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Solution of Inverse Euler-Bernoulli Problem with Integral Overdetermination and Periodic Boundary Conditions

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Abstract. In this work, we tried to find the inverse coefficient in the Euler problem with over determination conditions. It showed the existence, stability of the solution by iteration method and linearization method was used for this problem in numerical part. Also two examples are presented with figures.

Keywords: Inverse Coefficient Problem, Periodic boundary condition, Euler-Bernoulli equation, Fourier method.

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1. Introduction

Mathematical modeling of sound wave distribution problems are denote following equation

$$\frac{\partial^2 v}{\partial t^2}+\frac{\partial^4 v}{\partial x^4}=g(x,t,v)$$

is called the Euler-Bernoulli equations. The vibration, buckling and dynamic behavior of various building elements widely used in nano-technology are represented by the Euler-Bernoulli equations.Due to the new and exceptionally its electronic and mechanical properties, carbon nanotubes are considered to be one of the most useful material in future [4, 8]. These elements are tackled by different boundary conditions depending on different loading conditions. Therefore, investigation of the solution of Euler-Bernoulli equations with different boundary conditions used in the mathematical modeling of the structural components of nano-materials continues to be a focus of interest amongst mathematicians.

In mathematics, the classical statement of Euler-Bernoulli equation is in the following form:

$$\frac{\partial^2 v}{\partial t^2}+\frac{\partial^4 v}{\partial x^4}=0.$$

The inverse problem have been worked by many authors [2, 1, 3, 7]. It will be examined to inverse Euler Bernoulli equations in this article.

The periodic boundary conditions arise from many important applications in heat transfer, life sciences[5].

Let $T>0$ and denote by $\Omega:=\{0<x<\pi, 0<t<T\}$.

The quasi-linear time-dependent equation

$$\frac{\partial^2 v}{\partial t^2}+\frac{\partial^4 v}{\partial x^4}-a(t)v=g(x,t,v),(x,t)\in\ \Omega, \tag{1.1}$$

with boundary conditions

$$\begin{aligned} v(0,t) &= v(\pi,t), \\ v_x(0,t) &= v_x(\pi,t), \\ v_{xx}(0,t) &= v_{xx}(\pi,t), \\ v_{xxx}(0,t) &= v_{xxx}(\pi,t), t\in[0,T], \end{aligned} \tag{1.2}$$

with initial conditions

$$v(x,0)=\varphi(x), v_t(x,0)=\psi(x)\ , x\in[0,\pi], \tag{1.3}$$

and integral overdetermination conditions

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$$E(t) = \int_0^\pi xu(x,t)dx, t \in [0,T], \qquad (1.4)$$

for nonlinear source term $g(x,t,v)$ and $\varphi(x)$, $\psi(x)$ and $E(t)$ are known functions which are positive and continuous,$v(x,t)$ and $a(t)$ are unknown functions. In heat propagation in a thin rod in which the law of variation $E(t)$ of the total quantity of heat in the rod is given in [6].

Definition 1.1. $\{a(t), v(x,t)\}$ is called the solution of the inverse problem (1.1)-(1.4).

Definition 1.2. $w(t,x) \in C(\overline{\Omega})$ is refereed test function that gives following conditions:

$w(T,x) = w_t(T,x) = 0, w(0,t) = w(\pi,t), w_x(0,t) = w_x(\pi,t), w_{xx}(0,t) = w_{xx}(\pi,t), w_{xxx}(0,t) = w_{xxx}(\pi,t), t\epsilon\,[0,T]$.

$v(x,t) \in C(\overline{\Omega})$ is named generalized solution that gives following equation:

$$\int_0^T \int_0^\pi \left(\left\{ \frac{\partial^2 w}{\partial t^2} + \frac{\partial^4 w}{\partial x^4} - a(t)w \right\} v - gw \right) dxdt - \int_0^\pi w(x,0)\psi(x)dx + \int_0^\pi w_t(x,0)\varphi(x)dx = 0.$$

Nomenclature

$\varphi(x)$ Initial function
$a(t)$ Unknown coefficient
$E(t)$ Energy
$v(x,t)$ Temperature distribution
$g(x,t,v)$ Source function
$v_0(t), v_{ck}(t), v_{ck}(t)$ Fourier coefficients
M Arbitrary constant
M_1, M_2 Dimensionless constants
$\Omega := \{0 < x < \pi,\ 0 < t < T\}$ Domain of x,t

2. Solution of the Inverse Problem

(S1) $E(t) \in C^2[0,T]$.

(S2) $\varphi(x) \in C^3[0,\pi], \psi(x) \in C^1[0,\pi],\ E(0) = \int_0^\pi x\varphi(x)dx$,

(S3) $g(x,t,v)$ is provided the following conditions in $\overline{\Omega} \times (-\infty,\infty)$,

(1)

$$\left| \frac{\partial^{(n)} g(x,t,v)}{\partial x^n} - \frac{\partial^{(n)} g(x,t,\widetilde{v})}{\partial x^n} \right| \le b(x,t)\,|v - \widetilde{v}|\ , n = 0,1,2,$$

where $b(x,t) \in L_2(\Omega),\ b(x,t) \ge 0$,

(2)$g(x,t,v) \in \overline{\Omega} \times (-\infty, \infty)$, $t \in [0,T], |g(x,t,v)| \leq M$,
(3) $a(t) \in C[0,T]$.

