

On a New Reverse Hilbert's Type Inequality

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ABSTRACT. In this paper, by using the Euler-Maclaurin expansion for the Riemann- ζ function, we establish an inequality of a weight coefficient. Using this inequality, we derive a new reverse Hilbert's type inequality. As an applications, an equivalent form is obtained.

Keywords: Hilbert's type inequality, Weight coefficient, Hölder inequality, Riemann- ζ function, Reverse.

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1. INTRODUCTION

If $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_n \geq 0$, $b_n \geq 0$, for $n \geq 1$, $n \in N$ and $0 < \sum_{n=1}^{\infty} a_n^p < \infty$,
 $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{m+n} < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left\{ \sum_{n=1}^{\infty} a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} b_n^q \right\}^{\frac{1}{q}}, \quad (1.1)$$

and

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{m, n\}} < pq \left\{ \sum_{n=1}^{\infty} a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} b_n^q \right\}^{\frac{1}{q}}, \quad (1.2)$$

where the constant $\frac{\pi}{\sin \frac{\pi}{p}}$ and pq is best possible for each inequality respectively. Inequality (1.1) is Hardy-Hilbert's inequality. Inequality (1.2) is a Hilbert's type inequality [1].

In [5], [10] and [9], Krnic, Pecaric and Yang gave some generalization and reinforcement of inequality (1.1). In [3], Kuang and Debnath gave a reinforcement of inequality (1.2):

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{m, n\}} < \left\{ \sum_{n=1}^{\infty} [pq - G(p, n)] a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} [pq - G(q, n)] b_n^q \right\}^{\frac{1}{q}} \quad (1.3)$$

where $G(r, n) = \frac{r + \frac{1}{3r} - \frac{4}{3}}{(2n+1)^{\frac{1}{r}}} > 0$ ($r = p, q$).

In [6] and [7], Xi gave a generalization and reinforcement of inequalities (1.2) and (1.3):

$$\begin{aligned} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{m^\lambda, n^\lambda\}} < \left\{ \sum_{n=1}^{\infty} \left[\kappa(\lambda) - \frac{1}{3qn^{\frac{q+\lambda-2}{q}}} \right] n^{1-\lambda} a_n^p \right\}^{\frac{1}{p}} \\ \times \left\{ \sum_{n=1}^{\infty} \left[\kappa(\lambda) - \frac{1}{3pn^{\frac{p+\lambda-2}{p}}} \right] n^{1-\lambda} b_n^q \right\}^{\frac{1}{q}}, \quad (1.4) \end{aligned}$$

where $\kappa(\lambda) = \frac{pq\lambda}{(p+\lambda-2)(q+\lambda-2)} > 0$, $2 - \min\{p, q\} < \lambda \leq 2$.

$$\begin{aligned} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{m^\lambda + A, n^\lambda + B\}} < \left\{ \sum_{n=1}^{\infty} \left[\kappa(\lambda) - \frac{1}{n^{\frac{q+\lambda-2}{q}}} \left(\frac{1}{3q} - \frac{B}{1+B} \right) \right] \right. \\ \left. \times n^{1-\lambda} a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} \left[\kappa(\lambda) - \frac{1}{n^{\frac{p+\lambda-2}{p}}} \left(\frac{1}{3p} - \frac{A}{1+A} \right) \right] n^{1-\lambda} b_n^q \right\}^{\frac{1}{q}}, \quad (1.5) \end{aligned}$$

For the reverse Hilbert's type inequality, In [8], Xi and Wang gave a reverse Hilbert's type inequality:

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{m^2, n^2\}} > 2 \left[\sum_{n=1}^{\infty} \left(1 - \frac{1}{2n} \right) \frac{1}{n} a_n^p \right]^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} \frac{1}{n} b_n^q \right]^{\frac{1}{q}}. \quad (1.6)$$

In this paper, by introducing a parameter λ and using the Euler-Maclaurin expansion for the Riemann- ζ function, we establish an inequality of a weight coefficient. Using this inequality, we derive a reverse of the Hilbert's type inequality (1.4).

