^{p^{\gamma_1}}$  and consequently  $Z(G) = L(G)G^{p^{\gamma_1}}$ . Therefore by Theorem 3.2,  $\text{Aut}^L(G) = \text{Inn}(G)$ . This completes the proof.  $\square$

As an application of Theorem 3.2, we get another proof of the main result of [15].

**Theorem 3.4.** [15, Theorem 3.2] *Let  $G$  be a non-abelian autonilpotent finite  $p$ -group of class 2. Then  $\text{Aut}^L(G) = \text{Inn}(G)$  if and only if  $L(G) = Z(G)$  and  $L(G)$  is cyclic.*

*Proof.* Suppose that  $\text{Aut}^L(G) = \text{Inn}(G)$ . Hence  $L(G)$  is cyclic and  $Z(G) = L(G)G^{p^n}$ , where  $\exp(L(G)) = p^n$ . Now by [15, Proposition 2.13],  $\exp(G/L(G))$  divides  $\exp(L(G))$  and so  $Z(G) = L(G)G^{p^n} = L(G)$ . Conversely, assume that  $L(G) = Z(G)$  and  $L(G)$  is cyclic. Since  $G$  be a non-abelian autonilpotent  $p$ -group of class 2,  $\text{Aut}^L(G) = \text{Aut}(G)$ , by [15, Lemma 2.11]. Therefore  $\text{Inn}(G) \leq \text{Aut}^L(G)$ ,  $G' \leq L(G)$  and  $G/L(G)$  is abelian. Obviously,  $Z(G) = L(G) = L(G)G^{p^n}$ , where  $\exp(L(G)) = p^n$ , and so  $\text{Aut}^L(G) = \text{Inn}(G)$ , by Theorem 3.2, as required.  $\square$

**Corollary 3.5.** *Let  $G$  be an extraspecial  $p$ -group.*

- (i) *If  $p > 2$ , then  $L(G)$  and  $\text{Aut}^L(G)$  is trivial.*
- (ii) *If  $p = 2$ , then  $L(G) \cong C_2$  and  $\text{Aut}^L(G) = \text{Inn}(G)$ .*

*Proof.* Let  $G$  be an extraspecial  $p$ -group. First assume that  $p > 2$ . By [10, Theorem 3],  $L(G)$  is trivial and so  $\text{Aut}^L(G) = 1$ .

To prove (ii), since  $|G'| = 2$ , and  $G'$  is a characteristic subgroup of  $G$ , we have  $G' \leq L(G) \leq Z(G)$ . Thus  $G' = L(G) = Z(G) = \Phi(G)$  is cyclic of order 2. Now by Theorem 3.2,  $\text{Aut}^L(G) = \text{Inn}(G)$ .  $\square$

Let  $G$  be a finite non-abelian  $p$ -group such that  $G/L(G)$  is abelian. Then  $G$  is of class 2 and  $\text{Aut}^{G'}(G) \leq \text{Aut}^L(G)$ . Let  $G/G' = C_{p^{a_1}} \times C_{p^{a_2}} \times \cdots \times C_{p^{a_k}}$ , where  $a_1 \geq a_2 \geq \cdots \geq a_k \geq 1$ . Also let  $L(G) = C_{p^{b_1}} \times C_{p^{b_2}} \times \cdots \times C_{p^{b_l}}$ , where  $b_1 \geq b_2 \geq \cdots \geq b_l \geq 1$  and  $G' = C_{p^{e_1}} \times C_{p^{e_2}} \times \cdots \times C_{p^{e_n}}$ , where  $e_1 \geq e_2 \geq \cdots \geq e_n \geq 1$ . Since  $G' \leq L(G)$ , by [2, Section 25] we have  $n \leq l$  and  $e_j \leq b_j$  for all  $1 \leq j \leq n$ . By the above notation, we prove the following theorem:

**Theorem 3.6.** *Let  $G$  be a finite non-abelian  $p$ -group. Then  $\text{Aut}^L(G) = \text{Aut}^{G'}(G)$  if and only if  $G' = L(G)$  or  $G' < L(G)$ ,  $d(G') = d(L(G))$  and  $a_1 = e_t$ , where  $t$  is the largest integer between 1 and  $n$  such that  $b_t > e_t$ .*