By Fourier method, we have

$$\begin{aligned} v(x,t) &= \frac{1}{2}\left[\varphi_0 + \psi_0 t + \frac{2}{\pi}\int_0^t\int_0^\pi (t-\tau)\left(a(\tau)v(\xi,\tau) + g(\xi,\tau,v)\right)d\xi d\tau\right] \\ &+ \sum_{k=1}^{\infty}\cos 2kx\left(\varphi_{ck}\cos(2k)^2 t + \frac{\psi_{ck}}{(2k)^2}\sin(2k)^2 t\right) \\ &+ \sum_{k=1}^{\infty}\cos 2kx\left(\frac{2}{\pi(2k)^2}\int_0^t\int_0^\pi \left(a(\tau)v(\xi,\tau) + g(\xi,\tau,v)\right)\sin(2k)^2(t-\tau)\cos 2k\xi d\xi d\tau\right) \\ &+ \sum_{k=1}^{\infty}\sin 2kx\left(\varphi_{sk}\cos(2k)^2 t + \frac{\psi_{sk}}{(2k)^2}\sin(2k)^2 t\right) \\ &+ \sum_{k=1}^{\infty}\sin 2kx\left(\frac{2}{\pi(2k)^2}\int_0^t\int_0^\pi \left(a(\tau)v(\xi,\tau) + g(\xi,\tau,v)\right)\sin(2k)^2(t-\tau)\sin 2k\xi d\xi d\tau\right), \end{aligned} \tag{2.1}$$

where $\varphi_0 = \frac{2}{\pi}\int_0^\pi \varphi(x)dx, \varphi_{ck} = \frac{2}{\pi}\int_0^\pi \varphi(x)\cos 2kxdx, \varphi_{sk} = \frac{2}{\pi}\int_0^\pi \varphi(x)\sin 2kxdx,$
$\psi_0 = \frac{2}{\pi}\int_0^\pi \psi(x)dx, \psi_{ck} = \frac{2}{\pi}\int_0^\pi \psi(x)\cos 2kxdx, \psi_{sk} = \frac{2}{\pi}\int_0^\pi \psi(x)\sin 2kxdx,$
$g_0(t,v) = \frac{2}{\pi}\int_0^\pi g(x,t,v)dx,\ g_{ck}(t,v) = \frac{2}{\pi}\int_0^\pi g(x,t,v)\cos 2kxdx,\ g_{sk}(t,v) = \frac{2}{\pi}\int_0^\pi g(x,t,v)\sin 2kxdx,\ k = 1,2,3,\ldots$.

Using (S1)-(S3), differentiating (1.4), we have

$$\int_0^\pi x v_t(x,t)dx = E^{'}(t), 0 \leq t \leq T. \tag{2.2}$$

From (2.1) and (2.2),we obtain

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$$
\begin{aligned}
a(t) = {} & \frac{E''(t)}{E(t)} \\
& -\frac{\pi \sum\limits_{k=1}^{\infty} (2k)^3 \left(\varphi_{sk} \cos(2k)^2 t + \frac{\psi_{ck}}{(2k)^2} \sin(2k)^2 t\right)}{E(t)} \\
& -\frac{\pi \sum\limits_{k=1}^{\infty} (2k)^3 \frac{2}{\pi(2k)^2} \int\limits_0^t \int\limits_0^\pi (a(\tau)v(\xi,\tau) + g(\xi,\tau,v)) \sin\left((2k)^2(t-\tau)\right) \sin 2k\xi d\xi d\tau}{E(t)} \\
& -\frac{\int\limits_0^\pi \xi g(\xi,t,v)\, d\xi}{E(t)}.
\end{aligned}
\tag{2.3}
$$

Definition 2.1. Denote the following set;

Let $\{v(t)\} = \{v_0(t), v_{ck}(t), v_{sk}(t), k = 1, ..., n\}$ satisfy such that

$$\max_{0\le t\le T} \frac{|v_0(t)|}{2} + \sum_{k=1}^{\infty} \left(\max_{0\le t\le T} |v_{ck}(t)| + \max_{0\le t\le T} |v_{sk}(t)| \right) < \infty.$$

$$\|v(t)\| = \max_{0\le t\le T} \frac{|v_0(t)|}{2} + \sum_{k=1}^{\infty} \left(\max_{0\le t\le T} |v_{ck}(t)| + \max_{0\le t\le T} |v_{sk}(t)| \right)$$

is the norm of $\mathbf{B}_1$ Banach space.

Theorem 2.2. *If the conditions (S1)-(S3) be ensured. Then the Euler-Bernoulli problem has a unique solution.*

Proof. Let, iteration to equation (2.1)

$$
\begin{aligned}
v_0^{(N+1)}(t) &= v_0^{(0)}(t) + \frac{2}{\pi} \int_0^t \int_0^\pi (t-\tau) \left(a^{(N)}(\tau) v^{(N)}(\xi,\tau) + g(\xi,\tau,v^{(N)}) \right) d\xi d\tau \\
v_{ck}^{(N+1)}(t) &= v_{ck}^{(0)}(t) + \frac{2}{\pi(2k)^2} \int_0^t \int_0^\pi \left(a^{(N)}(\tau) v^{(N)}(\xi,\tau) + g(\xi,\tau,v^{(N)}) \right) \sin(2k)^2(t-\tau) \cos 2k\xi \, d\xi d\tau, \\
v_{sk}^{(N+1)}(t) &= v_{sk}^{(0)}(t) + \frac{2}{\pi(2k)^2} \int_0^t \int_0^\pi \left(a^{(N)}(\tau) v^{(N)}(\xi,\tau) + g(\xi,\tau,v^{(N)}) \right) \sin(2k)^2(t-\tau) \sin 2k\xi \, d\xi d\tau,
\end{aligned}
\tag{2.4}
$$