2. A LEMMA

First, we need the following formula of the Riemann- ζ function (see [4], [12] and [11]):

$$\zeta(\sigma) = \sum_{k=1}^n \frac{1}{k^\sigma} - \frac{n^{1-\sigma}}{1-\sigma} - \frac{1}{2n^\sigma} - \sum_{k=1}^{l-1} \frac{B_{2k}}{2k} \binom{-\sigma}{2k-1} \frac{1}{n^{\sigma+2k-1}} - \frac{B_{2l}}{2l} \binom{-\sigma}{2l-1} \frac{\varepsilon}{n^{\sigma+2l-1}}, \tag{2.1}$$

where $\sigma > 0, \sigma \neq 1, n, l \geq 1, n, l \in N, 0 < \varepsilon = \varepsilon(\sigma, l, n) < 1$. The numbers $B_1 = -1/2, B_2 = 1/6, B_3 = 0, B_4 = -1/30, \dots$ are Bernoulli numbers. In particular, $\zeta(\sigma) = \sum_{k=1}^\infty \frac{1}{k^\sigma} (\sigma > 1)$.

Since $\zeta(0) = -1/2$, then the formula of the Riemann- ζ function (2.1) is also true for $\sigma = 0$.

Lemma 2.1. *If $0 < p < 1, \frac{1}{p} + \frac{1}{q} = 1, 2 - p < \lambda \leq 2, n \geq 1$ and $n \in N$, then*

$$\omega(n, \lambda, q) = \sum_{k=1}^\infty \frac{1}{\max\{k^\lambda, n^\lambda\}} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{q}} > \frac{qn^{1-\lambda}}{q + \lambda - 2}, \tag{2.2}$$

and

$$\omega(n, \lambda, p) = \sum_{k=1}^\infty \frac{1}{\max\{k^\lambda, n^\lambda\}} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}} < n^{1-\lambda} \left[\kappa(\lambda) - \frac{p+2}{2(p+\lambda-2)n^{\frac{p+\lambda-2}{p}}} \right], \tag{2.3}$$

where $\kappa(\lambda) = \frac{pq\lambda}{(p+\lambda-2)(q+\lambda-2)}$.

Proof. Equalities (2.2) and (2.3) define the weight coefficient. When $2 - p < \lambda \leq 2$, taking $\sigma = \frac{2-\lambda}{p} \geq 0, l = 1$, in (2.1), we obtain

$$\zeta\left(\frac{2-\lambda}{p}\right) = \sum_{k=1}^n \frac{1}{k^{\frac{2-\lambda}{p}}} - \frac{pn^{\frac{p+\lambda-2}{p}}}{p+\lambda-2} - \frac{1}{2n^{\frac{2-\lambda}{p}}} + \frac{2-\lambda}{12pn^{1+\frac{2-\lambda}{p}}} \varepsilon_1, \tag{2.4}$$

where $0 < \varepsilon_1 < 1$.

Since $\frac{2}{q} + \frac{\lambda}{p} = 2 + \frac{\lambda-2}{p} = \frac{2p+\lambda-2}{p} > 0 (p+\lambda-2 > 0)$. Taking $\sigma = \frac{\lambda}{p} + \frac{2}{q}, l = 1$, we obtain

$$\zeta\left(\frac{2}{q} + \frac{\lambda}{p}\right) = \sum_{k=1}^{n-1} \frac{1}{k^{\frac{2}{q} + \frac{\lambda}{p}}} + \frac{pn^{-\frac{p+\lambda-2}{p}}}{p+\lambda-2} + \frac{1}{2n^{\frac{2}{q} + \frac{\lambda}{p}}} + \frac{q\lambda + 2p}{12pqn^{1+\frac{2}{q} + \frac{\lambda}{p}}} \varepsilon_2, \tag{2.5}$$

where $0 < \varepsilon_2 < 1$.

Since $\frac{2}{p} + \frac{\lambda}{q} = 2 + \frac{\lambda-2}{q} = \frac{2q+\lambda-2}{q} > 0 (q+\lambda-2 < 0, q < 0)$. Taking $\sigma = \frac{2}{p} + \frac{\lambda}{q}, l = 1$, we obtain

$$\zeta\left(\frac{2}{p} + \frac{\lambda}{q}\right) = \sum_{k=1}^{n-1} \frac{1}{k^{\frac{2}{p} + \frac{\lambda}{q}}} + \frac{qn^{-\frac{q+\lambda-2}{q}}}{q+\lambda-2} + \frac{1}{2n^{\frac{2}{p} + \frac{\lambda}{q}}} + \frac{p\lambda + 2q}{12pqn^{1+\frac{2}{p} + \frac{\lambda}{q}}} \varepsilon_3, \tag{2.6}$$

where $0 < \varepsilon_3 < 1$.