*Proof.* Suppose that  $\text{Aut}^L(G) = \text{Aut}^{G'}(G)$  and  $G' \neq L(G)$ . By Theorem 2.5 and Lemma 2.3, we have  $|\text{Hom}(G/G', L(G))| = |\text{Hom}(G/G', G')|$ . First, we claim that  $d(G') = d(L(G))$ . Suppose, for a contradiction, that  $d(G') = n < l = d(L(G))$ . Since  $b_j \geq e_j$  for all  $j$  such that  $1 \leq j \leq n$ , by Lemma 2.1,

$$\begin{aligned} |\text{Aut}^{G'}(G)| &= |\text{Hom}(G/G', G')| = |\text{Hom}(G/G', C_{p^{e_1}} \times C_{p^{e_2}} \times \cdots \times C_{p^{e_n}})| \\ &\leq |\text{Hom}(G/G', C_{p^{b_1}} \times C_{p^{b_2}} \times \cdots \times C_{p^{b_n}})| < |\text{Hom}(G/G', C_{p^{b_1}} \times C_{p^{b_2}} \times \cdots \times C_{p^{b_n}})| \\ &\times |\text{Hom}(G/G', C_{p^{b_{n+1}}} \times \cdots \times C_{p^{b_l}})| = |\text{Hom}(G/G', C_{p^{b_1}} \times C_{p^{b_2}} \times \cdots \times C_{p^{b_l}})| \\ &= |\text{Hom}(G/G', L(G))| = |\text{Aut}^L(G)|, \end{aligned}$$

which is a contradiction. So  $n = l$ , as required. Next, since  $|\text{Aut}^L(G)| = |\text{Aut}^{G'}(G)|$ , we have

$$\prod_{1 \leq i \leq k, 1 \leq j \leq l} p^{\min\{a_i, b_j\}} = \prod_{1 \leq i \leq k, 1 \leq j \leq l} p^{\min\{a_i, e_j\}}.$$

Since  $b_j \geq e_j$  for all  $j$  such that  $1 \leq j \leq l$ , we have  $\min\{a_i, b_j\} \geq \min\{a_i, e_j\}$ , where  $1 \leq i \leq k, 1 \leq j \leq l$ . Thus  $\min\{a_i, b_j\} = \min\{a_i, e_j\}$ , for all  $1 \leq i \leq k, 1 \leq j \leq l$ . Next, since  $G' < L(G)$ , there exists some  $1 \leq j \leq l$  such that  $e_j < b_j$ . Let  $t$  be the largest integer between 1 and  $n$  such that  $e_t < b_t$ . We show that  $a_1 \leq e_t$ . Suppose, on the contrary, that  $a_1 > e_t$ . Then by the above equality, we must have  $\min\{a_1, b_t\} = \min\{a_1, e_t\} = e_t$ , which is impossible. Hence  $a_1 \leq e_t$ . Let  $\exp(G/Z(G)) = p^f$ , where  $f \in \mathbb{N}$ . Since  $\text{cl}(G) = 2$ , by [13, Lemma 0.4],  $f = e_1$ . But  $a_1 \leq e_t \leq e_{t-1} \leq \dots \leq e_1 = f \leq a_1$ . Whence  $a_1 = e_t$ .

Conversely, if  $G' = L(G)$ , then  $\text{Aut}^{G'}(G) = \text{Aut}^L(G)$ . Assume that  $G' < L(G)$ ,  $d(G') = n = d(L(G)) = l$  and  $a_1 = e_t$ , where  $t$  is the largest integer between 1 and  $n$  such that  $b_t > e_t$ . Now by Lemma 2.3,

$$|\text{Aut}^{G'}(G)| = |\text{Hom}(G/G', G')| = \prod_{1 \leq i \leq k, 1 \leq j \leq l} p^{\min\{a_i, e_j\}},$$

and by Theorem 2.5,

$$|\text{Aut}^L(G)| = |\text{Hom}(G/G', L(G))| = \prod_{1 \leq i \leq k, 1 \leq j \leq l} p^{\min\{a_i, b_j\}}.$$

