

## Spaceability on Morrey Spaces

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ABSTRACT. In this paper, as a main result for Morrey spaces, we prove that the set  $\mathcal{M}_q^p(\mathbb{R}^n) \setminus \bigcup_{q < r \leq p} \mathcal{M}_r^p(\mathbb{R}^n)$  is spaceable in  $\mathcal{M}_q^p(\mathbb{R}^n)$ , where  $0 < q < p < \infty$ .

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### 1. INTRODUCTION

Let  $0 < q \leq p < \infty$ . Define the *Morrey (quasi-)norm*  $\|\cdot\|_{\mathcal{M}_q^p}$  by

$$\|f\|_{\mathcal{M}_q^p} \equiv \sup \left\{ |Q|^{\frac{1}{p} - \frac{1}{q}} \|f\|_{L^q(Q)} : Q \text{ is a cube in } \mathbb{R}^n \right\}$$

for a measurable function  $f$ , where a cube is defined to be the set of the form  $\{a + y : a \in \mathbb{R}^n, y \in [0, t]^n\}$  for some  $a \in \mathbb{R}^n$  and  $t > 0$ . The *Morrey space*  $\mathcal{M}_q^p$  is the set of all measurable functions  $f$  for which  $\|f\|_{\mathcal{M}_q^p}$  is finite. Morrey spaces date back to 1938, when C.B. Morrey considered elliptic differential equations and discussed continuity of the solutions using a lemma [7]. His lemma was refined by Peetre [8]. Morrey's lemma gave rise to the theory of function spaces;

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see [8] and [10]. The function spaces dealt with are called Morrey spaces. Let  $0 < q < p < \infty$ . For each  $r \in (q, p]$  the set  $\mathcal{M}_r^p$  is a proper subset of  $\mathcal{M}_q^p$  [4, 9]. In this paper, we show that the difference of these two sets is large enough. Precisely, by a technical lemma we prove:

**Theorem 1.1.** *If  $0 < q < p$ , then  $\mathcal{M}_q^p \setminus \bigcup_{q < r \leq p} \mathcal{M}_r^p$  is a spaceable subset of  $\mathcal{M}_q^p$ .*

Another different proof will also be given via [6, Theorem 3.3]. We recall that a subset  $S$  of a topological vector space  $X$  is called *spaceable* if  $S \cup \{0\}$  contains a closed infinite-dimensional linear subspace of  $X$ . The concepts spaceability and lineability were introduced by the paper [1] and then have been studied on different kinds of function or sequence spaces (see [5, 6]). In particular, for spaceability of the difference of Lebesgue spaces (see [2, 3, 11]).

## 2. MAIN RESULT

Let  $p > q > 0$  and  $R > 1$  be fixed so that

$$(R+1)^{-\frac{1}{p}} = 2^{\frac{1}{q}}(1+R)^{-\frac{1}{q}}. \quad (2.1)$$

For a vector  $\varepsilon \in \{0, 1\}^n$ , we define an affine transformation  $T_\varepsilon$  by

$$T_\varepsilon(x) \equiv \frac{1}{R+1}x + \frac{R}{R+1}\varepsilon \quad (x \in \mathbb{R}^n).$$

Let  $E_0 := [0, 1]^n$ . Suppose that we have defined  $E_0, E_1, E_2, \dots, E_j$ . Define

$$E_{j+1} := \bigcup_{\varepsilon \in \{0, 1\}^n} T_\varepsilon(E_j)$$

and

$$E_{j,0} := [0, (1+R)^{-j}]^n.$$

The following technical lemma is proved in [10, Proposition 4.1]. Here, for the sake of convenience of readers, we reproduce the proof.

**Lemma 2.1.** *Under above notations we have*

$$\|\chi_{E_j}\|_{\mathcal{M}_q^p} \sim (1+R)^{-j\frac{n}{p}} = \|\chi_{E_{j,0}}\|_{\mathcal{M}_q^p} = \|\chi_{E_{j,0}}\|_p = \|\chi_{E_j}\|_q, \quad (2.2)$$

where the implicit constants in  $\sim$  does not depend on  $j$  but can depend on  $p$  and  $q$  and  $\|\cdot\|_{\mathcal{M}_q^p}$  is the Morrey norm.