$$u_0^{(0)}(t) = \varphi_0 + \psi_0 t, u_{ck}^{(0)}(t) = \varphi_{ck} \cos(2k)^2 t + \frac{\psi_{ck}}{(2k)^2} \sin(2k)^2 t, u_{sk}^{(0)}(t) = \varphi_{sk} \cos(2k)^2 t + \frac{\psi_{sk}}{(2k)^2} \sin(2k)^2 t.$$

$$
\begin{aligned}
a^{(N+1)}(t) &= \frac{E^{''}(t)}{E(t)} - \\
&-\frac{\pi\sum\limits_{k=1}^{\infty}(2k)^3\left(\varphi_{sk}\cos(2k)^2t+\frac{\psi_{ck}}{(2k)^2}\sin(2k)^2t)\right)}{E(t)} \\
&-\frac{\pi\sum\limits_{k=1}^{\infty}(2k)^3\frac{2}{\pi(2k)^2}\int\limits_0^t\int\limits_0^\pi\left(a(\tau)v(\xi,\tau)+g(\xi,\tau,v^{(N)})\right)\sin\left((2k)^2(t-\tau)\right)\sin 2k\xi d\xi d\tau}{E(t)} \\
&-\frac{\int\limits_0^\pi \xi g\left(\xi,t,v^{(N)}\right)d\xi}{E(t)}.
\end{aligned}
$$

According to the theorem, we obtain $v^{(0)}(t)\in\mathbf{B}_1$, $t\in[0,T]$.

$$
v_0^{(1)}(t)=v_0^{(0)}(t)+\frac{2}{\pi}\int\limits_0^t\int\limits_0^\pi(t-\tau)\left(a^{(0)}(\tau)v^{(0)}(\xi,\tau)+g(\xi,\tau,v^{(0)})\right)d\xi d\tau
$$

Adding and subtracting $\int\limits_0^t\int\limits_0^\pi g(\xi,\tau,0)d\xi d\tau$ and after applying Cauchy, Bessel, Lipschitz inequalities, we obtain

$$
\begin{aligned}
\max_{0\le t\le T}\left|v_0^{(1)}(t)\right| \le\ & |\varphi_0|+T|\psi_0|+2\sqrt{\frac{T^3}{3\pi}}\left\|v^{(0)}(t)\right\|_{B_1}\left\|a^{(0)}(t)\right\|_{C[0,T]} \\
&+2\sqrt{\frac{T^3}{3\pi}}\left\|v^{(0)}(t)\right\|_{B_1}\|b(x,t)\|_{L_2(\Omega)}+2\sqrt{\frac{T^3}{3\pi}}\|g(x,t,0)\|_{L_2(\Omega)}.
\end{aligned}
$$

$$
v_{ck}^{(1)}(t)=v_{ck}^{(0)}(t)+\frac{2}{\pi(2k)^2}\int\limits_0^t\int\limits_0^\pi\left(a^{(0)}(\tau)v^{(0)}(\xi,\tau)+g(\xi,\tau,v^{(0)})\right)\sin(2k)^2(t-\tau)\cos 2k\xi d\xi d\tau.
$$

After applying Cauchy, Bessel, Lipschitz ,Hölder inequalities, we have

$$
\begin{aligned}
\sum_{k=1}^{\infty}\max_{0\le t\le T}\left|v_{ck}^{(1)}(t)\right| \le\ & \sum_{k=1}^{\infty}|\varphi_{ck}|+\frac{\pi^2}{24}\sum_{k=1}^{\infty}|\psi_{ck}| \\
&+\frac{\pi\sqrt{T}}{12}\left\|v^{(0)}(t)\right\|_{B_1}\left\|a^{(0)}(t)\right\|_{C[0,T]}+\frac{\pi\sqrt{T}}{12}\left\|v^{(0)}(t)\right\|_{B_1}\|b(x,t)\|_{L_2(\Omega)} \\
&+\frac{\pi\sqrt{T}}{12}\|g(x,t,0)\|_{L_2(\Omega)},
\end{aligned}
$$

and by the same approaches,

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$$\sum_{k=1}^{\infty}\max_{0\le t\le T}\left|v_{sk}^{(1)}(t)\right| \le \sum_{k=1}^{\infty}|\varphi_{sk}|+\frac{\pi^2}{24}\sum_{k=1}^{\infty}|\psi_{sk}| +\frac{\pi\sqrt{T}}{12}\left\|v^{(0)}(t)\right\|_{B_1}\left\|a^{(0)}(t)\right\|_{C[0,T]}+\frac{\pi\sqrt{T}}{12}\left\|v^{(0)}(t)\right\|_{B_1}\|b(x,t)\|_{L_2(\Omega)} +\frac{\pi\sqrt{T}}{12}\|g(x,t,0)\|_{L_2(\Omega)}.$$