In addition,

$$\begin{aligned}
 \omega(n, \lambda, q) &= \sum_{k=1}^{\infty} \frac{1}{\max\{k^\lambda, n^\lambda\}} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{q}} \\
 &= \sum_{k=1}^n \frac{1}{\max\{k^\lambda, n^\lambda\}} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{q}} - \frac{1}{n^\lambda} + \sum_{k=n}^{\infty} \frac{1}{\max\{k^\lambda, n^\lambda\}} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{q}} \\
 &= \sum_{k=1}^n \frac{1}{n^\lambda} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{q}} - \frac{1}{n^\lambda} + \sum_{k=n}^{\infty} \frac{1}{k^\lambda} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{q}} \\
 &= \frac{1}{n^{\frac{(q+1)\lambda-2}{q}}} \sum_{k=1}^n \frac{1}{k^{\frac{2-\lambda}{q}}} - \frac{1}{n^\lambda} + n^{\frac{2-\lambda}{q}} \sum_{k=n}^{\infty} \frac{1}{k^{\frac{\lambda}{p} + \frac{2}{q}}} \\
 &> \frac{1}{n^{\frac{(q+1)\lambda-2}{q}}} - \frac{1}{n^\lambda} + n^{\frac{2-\lambda}{q}} \sum_{k=n}^{\infty} \frac{1}{k^{\frac{\lambda}{p} + \frac{2}{q}}}.
 \end{aligned}$$

By (2.5) and $\frac{2}{q} + \frac{\lambda}{p} = \frac{q\lambda+2p}{pq} > 0$

$$\begin{aligned}
 \omega(n, \lambda, q) &> \frac{1}{n^{\frac{(p+1)\lambda-2}{p}}} - \frac{1}{n^\lambda} + n^{\frac{2-\lambda}{q}} \left[\frac{pn^{-\frac{p+\lambda-2}{p}}}{p+\lambda-2} + \frac{1}{2n^{\frac{2}{q} + \frac{\lambda}{p}}} + \frac{q\lambda+2p}{12pq n^{1+\frac{2}{q} + \frac{\lambda}{p}}} \varepsilon_2 \right] \\
 &> \frac{1}{n^{\frac{(p+1)\lambda-2}{p}}} - \frac{1}{n^\lambda} + n^{\frac{2-\lambda}{q}} \left[\frac{pn^{-\frac{p+\lambda-2}{p}}}{p+\lambda-2} + \frac{1}{2n^{\frac{2}{q} + \frac{\lambda}{p}}} \right] \\
 &= \frac{1}{n^{\frac{(p+1)\lambda-2}{p}}} - \frac{1}{n^\lambda} + \frac{qn^{1-\lambda}}{q+\lambda-2} + \frac{1}{2n^\lambda} \\
 &= \frac{1}{n^{\frac{(p+1)\lambda-2}{p}}} - \frac{1}{2n^\lambda} + \frac{qn^{1-\lambda}}{q+\lambda-2} \\
 &> \frac{qn^{1-\lambda}}{q+\lambda-2}.
 \end{aligned}$$

Using the last result and the inequality for $\omega(n, \lambda, q)$ above, we obtain (2.2).

$$\begin{aligned}
 \omega(n, \lambda, p) &= \sum_{k=1}^{\infty} \frac{1}{\max\{k^\lambda, n^\lambda\}} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}} \\
 &= \sum_{k=1}^n \frac{1}{\max\{k^\lambda, n^\lambda\}} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}} - \frac{1}{n^\lambda} + \sum_{k=n}^{\infty} \frac{1}{\max\{k^\lambda, n^\lambda\}} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}} \\
 &= \sum_{k=1}^n \frac{1}{n^\lambda} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}} - \frac{1}{n^\lambda} + \sum_{k=n}^{\infty} \frac{1}{k^\lambda} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}} \\
 &= \frac{1}{n^{\frac{(p+1)\lambda-2}{p}}} \sum_{k=1}^n \frac{1}{k^{\frac{2-\lambda}{p}}} - \frac{1}{n^\lambda} + n^{\frac{2-\lambda}{p}} \sum_{k=n}^{\infty} \frac{1}{k^{\frac{2}{p} + \frac{\lambda}{q}}}.
 \end{aligned}$$