Since  $a_1 = e_t$ , we have  $1 \leq a_k \leq \dots \leq a_2 \leq a_1 = e_t \leq e_{t-1} \leq \dots \leq e_2 \leq e_1$ . Thus  $b_j \geq e_j \geq a_i$  for all  $1 \leq i \leq k$  and  $1 \leq j \leq t$ , which shows that  $\min\{a_i, e_j\} = a_i = \min\{a_i, b_j\}$  for  $1 \leq i \leq k$  and  $1 \leq j \leq t$ . Since  $e_j = b_j$  for all  $j \geq t+1$ , we have  $\min\{a_i, e_j\} = \min\{a_i, b_j\}$  for all  $1 \leq i \leq k$  and  $t+1 \leq j \leq l$ . Thus  $\min\{a_i, e_j\} = \min\{a_i, b_j\}$  for all  $1 \leq i \leq k$  and  $1 \leq j \leq l$ . Therefore  $|\text{Aut}^{G'}(G)| = |\text{Aut}^L(G)|$ . Since  $G' < L(G)$  we have  $\text{Aut}^{G'}(G) = \text{Aut}^L(G)$ , which completes the proof.  $\square$

In [11], Meng and Guo proved that for a finite group  $G$ , if  $C_2$  is not a direct factor of  $G$ , then  $L(G) \leq \Phi(G)$ . We end this section by characterizing the finite non-abelian  $p$ -groups  $G$  with cyclic Frattini subgroup for which  $\text{Aut}^L(G) = \text{Aut}^\Phi(G)$ .

First, we give some basic results about the finite non-abelian  $p$ -groups  $G$  with cyclic Frattini subgroup.

Let  $n > 1$ . Following [1], we denote by  $D_{2^{n+3}}^+$  and  $Q_{2^{n+3}}^+$  the 2-groups of order  $2^{n+3}$  defined by the following presentations.

$$D_{2^{n+3}}^+ = \langle a, b, c \mid a^{2^{n+1}} = b^2 = c^2 = 1, a^b = a^{-1+2^n}, a^c = a^{1+2^n}, [b, c] = 1 \rangle,$$

$Q_{2^{n+3}}^+ = \langle a, b, c \mid a^{2^{n+1}} = b^2 = 1, a^b = a^{-1+2^n}, a^c = a^{1+2^n}, a^{2^n} = c^2, [b, c] = 1 \rangle$ .  
 Note that if  $G$  is either  $D_{2^{n+3}}^+$  or  $Q_{2^{n+3}}^+$ , then  $\text{cl}(G) = n + 1$ .

In [1], Berger, Kovács and Newman proved the following result.

**Theorem 3.7.** [1, Theorem 2] *If  $G$  is a finite  $p$ -group with  $Z(\Phi(G))$  cyclic, then*

$$G = E \times (G_0 * G_1 * \cdots * G_s),$$

where  $E$  is an elementary abelian,  $G_1, \dots, G_s$  are non-abelian of order  $p^3$ , of exponent  $p$  for  $p$  odd and dihedral for  $p = 2$ , while  $G_0 > 1$  if  $E > 1$ ,  $|G_0| > 2$  if  $s > 0$ , and  $G_0$  is one of the following types: cyclic, non-abelian with a cyclic maximal subgroup,  $D_{2^{n+2}} * \mathbb{Z}_4, S_{2^{n+2}} * \mathbb{Z}_4, D_{2^{n+3}}^+, Q_{2^{n+3}}^+, D_{2^{n+3}}^+ * \mathbb{Z}_4$ , all with  $n > 1$ . Conversely, every such group has cyclic Frattini subgroup.

**Theorem 3.8.** [20, Theorem 2.3] *Let  $G$  be a finite non-abelian  $p$ -group with cyclic Frattini subgroup  $\Phi(G)$ .*

- (i) *If  $p > 2$ , or  $p = 2$  and  $\text{cl}(G) = 2$ , then  $\Phi(G) \leq Z(G)$ .*
- (ii) *If  $\text{cl}(G) > 2$ , then  $G' = \Phi(G)$ .*

**Lemma 3.9.** [20, Lemma 2.4] *Let  $G$  be a finite group with  $\Phi(G) \leq Z(G)$ . Then there is a bijection from  $\text{Hom}(G/G', \Phi(G))$  onto  $\text{Aut}^\Phi(G)$  associating to every homomorphism  $f : G \rightarrow \Phi(G)$  the automorphism  $x \mapsto xf(x)$  of  $G$ . In particular, if  $G$  is a  $p$ -group and  $\exp(\Phi(G)) = p$ , then  $\text{Aut}^\Phi(G) \cong \text{Hom}(G/G', \Phi(G))$ .*

In the following theorem, we will make use Theorem 3.7, which is the structural theorem for  $p$ -groups with cyclic Frattini subgroup.