*Proof.* A direct calculation shows that

$$\|\chi_{E_j}\|_{\mathcal{M}_q^p} \geq \|\chi_{E_{j,0}}\|_{\mathcal{M}_q^p} = (1+R)^{-j\frac{n}{p}} = \|\chi_{E_{j,0}}\|_p = \|\chi_{E_j}\|_q,$$

Thus, we need to show

$$\|\chi_{E_j}\|_{\mathcal{M}_q^p} \lesssim \|\chi_{E_{j,0}}\|_p.$$

Let us calculate

$$\|\chi_{E_j}\|_{\mathcal{M}_q^p} \sim \sup_{S \in \mathcal{Q}} |S|^{\frac{1}{p}-\frac{1}{q}} |S \cap E_j|^{\frac{1}{q}},$$

where  $\mathcal{Q}$  denotes the set of all cubes. Fix  $j \in \mathbb{N}$ . Let us temporarily say that  $Q \in \mathcal{Q}$  is wasteful, if there exists a cube  $S \in \mathcal{Q}$  such that

$$|Q|^{\frac{1}{p}-\frac{1}{q}}|Q \cap E_j|^{\frac{1}{q}} < |S|^{\frac{1}{p}-\frac{1}{q}}|S \cap E_j|^{\frac{1}{q}}.$$

Thus by definition, if  $\ell(Q) := |Q|^{\frac{1}{n}} > 1$ , then

$$|Q|^{\frac{1}{p}-\frac{1}{q}}|Q \cap E_j|^{\frac{1}{q}} < |E_j|^{\frac{1}{q}} = |[0, 1]^n|^{\frac{1}{p}-\frac{1}{q}}|[0, 1]^n \cap E_j|^{\frac{1}{q}}.$$

In addition, if the side-length of a cube  $Q$  is less than  $(R+1)^{-j}$ , then it is wasteful. Indeed, then the equalities

$$\begin{aligned} & \sup \left\{ |Q|^{\frac{1}{p}-\frac{1}{q}}|Q \cap E_j|^{\frac{1}{q}} : Q \in \mathcal{Q}, |Q| \leq (R+1)^{-jn} \right\} \\ &= \sup \{ |Q|^{\frac{1}{p}-\frac{1}{q}}|Q \cap E_j|^{\frac{1}{q}} : Q \in \mathcal{Q}, Q \subset E_j \} \\ &= \sup \left\{ |Q|^{\frac{1}{p}} : Q \in \mathcal{Q}, Q \subset E_j \right\} \\ &= |E_{j,0}|^{\frac{1}{p}} \end{aligned}$$

hold. Here, we obtain the first equality by translating  $Q$  so that  $Q$  which is included in  $E_{j,0}$ . This calculation shows that

$$|Q|^{\frac{1}{p}-\frac{1}{q}}|Q \cap E_j|^{\frac{1}{q}} < |E_{j,0}|^{\frac{1}{p}}$$

for any such cube. Thus, if the cube  $Q$  is not wasteful, then  $(R+1)^{-j} \leq \ell(Q) \leq 1$ . So, there exists  $k \in \{1, 2, \dots, n\}$  such that  $(R+1)^{-kn} \leq |Q| \leq (R+1)^{-(k-1)n}$ . In this case, since any connected component  $P$  of  $E_k$  satisfies  $(R+1)^{-kn} = |P| \leq |Q| \leq (R+1)^{-(k-1)n}$ ,  $3Q$  contains a connected component of  $E_k$ . Hence, it follows that

$$\begin{aligned} & \|\chi_{E_j}\|_{\mathcal{M}_q^p} \\ & \sim \sup \{ |Q|^{\frac{1}{p}-\frac{1}{q}}|Q \cap E_j|^{\frac{1}{q}} : Q \text{ contains a connected component of } E_j \}. \end{aligned}$$

Let  $S$  be a cube which contains a connected component of  $E_j$  and is not wasteful. By symmetry, we may assume  $S = I \times I \times \dots \times I$  for some interval  $I$ . We define

$$S^* := \text{co} \left( \bigcup \{W : W \text{ is a connected component of } E_j \text{ intersecting } S\} \right),$$

where  $\text{co}(A)$  stands for the smallest convex set containing a set  $A$ . Then a geometric observation shows that  $S^*$  engulfs  $k^n$  connected component of  $E_j$  for some  $1 \leq k \leq 2^j$ . Take an integer  $l$  such that  $2^{l-1} \leq k \leq 2^l$ . Then we have