By (2.4) and (2.6)

$$\begin{aligned} \omega(n, \lambda, p) &< \frac{1}{n^{\frac{(p+1)\lambda-2}{p}}} \left[\zeta\left(\frac{2-\lambda}{p}\right) + \frac{pn^{\frac{p+\lambda-2}{p}}}{p+\lambda-2} + \frac{1}{2n^{\frac{2-\lambda}{p}}} \right] - \frac{1}{n^\lambda} \\ &\quad + n^{\frac{2-\lambda}{p}} \left[\frac{qn^{-\frac{q+\lambda-2}{q}}}{q+\lambda-2} + \frac{1}{2n^{\frac{2}{p}+\frac{\lambda}{q}}} + \frac{p\lambda+2q}{12pqn^{1+\frac{2}{p}+\frac{\lambda}{q}}} \right] \\ &= \frac{1}{n^{\frac{(p+1)\lambda-2}{p}}} \zeta\left(\frac{2-\lambda}{p}\right) + \frac{pn^{1-\lambda}}{p+\lambda-2} + \frac{1}{2n^\lambda} - \frac{1}{n^\lambda} + \frac{qn^{1-\lambda}}{q+\lambda-2} \\ &\quad + \frac{1}{2n^\lambda} + \frac{p\lambda+2q}{12pqn^{1+\lambda}} \\ &= \frac{1}{n^{\frac{(p+1)\lambda-2}{p}}} \zeta\left(\frac{2-\lambda}{p}\right) + \frac{pq\lambda n^{1-\lambda}}{(p+\lambda-2)(q+\lambda-2)} + \frac{p\lambda+2q}{12pqn^{1+\lambda}} \\ &= n^{1-\lambda} \left\{ \frac{pq\lambda}{(p+\lambda-2)(q+\lambda-2)} - \frac{1}{n^{\frac{p+\lambda-2}{p}}} \left[-\zeta\left(\frac{2-\lambda}{p}\right) \right. \right. \\ &\quad \left. \left. - \frac{p\lambda+2q}{12pqn^{\frac{p-\lambda+2}{p}}} \right] \right\}. \end{aligned}$$

In (2.4), taking $n = 1$, by $2 - p < \lambda \leq 2$, we obtain

$$\begin{aligned} \zeta\left(\frac{2-\lambda}{p}\right) &= 1 - \frac{p}{p+\lambda-2} - \frac{1}{2} + \frac{(2-\lambda)\varepsilon_1}{12p} \\ &< \frac{1}{2} - \frac{p}{p+\lambda-2} + \frac{2-\lambda}{12p} \\ &= -\frac{(\lambda-2-3p)(\lambda-2-2p)}{12p(p+\lambda-2)} \\ &< 0. \end{aligned}$$

So for $n \geq 1$, $n \in N$, $2 - p < \lambda \leq 2$, we have

$$\begin{aligned} &-\zeta\left(\frac{2-\lambda}{p}\right) + \frac{2-\lambda}{12pn^{1+\frac{2-\lambda}{p}}} \\ &\leq -\zeta\left(\frac{2-\lambda}{p}\right) + \frac{2-\lambda}{12p} \\ &= \frac{(\lambda-2-3p)(\lambda-2-2p)}{12p(p+\lambda-2)} + \frac{2-\lambda}{12p} \end{aligned}$$

$$\begin{aligned}
&= \frac{(\lambda - 2 - 3p)(\lambda - 2 - 2p) + (2 - \lambda)(p + \lambda - 2)}{12p(p + \lambda - 2)} \\
&= \frac{(\lambda - 2)^2 + (-5p - p - \lambda + 2)(\lambda - 2) + 6p^2}{12p(p + \lambda - 2)} \\
&= \frac{6p(2 - \lambda) + 6p^2}{12p(p + \lambda - 2)} \\
&= \frac{(2 - \lambda) + p}{2(p + \lambda - 2)} \\
&\leq \frac{2 + p}{2(p + \lambda - 2)}.
\end{aligned}$$