we get

$$\begin{aligned}\left\|v^{(1)}(t)\right\|_{\mathbf{B}_1} &= \max_{0\le t\le T}\frac{\left|v_0^{(1)}(t)\right|}{2}+\sum_{k=1}^{\infty}\left(\max_{0\le t\le T}\left|v_{ck}^{(1)}(t)\right|+\max_{0\le t\le T}\left|v_{sk}^{(1)}(t)\right|\right)\\ &\le \frac{|\varphi_0|}{2}+\sum_{k=1}^{\infty}(|\varphi_{ck}|+|\varphi_{sk}|)+\frac{\pi^2}{24}\sum_{k=1}^{\infty}(|\psi_{ck}|+|\psi_{sk}|)\\ &+(2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6})\left\|v^{(0)}(t)\right\|_{B_1}\left\|a^{(0)}(t)\right\|_{C[0,T]}\\ &+(2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6})\left\|v^{(0)}(t)\right\|_{B_1}\|b(x,t)\|_{L_2(D)}\\ &+(2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6})\|g(x,t,0)\|_{L_2(\Omega)}.\end{aligned}$$

According to theorem , $v^{(1)}(t)\in\mathbf{B}_1$.

Same estimations for the step N,

$$\begin{aligned}\left\|v^{(N+1)}(t)\right\|_{\mathbf{B}_1} &= \max_{0\le t\le T}\frac{\left|v_0^{(N)}(t)\right|}{2}+\sum_{k=1}^{\infty}\left(\max_{0\le t\le T}\left|v_{ck}^{(N)}(t)\right|+\max_{0\le t\le T}\left|v_{sk}^{(N)}(t)\right|\right)\\ &\le \frac{|\varphi_0|}{2}+\sum_{k=1}^{\infty}(|\varphi_{ck}|+|\varphi_{sk}|)+\frac{\pi^2}{24}\sum_{k=1}^{\infty}(|\psi_{ck}|+|\psi_{sk}|)\\ &+(2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6})\left\|v^{(N)}(t)\right\|_{B_1}\left\|a^{(N)}(t)\right\|_{C[0,T]}\\ &+(2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6})\left\|v^{(N)}(t)\right\|_{B_1}\|b(x,t)\|_{L_2(\Omega)}\\ &+(2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6})\|g(x,t,0)\|_{L_2(\Omega)}.\end{aligned}$$

According to $v^{(N)}(t)\in\mathbf{B}_1$ and theorem we obtain $v^{(N+1)}(t)\in\mathbf{B}_1$,

$$\{v(t)\}=\{v_0(t),v_{ck}(t),v_{sk}(t),\ k=1,2,...\}\in\mathbf{B}_1.$$

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Same estimations for the step N,

$$\begin{aligned} &\left\| a^{(N+1)}(t) \right\|_{C[0,T]} \\ \leq\ & \frac{1}{E(t)}(\left| E^{''}(t) \right| + 2\sum_{k=1}^{\infty}\left(\left| \varphi_{ck}^{'''} \right| + \left| \psi_{ck}^{'} \right| \right) \\ &+\frac{\pi}{\sqrt{6}} \left\| v_{xx}^{(N)}(t) \right\|_{B_1} \left\| a^{(N)}(t) \right\|_{C[0,T]} \\ &+\frac{\pi}{\sqrt{6}} \left\| v^{(N)}(t) \right\|_{B_1} \left\| b(x,t) \right\|_{L_2(\Omega)} \\ &+\pi \left\| v^{(N)}(t) \right\|_{B_1} \left\| b(x,t) \right\|_{L_2(\Omega)} \\ &+\pi \left\| g(x,t,0) \right\|_{L_2(\Omega)}). \end{aligned}$$

We have $a^{(N+1)}(t) \in \mathbf{B}_1$.
For $N \to \infty$,$v^{(N+1)}(t), a^{(N+1)}$ are converged.
After applying Cauchy, Bessel ,Lipschitz, Hölder inequalities, we obtain

$$\begin{aligned} \left\| v^{(1)}(t) - v^{(0)}(t) \right\|_{B_1} \leq\ & (2\sqrt{\frac{T^3}{3\pi}} + \frac{\pi\sqrt{T}}{6}) \left\| v^{(0)}(t) \right\|_{B_1} \left\| a^{(0)}(t) \right\|_{C[0,T]} \\ &+(2\sqrt{\frac{T^3}{3\pi}} + \frac{\pi\sqrt{T}}{6}) \left\| v^{(0)}(t) \right\|_{B_1} \left\| b(x,t) \right\|_{L_2(\Omega)} \\ &+(2\sqrt{\frac{T^3}{3\pi}} + \frac{\pi\sqrt{T}}{6}) \left\| g(x,t,0) \right\|_{L_2(\Omega)}. \end{aligned}$$

$$\begin{aligned} A =\ & (2\sqrt{\frac{T^3}{3\pi}} + \frac{\pi\sqrt{T}}{6}) \left\| v^{(0)}(t) \right\|_{B_1} \left\| a^{(0)}(t) \right\|_{C[0,T]} + \\ &+(2\sqrt{\frac{T^3}{3\pi}} + \frac{\pi\sqrt{T}}{6}) \left\| v^{(0)}(t) \right\|_{B_1} \left\| b(x,t) \right\|_{L_2(\Omega)} + M). \end{aligned}$$

$$\begin{aligned} \left\| a^{(1)}(t) - a^{(0)}(t) \right\|_{C[0,T]} \leq\ & \frac{\pi}{\sqrt{6}E(t)B} \left\| v_{xx}^{(1)} - v_{xx}^{(0)} \right\|_{B_1} \left\| a^{(0)}(t) \right\|_{C[0,T]} \\ &+\frac{\pi}{E(t)B} \left\| v^{(1)} - v^{(0)} \right\|_{B_1} \left\| b(x,t) \right\|_{L_2(\Omega)}. \end{aligned}$$