**Theorem 3.10.** *Let  $G$  be a finite non-abelian  $p$ -group with cyclic Frattini subgroup. Then  $\text{Aut}^L(G) = \text{Aut}^\Phi(G)$  if and only if  $G$  is one of the following types:  $C_2^m \times D_8^{*(s+1)}$  or  $C_2^m \times (D_8^{*s} * Q_8)$ , where  $s, m \geq 0$ .*

*Proof.* Let  $\text{Aut}^L(G) = \text{Aut}^\Phi(G)$ . Hence  $\text{Aut}^\Phi(G)$  is abelian,  $G$  is of class 2 and by Theorem 3.8,  $\Phi(G) \leq Z(G)$ . It follows that  $\exp(G') = \exp(G/Z(G)) = p$  and so  $|G'| = p$ . Assume that  $|\Phi(G) : G'| = p^a$ . Then  $\Phi(G) \cong C_{p^{a+1}}$  and we observe that  $\exp(G/G') \leq p^{a+1} = |\Phi(G)|$ . Together with Lemma 3.9, we have  $|\text{Aut}^\Phi(G)| = |\text{Hom}(G, \Phi(G))| = |G|/p$ . Next, we note that  $G' \cap L(G) \neq 1$ ; otherwise,  $G' \cap L(G) = 1$  and  $G' \times L(G)$  would be a subgroup of  $\Phi(G)$ . Hence either  $G' = 1$  or  $L(G) = 1$ , a contradiction. Whence  $G' \leq L(G)$ . Now we are able to show that  $G' = L(G) \cong C_p$ . To do this, first assume that  $L(G) \neq \Phi(G)$ . By similar argument that was applied for Theorem 3.6, we have  $\exp(G/G') \leq \exp(L(G))$ , which implies that  $\exp(G/L(G)) \leq \exp(G/G') \leq \exp(L(G)) = |L(G)|$ . If  $L(G) = \Phi(G)$ , then  $\exp(G/L(G)) = \exp(G/\Phi(G)) \leq \exp(L(G)) = |L(G)|$ . Thus  $|\text{Aut}^L(G)| = |G/L(G)| = |\text{Aut}^\Phi(G)| = |G/G'|$ , by [12, Proposition 1] and so  $G' = L(G) \cong C_p$ . Now, we will make use of the notation of Theorem 3.7.

Since  $\text{cl}(G) = 2$ , by Theorem 3.7 and [5, Theorems 5.4.3 and 5.4.4],  $G_0$  is one of the groups  $M_p(n, 1)$ , where  $n \geq 3$ , if  $p = 2$ ;  $D_8$  or  $Q_8$ .

We claim that  $G' = G'_0$  and  $\Phi(G) = \Phi(G_0)$ . To see this, since  $G'_0 \cap G'_i \neq 1$  for  $1 \leq i \leq s$  and  $|G'_i| = p$ , we have  $G'_i \leq G'_0$  and so  $G' = G'_0$ . Also  $\Phi(G) = G'G^p = G'_0 E^p G_0^p G_1^p \cdots G_s^p = G'_0 G_0^p = \Phi(G_0)$ . To continue the proof, we may consider two cases:

Case I.  $E = 1$ .