Using the last result and the inequality for $\omega(n, \lambda, p)$ above, we obtain (2.3). \square

3. MAIN RESULTS

Theorem 3.1. *If $0 < p < 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $2 - p < \lambda \leq 2$, $a_n \geq 0$, $b_n \geq 0$, for $n \geq 1, n \in N$ and $0 < \sum_{n=1}^{\infty} a_n^p < \infty$, $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then*

$$\begin{aligned}
\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{m^\lambda, n^\lambda\}} &> \left\{ \sum_{n=1}^{\infty} \frac{q}{q + \lambda - 2} n^{1-\lambda} a_n^p \right\}^{\frac{1}{p}} \\
&\times \left\{ \sum_{n=1}^{\infty} \left[\kappa(\lambda) - \frac{p+2}{2(p+\lambda-2)n^{\frac{p+\lambda-2}{p}}} \right] n^{1-\lambda} b_n^q \right\}^{\frac{1}{q}}, \quad (3.1)
\end{aligned}$$

where $\kappa(\lambda) = \frac{p q \lambda}{(p+\lambda-2)(q+\lambda-2)} > 0$.

Proof. By the reverse Hölder's inequality [2], we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{m^\lambda, n^\lambda\}} &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\frac{a_m}{\max\{m^\lambda, n^\lambda\}^{\frac{1}{p}}} \left(\frac{m}{n}\right)^{\frac{2-\lambda}{pq}} \right] \\
&\times \left[\frac{b_n}{\max\{m^\lambda, n^\lambda\}^{\frac{1}{q}}} \left(\frac{n}{m}\right)^{\frac{2-\lambda}{pq}} \right] \\
&\geq \left\{ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\frac{a_m^p}{\max\{m^\lambda, n^\lambda\}} \left(\frac{m}{n}\right)^{\frac{2-\lambda}{q}} \right] \right\}^{\frac{1}{p}} \\
&\times \left\{ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\frac{b_n^q}{\max\{m^\lambda, n^\lambda\}} \left(\frac{n}{m}\right)^{\frac{2-\lambda}{p}} \right] \right\}^{\frac{1}{q}} \\
&= \left\{ \sum_{m=1}^{\infty} \omega(m, \lambda, q) a_m^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} \omega(n, \lambda, p) b_n^q \right\}^{\frac{1}{q}}.
\end{aligned}$$

Since $0 < p < 1$ and $q < 0$. By (2.2), (2.3), we obtain (3.1). Theorem 3.1 is proved. \square

Theorem 3.2. *If $0 < p < 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $2 - p < \lambda \leq 2$, $a_n \geq 0$, $b_n \geq 0$, for $n \geq 1, n \in N$ and $0 < \sum_{n=1}^{\infty} n^{1-\lambda} b_n^q < \infty$, then*

$$\begin{aligned} & \sum_{n=1}^{\infty} \left(\frac{q}{q + \lambda - 2} n^{1-\lambda} \right)^{1-q} \left(\sum_{m=1}^{\infty} \frac{b_m}{\max\{m^\lambda, n^\lambda\}} \right)^q \\ & > \sum_{n=1}^{\infty} \left[\kappa(\lambda) - \frac{p + 2}{2(p + \lambda - 2)n^{\frac{p+\lambda-2}{p}}} \right] n^{1-\lambda} b_n^q. \end{aligned} \tag{3.2}$$

where $\kappa(\lambda) = \frac{p q \lambda}{(p+\lambda-2)(q+\lambda-2)} > 0$.

Inequalities (3.2) and (3.1) are equivalent.