For the step N :

$$\begin{aligned} \left\| a^{(N+1)}(t) - a^{(N)}(t) \right\|_{C[0,T]} \leq\ & \frac{\pi}{\sqrt{6}E(t)B} \left\| v_{xx}^{(N+1)} - v_{xx}^{(N)} \right\|_{B_1} \left\| a^{(N)}(t) \right\|_{C[0,T]} \\ &+\frac{\pi}{E(t)B} \left\| v^{(N+1)} - v^{(N)} \right\|_{B_1} \left\| b(x,t) \right\|_{L_2(\Omega)}. \end{aligned}$$

$$\left\| v^{(N+1)}(t) - v^{(N)}(t) \right\|_{B_1} \leq \frac{A}{D\sqrt{N!}} E \left\| b(x,t) \right\|_{L_2(\Omega)}^{N}. \qquad (2.5)$$

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$$\begin{aligned}
B &= 1-\frac{\pi}{\sqrt{6}E(t)}\left\|v^{(N)}\right\|_{B_1},\\
D &= 1-\frac{\pi}{\sqrt{6}E(t)B}\left\|v^{(N)}\right\|_{B_1}-\frac{\pi}{3E(t)}\left\|a^{(N)}(t)\right\|_{C[0,T]}\left\|v_x^{(N)}\right\|_{B_1}\\
&\quad -\frac{\sqrt{6}\pi}{E(t)9}\left\|a^{(N)}(t)\right\|^2_{C[0,T]}\left\|v^{(N)}\right\|_{B_1}.
\end{aligned}$$

$v^{(N+1)} \to v^{(N)}$, $N \to \infty$, then $a^{(N+1)} \to a^{(N)}$, $N \to \infty$.
Let us show

$$\lim_{N\to\infty} v^{(N+1)}(t) = v(t),\ \lim_{N\to\infty} a^{(N+1)}(t) = a(t).$$

$$\begin{aligned}
\left\|v-v^{(N+1)}\right\|_{B_1} &\le (2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6})\left\|a(t)-a^{(N+1)}(t)\right\|_{C[0,T]}\left\|v^{(N+1)}(t)\right\|_{B_1}\\
&\quad +(2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6})\left\|v-v^{(N+1)}\right\|_{B_1}\left\|a^{(N+1)}(t)\right\|_{C[0,T]}\\
&\quad +(2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6})\left\|v-v^{(N+1)}\right\|_{B_1}\|b(x,t)\|_{L_2(\Omega)}\\
&\quad +(2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6})\left\|v^{(N+1)}-v^{(N)}\right\|_{B_1}\|b(x,t)\|_{L_2(\Omega)}.
\end{aligned}$$

Let us consider $\left\|v^{(N+1)}(t)-v^{(N)}(t)\right\|_{B_1}$ and applying Gronwall's inequality

$$\begin{aligned}
\left\|v(t)-v^{(N+1)}(t)\right\|^2_{\mathbf{B}_1} &\le \\
& 2\left[\frac{A}{\sqrt{N!}}\frac{F}{G}\|b(x,t)\|_{L_2(\Omega)}\right]^2 \qquad (2.6)\\
&\times \exp 2\left(\frac{H}{G}\right)^2\left(\|b(x,t)\|^{N+1}_{L_2(\Omega)}\right)^2.
\end{aligned}$$

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$$
\begin{aligned}
C &= 1-\frac{\pi}{3E(t)B}\left\|a^{(N+1)}\right\|_{C[0,T]}\left\|v_x^{(N)}\right\|_{B_1}-\frac{\pi}{3BE(t)}\left\|a^{(N+1)}(t)\right\|^2_{C[0,T]}\left\|v^{(N+1)}\right\|_{B_1},\\
F &= \frac{(2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6})}{C}\left\|v^{(N+1)}\right\|_{B_1}\left\|a^{(N+1)}(t)\right\|^2_{C[0,T]}+\frac{\pi^2}{3BE(t)}\left\|v^{(N+1)}\right\|_{B_1}\left\|a^{(N+1)}(t)\right\|_{C[0,T]}\\
&\quad+\frac{\pi}{\sqrt{6}E(t)}\left\|v^{(N+1)}(t)\right\|_{B_1},\\
G &= 1-\frac{(2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6})}{C}\frac{\pi}{3BE(t)}\left\|v^{(N+1)}\right\|_{B_1}\left\|a^{(N+1)}\right\|^3_{C[0,T]}+\frac{(2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6})}{C}\frac{\pi}{\sqrt{6}E(t)}\left\|v^{(N+1)}(t)\right\|_{B_1},\\
H &= \frac{(2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6})\pi}{C}\left\|v^{(N+1)}\right\|_{B_1}+(2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6})\left\|a^{(N+1)}\right\|_{C[0,T]}+(2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6}).
\end{aligned}
$$

$v^{(N+1)}\to u$, $a^{(N+1)}\to a$, $N\to\infty$.