Let  $G = G_0 * T$ , where  $T$  be one of the groups  $M_p(1, 1, 1)^{*s}$ , while  $p > 2$  or  $D_8^{*s}$ , where all  $s \geq 0$ . Note that if  $s = 0$ , then  $G = G_0$  and  $Z(G) = Z(G_0) = \Phi(G_0) = \Phi(G)$ ; otherwise, since  $1 \neq G_0 \cap T = Z(T) \leq Z(G_0)$ , then  $Z(G) = Z(G_0)$ , because  $|Z(T)| = p$ , which implies that  $\Phi(G) = \Phi(G_0) = Z(G_0) = Z(G)$ . We claim that  $G$  is an extraspecial  $p$ -group. To see this, since  $G' = L(G) \cong C_p$ , by Theorem 3.2,  $\text{Aut}^\Phi(G) = \text{Aut}^L(G) = \text{Inn}(G)$ . This shows that  $G$  is an extraspecial  $p$ -group, by Theorem 2.2. If  $p > 2$ , then by Corollary 3.5,  $L(G) = 1$ , which is impossible. Whence  $p = 2$ . If  $G_0 \cong M_2(n, 1)$ ,  $n \geq 3$ , then by [5, Theorem 5.4.3],  $Z(G) = \Phi(G)$  is of order  $2^{n-1}$ . This yields that  $n = 2$ , since  $|Z(G)| = 2$ , a contradiction. Therefore  $G_0$  is isomorphic either to  $D_8$  or  $Q_8$ , and  $G$  be one of the groups:  $D_8^{*(s+1)}$  or  $Q_8 * D_8^{*s}$ , for some  $s \geq 0$ .

Case II.  $E \neq 1$ .

In this case  $G_0 > 1$  and  $G = E \times (G_0 * T)$ , where  $T$  be one of the groups lying in Case I.

We claim that  $\text{Aut}^{\Phi(G_0 * T)}(G_0 * T) = \text{Aut}^{L(G_0 * T)}(G_0 * T)$ . Choose a non-trivial element  $\sigma$  of  $\text{Aut}^{\Phi(G_0 * T)}(G_0 * T)$ . Then the map  $\bar{\sigma}$  defined by  $(ef)^{\bar{\sigma}} = e f^\sigma$ , for all  $e \in E, f \in G_0 * T$  denotes an automorphism of  $\text{Aut}^\Phi(G) = \text{Aut}^L(G)$ . Since  $G' \cap L(G_0 * T) \neq 1$ , then  $L(G) \leq L(G_0 * T)$  and so  $\sigma$  is in  $\text{Aut}^{L(G_0 * T)}(G_0 * T)$ . This shows that  $\text{Aut}^{\Phi(G_0 * T)}(G_0 * T) = \text{Aut}^{L(G_0 * T)}(G_0 * T)$ , as required. Next, by a similar argument as mentioned for the previous case,  $G_0$  be one of the groups:  $D_8$  or  $Q_8$ . Therefore  $G$  has one of the following types:  $C_2^m \times D_8^{*(s+1)}$  or  $C_2^m \times (D_8^{*s} * Q_8)$ , where  $s \geq 0, m > 0$ .

Conversely, assume that  $G$  be of the groups in Theorem 3.10. Hence  $G' = L(G) \cong C_2$ . Now the proof is complete, since  $|\text{Aut}^L(G)| = |\text{Aut}^\Phi(G)| = |G|/2$ .  $\square$

#### 4. CLASSIFY ALL FINITE $p$ -GROUPS $G$ OF ORDER $p^n$ ( $3 \leq n \leq 5$ ), SUCH THAT $\text{Aut}^L(G) = \text{Inn}(G)$

Let  $G$  be a non-abelian group of order  $p^3$ . Then by Corollary 3.5,  $\text{Aut}^L(G) = \text{Inn}(G)$  if and only if  $p = 2$ . In the following corollaries, we use Theorems 4.7 and 5.1 of [11] and classify all finite  $p$ -groups  $G$  of order  $p^n$  ( $4 \leq n \leq 5$ ), such that  $\text{Aut}^L(G) = \text{Inn}(G)$ . First we recall the following concept, which was introduced by Hall in [6].

**Definition 4.1.** Two finite groups  $G$  and  $H$  are said to be isoclinic if there exist isomorphisms  $\phi : G/Z(G) \rightarrow H/Z(H)$  and  $\theta : G' \rightarrow H'$  such that, if  $(x_1Z(G))^\phi = y_1Z(H)$  and  $(x_2Z(G))^\phi = y_2Z(H)$ , then  $[x_1, x_2]^\theta = [y_1, y_2]$ . Notice that isoclinism is an equivalence relation among finite groups and the equivalence classes are called isoclinism families.