Proof. Let

$$a_n = \left(\frac{q}{q + \lambda - 2} n^{1-\lambda} \right)^{1-q} \left[\sum_{m=1}^{\infty} \frac{b_m}{\max\{m^\lambda, n^\lambda\}} \right]^{q-1}, \quad n \in N.$$

By (3.1), we have

$$\begin{aligned} \left\{ \sum_{n=1}^{\infty} \frac{q}{q + \lambda - 2} n^{1-\lambda} a_n^p \right\}^q &= \left\{ \sum_{n=1}^{\infty} \left(\frac{q}{q + \lambda - 2} n^{1-\lambda} \right)^{1-q} \right. \\ & \quad \times \left. \left(\sum_{m=1}^{\infty} \frac{b_m}{\max\{m^\lambda, n^\lambda\}} \right)^q \right\}^q \\ &= \left\{ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_n b_m}{\max\{m^\lambda, n^\lambda\}} \right\}^q \\ &> \left\{ \sum_{n=1}^{\infty} \frac{q}{q + \lambda - 2} n^{1-\lambda} a_n^p \right\}^{q-1} \left\{ \sum_{n=1}^{\infty} [\kappa(\lambda) \right. \\ & \quad \left. - \frac{p + 2}{2(p + \lambda - 2)n^{\frac{p+\lambda-2}{p}}} n^{1-\lambda} b_n^q] \right\}. \end{aligned}$$

Then we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{q}{q + \lambda - 2} n^{1-\lambda} a_n^p &= \sum_{n=1}^{\infty} \left(\frac{q}{q + \lambda - 2} n^{1-\lambda} \right)^{1-q} \left(\sum_{m=1}^{\infty} \frac{b_m}{\max\{m^\lambda, n^\lambda\}} \right)^q \\ &> \sum_{n=1}^{\infty} \left[\kappa(\lambda) - \frac{p + 2}{2(p + \lambda - 2)n^{\frac{p+\lambda-2}{p}}} \right] n^{1-\lambda} b_n^q. \end{aligned}$$

On the other-hand, by the reverse Hölder 's inequality [2], we have

$$\begin{aligned} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_n b_m}{\max\{m^\lambda, n^\lambda\}} &= \sum_{n=1}^{\infty} \left[\left(\frac{q}{q+\lambda-2} n^{1-\lambda} \right)^{-\frac{1}{p}} \sum_{m=1}^{\infty} \frac{b_m}{\max\{m^\lambda, n^\lambda\}} \right] \\ &\quad \times \left[\left(\frac{q}{q+\lambda-2} n^{1-\lambda} \right)^{\frac{1}{p}} a_n \right] \\ &\geq \left[\sum_{n=1}^{\infty} \left(\frac{q}{q+\lambda-2} n^{1-\lambda} \right)^{1-q} \left(\sum_{m=1}^{\infty} \frac{b_m}{\max\{m^\lambda, n^\lambda\}} \right)^q \right]^{\frac{1}{q}} \\ &\quad \times \sum_{n=1}^{\infty} \left[\frac{q}{q+\lambda-2} n^{1-\lambda} a_n^p \right]^{\frac{1}{p}}. \end{aligned}$$

From (3.2), it follows that

$$\begin{aligned} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_n b_m}{\max\{m^\lambda, n^\lambda\}} &> \sum_{n=1}^{\infty} \left[\frac{q}{q+\lambda-2} n^{1-\lambda} a_n^p \right]^{\frac{1}{p}} \\ &\quad \times \left[\sum_{n=1}^{\infty} \left(\kappa(\lambda) - \frac{p+2}{2(p+\lambda-2)n^{\frac{p+\lambda-2}{p}}} \right) n^{1-\lambda} b_n^q \right]^{\frac{1}{q}}. \end{aligned}$$

Then, (3.2) and (3.1) are equivalent. Theorem 3.2 is proved. \square

In inequality (3.1), taking $\lambda = 2$, we obtain:

Corollary 3.3. *If $0 < p < 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_n \geq 0$, $b_n \geq 0$, for $n \geq 1$, $n \in N$ and $0 < \sum_{n=1}^{\infty} a_n^p < \infty$, $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then*

$$\begin{aligned} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{m^2, n^2\}} &> \left\{ \sum_{n=1}^{\infty} \frac{1}{n} a_n^p \right\}^{\frac{1}{p}} \\ &\quad \times \left\{ \sum_{n=1}^{\infty} \left[1 - \frac{p+2}{2pn} \right] \frac{1}{n} b_n^q \right\}^{\frac{1}{q}}. \end{aligned} \quad (3.3)$$

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