For the uniqueness, let (u,a), (v,b) are two solutions of (1.1)-(1.4). If we use the same approaches to $|u(t)-v(t)|$ and $|a(t)-b(t)|$, we have:

$$
\begin{aligned}
\|u(t)-v(t)\|_{B_1} &\le (2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6})\|a(t)-b(t)\|_{C[0,T]}\|u(t)\|_{B_1}\\
&\quad+(2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6})\left(\int_0^t\int_0^\pi b^2(\xi,\tau)\left|u(\tau)-v(\tau)\right|^2 d\xi d\tau\right)^{\frac{1}{2}},
\end{aligned}
$$

$$
\begin{aligned}
\|a(t)-b(t)\|_{C[0,T]} &\le \frac{\pi}{\sqrt{6}E(t)B}\|a(t)\|_{C[0,T]}\|u_{xx}(t)-v_{xx}(t)\|_{B_1}\\
&\quad+\frac{\pi}{BE(t)}\|b(x,t)\|_{L_2(\Omega)}\|u(t)-v(t)\|_{B_1},
\end{aligned}
$$

$$
\begin{aligned}
\|u(t)-v(t)\|_{B_1} &\le \left[(2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6})\frac{\pi}{MBE(t)}\|a(t)\|_{C[0,T]}\|u(t)\|_{B_1}+\frac{(2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6})}{M}\right]\times\\
&\quad\left(\int_0^t\int_0^\pi b^2(\xi,\tau)\left|u(\tau)-v(\tau)\right|^2 d\xi d\tau\right)^{\frac{1}{2}}. \qquad (2.7)
\end{aligned}
$$

If Gronwall inequality is applied to (2.7) , $u(t)=v(t)$ then $a(t)=b(t)$. □

The proof is completed.

3. Stability of the solution (a,u)

Theorem 3.1. *If the condition (S1)-(S3) are implemented then the solution (a,v) of the problem depends continuously on φ,ψ,E.*

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Proof. Suppose $\Psi = \{\varphi, \psi,\ E,\ f\}$ and $\overline{\Psi} = \left\{\overline{\varphi}, \overline{\psi},\ \overline{E},\ f\right\}$. For $M_i,\ i = 0, 1, 2$(positiveconstants) such that

$$\begin{aligned} \|E\|_{C^2[0,T]} &\leq M_0, \left\|\overline{E}\right\|_{C^2[0,T]} \leq M_0, \|\varphi\|_{C^3[0,\pi]} \leq M_1, \|\overline{\varphi}\|_{C^3[0,\pi]} \leq M_1, \\ \|\psi\|_{C^1[0,\pi]} &\leq M_2, \left\|\overline{\psi}\right\|_{C^1[0,\pi]} \leq M_2. \end{aligned}$$

Let us show $\|\Psi\| = (\|E\|_{C^2[0,T]} + \|\varphi\|_{C^3[0,\pi]} + \|\psi\|_{C^1[0,\pi]})$. Let (a, v) and $(\overline{a}, \overline{v})$ are solutions of problems (1.1)-(1.4) corresponding to the data $\Psi = \{\varphi, \psi,\ E,\ f\}$ and $\overline{\Psi} = \left\{\overline{\varphi}, \overline{\psi},\ \overline{E},\ f\right\}$ respectively.

$$\begin{aligned} \|v(t) - \overline{v}(t)\|_{B_1} \leq\ & \|\varphi - \overline{\varphi}\|_{C^3[0,\pi]} + \frac{\pi^2}{24}\left\|\psi - \overline{\psi}\right\|_{C^1[0,\pi]} + (2\sqrt{\frac{T^3}{3\pi}} + \frac{\pi\sqrt{T}}{6})\|a(t) - \overline{a}(t)\|_{C[0,T]}\|u(t)\|_{B_1} \\ & +(2\sqrt{\frac{T^3}{3\pi}} + \frac{\pi\sqrt{T}}{6})\|v(t) - \overline{v}(t)\|_{B_1}\|\overline{a}(t)\|_{C[0,T]} \\ & +(2\sqrt{\frac{T^3}{3\pi}} + \frac{\pi\sqrt{T}}{6})\|v(t) - \overline{v}(t)\|_{B_1}\|b(x,t)\|_{L_2(\Omega)}, \end{aligned}$$

$$\begin{aligned} \|a(t) - \overline{a}(t)\|_{C[0,T]} \leq\ & \left(\frac{1}{1 - \frac{\pi}{\sqrt{6}E(t)}}\right)\left(\|\varphi - \overline{\varphi}\|_{C^3[0,\pi]} + \left\|\psi - \overline{\psi}\right\|_{C^1[0,\pi]}\right) \\ & + \left(\frac{1}{1 - \frac{\pi}{\sqrt{6}E(t)}}\right)\|v(t) - \overline{v}(t)\|_{B_1}\|\overline{a}(t)\|_{C[0,T]} \\ & + \frac{\pi}{6E(t)}\|v(t) - \overline{v}(t)\|_{B_1}\|b(x,t)\|_{L_2(\Omega)}, \end{aligned}$$

where

$$\begin{aligned} \left\|\Psi - \overline{\Psi}\right\| =\ & \|\varphi - \overline{\varphi}\|_{C^3[0,\pi]} + \frac{\pi^2}{24}\left\|\psi - \overline{\psi}\right\|_{C^1[0,\pi]} \\ & + \left(\frac{1}{1 - \frac{\pi}{\sqrt{6}E(t)}}\right)\left(\|\varphi - \overline{\varphi}\|_{C^3[0,\pi]} + \left\|\psi - \overline{\psi}\right\|_{C^1[0,\pi]}\right). \end{aligned}$$