**Corollary 4.2.** *Let  $G$  be a non-abelian group of order  $p^4$ . Then  $\text{Aut}^L(G) = \text{Inn}(G)$  if and only if  $p = 2$  and  $G$  is one of the following types:  $M_2(3, 1)$  or  $M_2(2, 1, 1)$ .*

*Proof.* Assume that  $|G| = p^4$  and  $\text{Aut}^L(G) = \text{Inn}(G)$ . We claim that  $|Z(G)| = p^2$ . Suppose for a contradiction, that  $|Z(G)| = p$ . We observe that  $G' \leq Z(G) \cong C_p$ , by Theorem 3.2 and so  $G$  is an extraspecial  $p$ -group, a contradiction since the order of  $G$  is not of the form  $p^{2n+1}$ , for some natural number  $n$ . Therefore  $G/Z(G) \cong C_p^2$ , and hence  $|G'| = p$ . We consider two cases:

Case I.  $p$  an odd prime. It is straightforward to see that the map  $\sigma : G \rightarrow G$  by  $x^\sigma = x^{1+p}$ , is an automorphism of  $G$ . Hence for any element  $x$  of  $L(G)$ ,  $x = x^\sigma = x^{1+p}$ , and so  $x^p = 1$ . Thus  $\exp(L(G)) = p$  and so  $G' = L(G) \cong C_p$ , by Theorem 3.2. If  $G/L(G) \cong C_{p^3}$ , then by [3, Theorem 2.2],  $G$  is cyclic, a contradiction. Next, we assume that  $G/L(G) \cong C_{p^2} \times C_p$ . Then  $G$  is an abelian group by [11, Theorem 5.1], which is impossible. Finally, if  $G/L(G) \cong C_p^3$ , then  $L(G) = \Phi(G)$  and so  $\text{Aut}^\Phi(G) = \text{Inn}(G)$ . Therefore by Theorem 2.2,  $G$  is an extraspecial  $p$ -group, a contradiction.

Case II.  $p = 2$ . Since  $|G'| = 2$ , and  $G'$  be a characteristic subgroup of  $G$ , we have  $G' \leq L(G) \leq Z(G)$ . Thus  $|L(G)| = 2$  or  $4$ . If  $|L(G)| = 4$ , then  $L(G) = Z(G)$  and  $G/L(G) \cong C_2^2$ . Hence by [11, Theorems 5.1 and 4.7],  $G \cong M_2(2, 2)$ , and  $L(G) \cong C_2^2$ , which is a contradiction by Theorem 3.2. Next we assume that  $|L(G)| = 2$ . So  $G' = L(G)$  and  $|G/L(G)| = 8$ . By a similar argument,  $G$  is isomorphic to one of the following groups:  $M_2(3, 1)$  or  $M_2(2, 1, 1)$ . The converse follows at once from Theorem 3.2.  $\square$

**Corollary 4.3.** *Let  $G$  be a non-abelian group of order  $p^5$ . Then  $\text{Aut}^L(G) = \text{Inn}(G)$  if and only if  $p = 2$  and  $G$  is one of the following types:  $M_2(3, 2)$ ,  $M_2(4, 1)$ ,  $M_2(2, 2, 1)$ ,  $D_8^{*2}$  or  $D_8 * Q_8$ .*

*Proof.* Let  $G$  be a finite group such that  $|G| = p^5$  and  $\text{Aut}^L(G) = \text{Inn}(G)$ . We consider two cases:

Case I.  $p > 2$ . These groups lying in the isoclinism families (5), (4) or (2) of [8, 4.5] and we show that  $\text{Aut}^L(G) \neq \text{Inn}(G)$ .

First, let  $G$  denote one of the groups in the isoclinism family (5). Hence  $|Z(G)| = p$  and  $G' = Z(G) = \Phi(G) \cong C_p$ , by Theorem 3.2. So  $G$  is an extraspecial  $p$ -group and by Corollary 3.5,  $|L(G)| = 1$ , a contradiction.

Next, let  $G$  be one of the groups in the isoclinism family (4). Then  $G' \cong C_p^2$ , which is a contradiction, since  $G'$  is cyclic.