$$|S^* \cap E_j| = k^n(1+R)^{-jn}, \quad |S^*| \sim (1+R)^{-jn+ln}.$$

Consequently, from (2.1) we have

$$\begin{aligned} |S^{*|\frac{1}{p}-\frac{1}{q}}|S^{*} \cap E_j|^{\frac{1}{q}} &\sim 2^{\frac{ln}{q}}(1+R)^{-\frac{jn}{q}}(1+R)^{(-j+l)(\frac{n}{p}-\frac{n}{q})} \\ &= 2^{\frac{ln}{q}}(1+R)^{-j\frac{n}{p}+ln(\frac{1}{p}-\frac{1}{q})} \\ &= 2^{\frac{ln}{q}}(1+R)^{ln(\frac{1}{p}-\frac{1}{q})}(1+R)^{-j\frac{n}{p}} = (1+R)^{-j\frac{n}{p}}. \end{aligned}$$

Therefore, (2.2) is obtained.  $\square$

So, we can say that the Morrey norm  $\|\cdot\|_{\mathcal{M}_q^p}$  reflects local regularity of the functions more precisely than the Lebesgue norm  $\|\cdot\|_p$ .

A chain of equalities in (2.2) is the motivation of choosing  $R$  in (2.1).

*Remark 2.2.* In Lemma 2.1, if one defines  $F_j := \{x \in \mathbb{R}^n : (R+1)^{-j}x \in E_j\}$ , then  $\{F_j\}_{j=1}^{\infty}$  is an increasing sequence of sets and each  $F_j$  is made up of disjoint union of cubes of length 1,  $\|\chi_{F_j}\|_{\mathcal{M}_q^p} \sim 1$ , and each component of  $F_j$  is a cube of size 1.

**Proposition 2.3.** *Let  $0 < q < r \leq p$ . Then  $\|\chi_{F_j}\|_{\mathcal{M}_r^p} \geq (\frac{1+R}{2})^{(\frac{n}{q}-\frac{n}{r})j}$ , and so that  $\lim_{j \rightarrow \infty} \|\chi_{F_j}\|_{\mathcal{M}_r^p} = \infty$ .*

*Proof.* Simply use

$$\|\chi_{F_j}\|_{\mathcal{M}_r^p} \geq (1+R)^{\frac{jn}{p}-\frac{jn}{r}}|F_j|^{\frac{1}{r}} = (1+R)^{\frac{jn}{q}-\frac{jn}{r}}|F_j|^{\frac{1}{r}-\frac{1}{q}} = \left(\frac{1+R}{2}\right)^{\left(\frac{n}{q}-\frac{n}{r}\right)j}.$$

$\square$

Now, we prove the main result of this paper.

*Proof of Theorem 1.1.* Under above notations, put

$$F := \bigcup_{j=1}^{\infty} F_j. \quad (2.3)$$

Then  $\chi_F \in \mathcal{M}_q^p$ . For each  $j = 1, 2, 3, \dots$  let us define

$$g_j(x) := \chi_F((R+1)^{-j-1}x) - \chi_F((R+1)^{-j}x) \quad (x \in \mathbb{R}^n),$$

and

$$h_j(x) := \chi_F(R^{-j-1}x) \quad (x \in \mathbb{R}^n).$$

Then by Lemma 2.1 we have  $\|g_j\|_{\mathcal{M}_q^p} \sim \|h_j\|_{\mathcal{M}_q^p} < \infty$ , while  $\|g_j\|_{\mathcal{M}_r^p} \sim \|h_j\|_{\mathcal{M}_r^p} = \infty$  for all  $q < r \leq p$ . We set

$$V := \left\{ \sum_{j=1}^{\infty} \lambda_j g_j : \text{for all } j, \lambda_j \in \mathbb{C} \right\} \cap \mathcal{M}_q^p.$$

Since the convergence with norm of  $\mathcal{M}_q^p$  is stronger than the almost everywhere convergence,  $V$  is closed in  $\mathcal{M}_q^p$ . Also, note that

$$V \cap \bigcup_{q < r \leq p} \mathcal{M}_r^p = \{0\},$$

since the characteristic function  $\chi_F$  belongs to the set  $\mathcal{M}_q^p \setminus \bigcup_{q < r \leq p} \mathcal{M}_r^p$ . Therefore,  $\mathcal{M}_q^p \setminus \bigcup_{q < r \leq p} \mathcal{M}_r^p$  is spaceable in  $\mathcal{M}_q^p$ .  $\square$

Here is an alternative proof of Theorem 1.1 with  $q \geq 1$  using a result by Kitson and Timoney [6]. We invoke this result from [6].