where

$$\begin{aligned} M_3 &= (2\sqrt{\frac{T^3}{3\pi}} + \frac{\pi\sqrt{T}}{6})/1 - \frac{\pi}{\sqrt{6}E(t)}, \\ M_4 &= \left((2\sqrt{\frac{T^3}{3\pi}} + \frac{\pi\sqrt{T}}{6})/1 - \frac{\pi}{\sqrt{6}E(t)}\right)\|u(\tau)\|_{B_1}, \end{aligned}$$

$$\begin{aligned} \|v - \overline{v}\|_{B_1}^2 \leq\ & 2M_3^2\left\|\Psi - \overline{\Psi}\right\|^2 \\ & \times \exp 2M_4^2\left(\int_0^t\int_0^\pi b^2(\xi,\tau)d\xi d\tau\right). \end{aligned}$$

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For$\Psi \to \overline{\Psi}$ then $v \to \overline{v}$. Hence $a \to \overline{a}$. □

4. Numerical Method for Euler-Bernoulli Problem

We obtain the following problem after linearization:

$$\frac{\partial^2 v^{(n)}}{\partial t^2} + \frac{\partial^4 v^{(n)}}{\partial x^4} - a(t)v^{(n)} = g(x,t,v^{(n-1)}),\ (x,t) \in\ \Omega, \tag{4.1}$$

$$\begin{aligned} v^{(n)}(0,t) &= v^{(n)}(\pi,t), \\ v_x^{(n)}(0,t) &= v_x^{(n)}(\pi,t), \\ v_{xx}^{(n)}(0,t) &= v_{xx}^{(n)}(\pi,t), \\ v_{xxx}^{(n)}(0,t) &= v_{xxx}^{(n)}(\pi,t), t \in [0,T]\,, \end{aligned} \tag{4.2}$$

$$v^{(n)}(x,0) = \varphi(x), v_t^{(n)}(x,0) = \psi(x)\ , x \in [0,\pi]\,, \tag{4.3}$$

$$E(t) = \int_0^\pi x v^{(n)}(x,t)dx,\ t \in [0,T]\,. \tag{4.4}$$

Let $v^{(n)}(x,t) = w(x,t)$ and $g(x,t,v^{(n-1)}) = \widetilde{g}(x,t)$. Then the problem:

$$\frac{\partial^2 w}{\partial t^2} + \frac{\partial^4 w}{\partial x^4} - a(t)w = \widetilde{g}(x,t),\ (x,t) \in\ \Omega, \tag{4.5}$$

$$\begin{aligned} w(0,t) &= w(\pi,t), \\ w_x(0,t) &= w_x(\pi,t), \\ w_{xx}(0,t) &= w_{xx}(\pi,t), \\ w_{xxx}(0,t) &= w_{xxx}(\pi,t),\ t \in [0,T]\,, \end{aligned} \tag{4.6}$$

$$w(x,0) = \varphi(x), w_t(x,0) = \psi(x),\ x \in [0,\pi]\,, \tag{4.7}$$

$$E(t) = \int_0^\pi x w(x,t)dx,\ t \in [0,T]\,. \tag{4.8}$$

We use finite-difference method for numerical approximation (4.5)-(4.8):

$$\frac{1}{\tau^2}\left(w_i^{j+1} - 2w_i^j + w_i^{j-1}\right) + \frac{1}{h^4}\left(w_{i+2}^j - 4w_{i+1}^j + 6w_i^j - 4w_{i-1}^j + w_{i-2}^j\right) = a^j w_i^j + \widetilde{f}_i^j$$

$$w_i^0 = \phi_i, \frac{1}{\tau}\left(v_i^1 - v_i^0\right) = \psi_i \tag{4.9}$$

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$$w_0^j = v_{N_x+1}^j, \tag{4.10}$$

$$w_1^j = v_{N_x+2}^j, \tag{4.11}$$

$$w_{-1}^j = v_{N_x}^j, \tag{4.12}$$

$$w_2^j - w_{-2}^j = w_{N_x+3}^j - w_{N_x-1}^j. \tag{4.13}$$

The region $[0,\pi] \times [0,T]$ is divided into an $N_x \times N_t$ mesh with the spatial step size $h = \pi/N_x$ in x direction and the time step size $\tau = T/N_t$, respectively.

Grid points x_i, t_j are defined by
$x_i = ih$; $i = 0; 1; 2; ...; N_x$;
$t_j = j\tau$; $k = 0; 1; 2; ...; N_t$;
$w_i^j = w(x_i, t_j)$, $g_i^j = \widetilde{g}(x_i, t_j)$, $a^j = a(t_j)$.

Integrate (4.5) with respect to x from 0 to π and use (4.6) and (4.7), we obtain

$$a(t) = \frac{1}{E(t)} \left[E''(t) + \pi w_{xxx}(\pi, t) - \int_0^\pi x\widetilde{g}(x,t)dx \right]. \tag{4.14}$$

The discretization of (4.14) is

$$a^{j+1} = \frac{\left[\left(\left(E^{j+1} - 2E^j + E^{j-1} \right) / \tau^2 \right) + \pi \left(w_{N_x+3}^j - 2w_{N_x+2}^j + 2w_{N_x}^j - w_{N_x-1}^j \right) - \left(\int_0^\pi x\widetilde{g}_i^j dx \right) \right]}{E^j},$$

where $E^j = E(t_j)$, $j = 0, 1, ..., N_t$.