Finally, let  $G$  denote one of the groups in the isoclinism family (2). Then  $G/Z(G) \cong C_p^2$  and so  $d(G/L(G)) > 1$ . We observe that  $G' = L(G) \cong C_p$  and  $Z(G) = \Phi(G)$ , by using Theorems 2.2, 3.2, [3, Theorem 2.2] and [11, Theorem 5.1]. So  $d(G) = 2$  and by [16],  $G$  is a minimal non-abelian  $p$ -group. If  $G/L(G) \cong C_{p^3} \times C_p$ , then  $G$  is an abelian group, by [11, Theorem 5.1], a contradiction. If  $G/L(G) \cong C_{p^2}^2$ , then by [16],  $G \cong M_p(3, 2)$  or  $G \cong M_p(2, 2, 1)$ . Thus  $L(G) = 1$ , by [11, Theorem 4.7], a contradiction. Finally, assume that  $G/L(G) \cong C_{p^2} \times C_p^2$  or  $G/L(G) \cong C_p^4$ . In this cases,  $\text{Aut}^L(G) \neq \text{Inn}(G)$ , by Theorem 2.5.

Case II.  $p = 2$ . We can see that  $|L(G)| = 2, 4$ , by [3, Theorem 2.2] and [11, Theorem 5.1]. First, we assume that  $|L(G)| = 4$ . Since  $G$  is a non-cyclic group, by [3, Theorem 2.2],  $d(G/L(G)) > 1$ . It follows that  $G/L(G)$  is one of the groups  $C_2^3$  or  $C_4 \times C_2$ . Now in the first case,  $L(G) = \Phi(G)$  and so  $G$  is an extraspecial 2-group by Theorem 2.2. Hence  $G' = L(G) \cong C_2$ , a contradiction. Therefore  $G/L(G) \cong C_4 \times C_2$  and by [11, Theorems 5.1 and 4.7],  $G$  is one of the groups:  $M_2(2, 3)$  or  $M_2(3, 1, 1)$ , and  $L(G) \cong C_2^2$ , a contradiction by Theorem 3.2. Now we may suppose that  $|L(G)| = 2$ . So  $G' = L(G) \cong C_2$ . We discuss the following cases.

If  $G/L(G) \cong C_2^4$ , then  $L(G) = \Phi(G)$  and so  $\text{Aut}^\Phi(G) = \text{Inn}(G)$ . Therefore by Theorem 2.2,  $G$  is an extraspecial 2-group. Thus  $G$  is one of the groups  $D_8^{*2}$  or  $D_8 * Q_8$ , by [21]. Next, suppose that  $G/L(G) \cong C_4 \times C_2^2$ . Hence  $G/L(G) = \langle \bar{a}, \bar{b}, \bar{c} \rangle$ , where  $\bar{a} = aL(G)$ ,  $\bar{b} = bL(G)$ ,  $\bar{c} = cL(G)$  and  $o(\bar{a}) = 4$ ,  $o(\bar{b}) = o(\bar{c}) = 2$ . Therefore  $G = \langle a, b, c, L(G) \rangle = \langle a, b, c \rangle$ , by [11, Corollary 3.7]. Since  $\langle a^2 \rangle \times G' \leq Z(G)$ , we have either  $Z(G) \cong C_4 \times C_2$  or  $C_2^2$ . If  $Z(G) \cong C_4 \times C_2$ , then  $\text{Aut}^L(G) \neq \text{Inn}(G)$ , by Theorem 2.5. Therefore  $Z(G) \cong C_2^2$ . Now by using GAP [4], we find that there are no such groups. Next, if  $G/L(G) \cong C_8 \times C_2$ , then  $G \cong M_2(4, 1)$ , by [11, Theorem 5.1]. Finally, suppose that  $G/L(G) \cong C_4^2$ . Then  $d(G) = 2$ , by [11, Corollary 3.7] and  $G' = L(G) \cong C_2$ . Hence by [16],  $G$  is a minimal non-abelian 2-group. Thus  $G$  is isomorphic to the group  $M_2(3, 2)$  or  $M_2(2, 2, 1)$ . The converse follows at once from Theorem 3.2.  $\square$

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