**Theorem 2.4.** [6, Theorem 3.3]. *Let  $Z_m$  ( $m \in \mathbb{N}$ ) be Banach spaces and  $X$  a Fréchet space. Let  $T_m : Z_m \rightarrow X$  be continuous linear operators and  $Y$  the linear span of  $\bigcup_{m=1}^{\infty} T_m(Z_m)$ . If  $Y$  is not closed in  $X$ , then the complement  $X \setminus Y$  is spaceable.*

For a different proof of Theorem 1.1, simply apply Theorem 2.4 with

$$X = \mathcal{M}_q^p, \quad Z_m = \mathcal{M}_{q+\frac{p-q}{m}}^p, \quad T_m = \text{inclusion mapping from } Z_m \text{ to } \mathcal{M}_q^p.$$

We will check the following property of the Morrey space  $\mathcal{M}_q^p$  to see that Theorem 2.4 is applicable.

**Proposition 2.5.** *Let  $0 < q < p < \infty$ . Then  $\bigcup_{q < r \leq p} \mathcal{M}_r^p$  is not closed in  $\mathcal{M}_q^p$ .*

We relate the case  $0 < q < 1$  here. Once this is shown, Theorem 1.1 will be proved with the help of the aforementioned theorem.

*Proof.* Let  $F_j$  be as in Remark 2.2. For each  $k \in \mathbb{N}$  define

$$f_k := \sum_{j=1}^k \frac{1}{j^2 \|\chi_{F_j}\|_{\mathcal{M}_q^p}} \chi_{F_j}.$$

Then  $(f_k)_{k=1}^{\infty}$  is a Cauchy sequence in  $\mathcal{M}_q^p$ . So,  $(f_k)_{k=1}^{\infty}$  converges to a function  $f$  in  $\mathcal{M}_q^p$  since  $\mathcal{M}_q^p$  is a Banach space. Note that each  $f_k \in L^p \subset \mathcal{M}_r^p$  for all  $q < r \leq p$ . If  $f$  is a member in  $\mathcal{M}_r^p$  for some  $r \in (q, p]$ , then

$$\left\{ \frac{1}{j^2 \|\chi_{F_j}\|_{\mathcal{M}_q^p}} \chi_{F_j} \right\}_{j=1}^{\infty}$$

would form a bounded set in  $\mathcal{M}_r^p$  since  $\mathcal{M}_r^p$  enjoys the lattice property. This is a contradiction to Proposition 2.3.  $\square$

## 3. APPENDIX

The situation is different from the case of Lebesgue spaces.

**Proposition 3.1.** *Let  $0 < q < p < \infty$ . Then  $\bigcup_{q < r \leq p} \mathcal{M}_r^p$  is not dense in  $\mathcal{M}_q^p$ .*

*Proof.* Let  $F$  be as in (2.3). We prove that  $2\chi_F$  is not in the closure of  $\bigcup_{q < r \leq p} \mathcal{M}_r^p$  by showing that for every  $q < r \leq p$ ,  $f \notin \mathcal{M}_r^p$  if  $f \in \mathcal{M}_q^p$  satisfies  $\|2\chi_F - f\|_{\mathcal{M}_q^p} < 1$ . Indeed, if  $K$  is one of the connected components of  $F$ , then  $\|f\|_{L^r(K)} \geq \|f\|_{L^q(K)} > c_q = (2^{\min(1,q)} - 1)^{\frac{1}{\min(1,q)}}$  since

$$\begin{aligned} 1 &> (\|2\chi_F - f\|_{\mathcal{M}_q^p})^{\min(1,q)} \\ &\geq (\|2 - f\|_{L^q(K)})^{\min(1,q)} \\ &\geq 2^{\min(1,q)} - (\|f\|_{L^q(K)})^{\min(1,q)}. \end{aligned}$$

Thus,

$$\|f\|_{\mathcal{M}_r^p} \geq |[0, R^j]^n|^{\frac{1}{p} - \frac{1}{r}} \|f\|_{L^r([0, R^j]^n)} \geq c_q R^{\frac{jn}{p} - \frac{jn}{r}} |E_j|^{\frac{1}{r}} = c_q 2^{\frac{jn}{r} - \frac{jn}{q}} R^{\frac{jn}{q} - \frac{jn}{r}}$$

for all  $j \in \mathbb{N}$ . Hence,  $\|f\|_{\mathcal{M}_r^p} = \infty$ , or equivalently  $f \notin \mathcal{M}_r^p$ .  $\square$

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