We mention that the integral is numerically calculated using trapezoidal rule.

$a^{j(s)}$, $w_i^{j(s)}$ are the s-th iteration step of a^j, w_i^j , respectively. At each s-th iteration step, $a^{j(s)}$ is

$$a^{j(s)} = \frac{\left[\left(\left(E^{j+1(s+1)} - 2E^{j(s)} + E^{j-1(s-1)} \right) / \tau^2 \right) + \pi \left(w_{N_x+3}^{j(s)} - 2w_{N_x+2}^{j(s)} + 2w_{N_x}^{j(s)} - w_{N_x-1}^{j(s)} \right) - \left(\int_0^\pi x\widetilde{g}_i^{j(s)} dx \right) \right]}{E^j}.$$

The iteration of (4.9)-(4.13) is

$$\frac{1}{\tau^2} \left(w_i^{j+1(s+1)} - 2w_i^{j(s)} + w_i^{j-1(s-1)} \right) + \frac{1}{h^4} \left(w_{i+2}^{j(s)} - 4w_{i+1}^{j(s)} + 6w_i^{j(s)} - 4w_{i-1}^{j(s)} + w_{i-2}^{j(s)} \right) = a^{j(s)} w_i^{j(s)} + \widetilde{g}_i^{j(s)} \tag{4.15}$$

$$w_i^0 = \phi_i, \frac{1}{\tau} \left(w_i^1 - w_i^0 \right) = \psi_i \tag{4.16}$$

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$$w_0^{j(s)} = w_{N_x+1}^{j(s)}, \tag{4.17}$$

$$w_1^{j(s)} = w_{N_x+2}^{j(s)}, \tag{4.18}$$

$$w_{-1}^{j(s)} = w_{N_x}^{j(s)}, \tag{4.19}$$

$$w_2^{j(s)} - w_{-2}^{j(s)} = w_{N_x+3}^{j(s)} - w_{N_x-1}^{j(s)}, \tag{4.20}$$

System (4.16)-(4.20) is solved and $w_i^{j+1(s+1)}$ is determined.

Example 4.1. The analytical solution is

$$\{a(t),\ v(x,t)\} = \{1 + \exp(t),\ \ (1 + \cos 2x)\exp(t)\},$$

for the given functions

$$\begin{aligned} \varphi(x) &= (1 + \cos 2x),\ E(t) = \frac{\pi^2}{2}\exp(t), \\ g(x,t,v) &= (16\cos 2x - v)\exp(t). \end{aligned}$$

Here $h = 0.0393$, $\tau = 0.005$.

$\left|a^{k+1(s+1)} - a^{k+1(s)}\right| \le h/100$ is the convergence criterion for $a(t)$.

The analytical solution and the approximate solution can be seen Figures 1 and 2 when the last time $T = 2$.

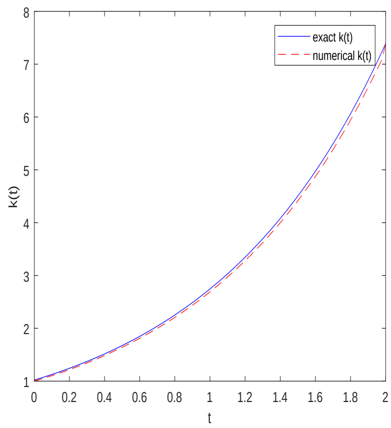

Figure 1. The exact and approximate solutions of a(t). The approximate solution is shown with dashed line.

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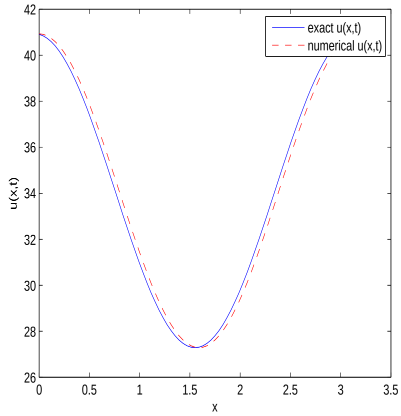

FIGURE 2. The exact and approximate solutions of u(x,2). The approximate solution is shown with dashed line.

EXAMPLE 4.2. (discontinuous coefficient)

In Example 1, a continuous function is given. Now, a more severe discontinuous function is considered:

$$a(t) = \begin{cases} 1, t \in [0,1) \\ -1, t \in [1,2] \end{cases}$$

The step sizes are $h = 0.0393$, $\tau = 0.005$. We obtain Figures 3 which shows the analytical and the approximate solutions of $a(t)$ when the last time $T = 2$.

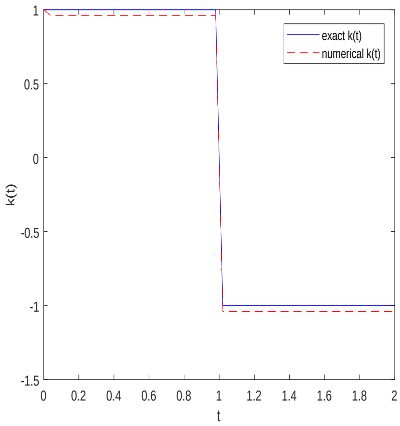

FIGURE 3. The exact and approximate solutions of a(t). The approximate solution is shown with dashed line.

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